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On Orbital Stability of Pendulum-like Satellite Rotations at the Boundaries of Stability Regions

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The motion of a rigid body satellite about its center of mass is considered. The problem of the orbital stability of planar pendulum-like rotations of the satellite is investigated. It is assumed that the satellite moves in a circular orbit and its geometry of mass corresponds to a plate. In unperturbed motion the minor axis of the inertia ellipsoid lies in the orbital plane.

A nonlinear analysis of the orbital stability for previously unexplored values of parameters corresponding to the boundaries of the stability regions is carried out. The study is based on the normal form technique. In the special case of fast rotations a normalization procedure is performed analytically. In the general case the coefficients of normal form are calculated numerically. It is shown that in the case under consideration the planar rotations of the satellite are mainly unstable, and only on one of the boundary curves there is a segment where the formal orbital stability takes place.

Keywords: satellite, rotations, orbital stability, Hamiltonian system, symplectic map, normal form, combinational resonance, resonance of essential type

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1. Introduction

We consider the motion of a satellite in the gravitational field. The satellite is modeled by a rigid body, whose size is small enough compared with the size of the orbit of its mass center. In this case it can be assumed that the satellite motion about its center of mass does not affect the motion of the mass center itself. We suppose that the satellite mass center moves in a circular orbit. Under the above assumptions the equations of motion of the satellite about its center of mass allow families of particular solutions corresponding to the planar pendulum-like motions, in which one of its principal axes of inertia is perpendicular to the plane of the orbit [1]. The planar motions represent either pendulum-like oscillations or rotations, or motions asymptotically approaching the unstable position of relative equilibrium. The above planar periodic motions are unstable with respect to perturbations of coordinates and velocities, but the problem of their orbital stability is of interest.

The problem of the orbital stability of the planar periodic satellite motions was repeatedly investigated earlier. In [2–4], comprehensive and rigorous analysis of the orbital stability of planar periodic motions were performed for the case of a dynamically symmetric satellite. The study of the orbital stability of the planar periodic motions of an unsymmetrical satellite is a more difficult problem. In linear approximation it was considered in [5–7]. The investigation of this problem in a rigorous nonlinear formulation was started in [8].

The most detailed stability analysis of planar periodic motions of an unsymmetrical satellite was performed when its geometry of mass corresponds to a plate [9–15]. In [9, 10] pendulum-like oscillations and rotations of such a satellite were considered under the assumption that the minor axis of the inertia ellipsoid is perpendicular to the orbital plane. In particular, a linear stability analysis of the above pendulum-like oscillations was carried out [9], and a nonlinear analysis of the pendulum-like rotations was performed [10]. In [11, 12], a rigorous study of the orbital stability of pendulum-like periodic oscillations and rotations of the satellite having the geometry of mass of a plate was performed under the assumption that in unperturbed motion the minor axis of the inertia ellipsoid lies in the orbital plane. For the values of the parameters lying inside the regions of stability in the first approximation, conclusions on stability for most initial conditions and formal stability were obtained. In [13] a rigorous analysis of the orbital stability of pendulum-like oscillations of the above satellite was performed at the boundaries of orbital stability regions, corresponding to the first and second order resonances of essential type. In [14, 15] the study of the orbital stability of planar oscillations was completed by rigorous analysis at the boundaries of regions of the stability in the first approximation, corresponding to the combinational resonance.

The purpose of this work is a rigorous study of the orbital stability of planar pendulum-like rotations of the satellite having the geometry of mass of a plate for the parameter values corresponding to the boundaries of the stability regions.

2. Statement of the problem

To describe the motion of the satellite about its center of mass, we introduce two coordinate systems: the orbital system $OXYZ$, whose axes OX , OY , OZ are directed along the radius vector of the center of mass O from the attracting center, along the transversal to the orbit and along the normal to the orbit plane; and the satellite-fixed coordinate system $Oxyz$, whose axes are directed along the principal inertia axes of the satellite. The orientation of the axes of the system $Oxyz$ in the orbital system $OXYZ$ is determined by the Euler angles ψ , θ , ϕ .

In what follows we assume that the principal moments of inertia J_x , J_y , J_z corresponding to the axes Ox , Oy , Oz satisfy the relation $J_x + J_y = J_z$, that is, the satellite has the geometry of mass of a plate. By introducing the corresponding dimensionless moments p_ψ , p_θ and p_φ , the equations of motion of the satellite about its center of mass can be written in canonical form with the Hamiltonian [1, 2]

$$\begin{aligned} H = & \frac{1}{2} \left[\frac{1}{\mu} (\sin^2 \varphi + \mu \cos^2 \varphi) \operatorname{ctg}^2 \theta + \frac{1}{\mu + 1} \right] p_\varphi^2 + \frac{\cos^2 \varphi + \mu \sin^2 \varphi}{2\mu} p_\theta^2 + \\ & + \frac{(\sin^2 \varphi + \mu \cos^2 \varphi)(p_\psi^2 - 2p_\psi p_\varphi \cos \theta)}{2\mu \sin^2 \theta} + \\ & + \frac{(\mu - 1) \sin 2\varphi}{2\mu \sin \theta} p_\theta (p_\varphi \cos \theta - p_\psi) - p_\psi + \frac{3}{2} [(\mu - 1)a_{11}^2 + \mu a_{13}^2], \end{aligned} \quad (2.1)$$

where a_{11} and a_{13} are calculated by the formulas

$$\begin{aligned} a_{11} &= \cos \psi \cos \varphi - \sin \psi \cos \theta \sin \varphi, \\ a_{13} &= \sin \psi \sin \theta \end{aligned}$$

and $\mu = J_x/J_y$. The true anomaly $\nu = \omega_0 t$ is the independent variable and ω_0 is an angular velocity of the radius vector of the satellite mass center.

The equations of motion allow a particular solution in which

$$\theta = \frac{\pi}{2}, \quad \varphi = 0, \quad p_\theta = p_\varphi = 0 \quad (2.2)$$

and the evolution of variables ψ and p_ψ is described by canonical equations with the Hamiltonian

$$H_0 = \frac{1}{2}(p_\psi - 1)^2 + \frac{3}{2} \sin^2 \psi. \quad (2.3)$$

This solution describes the planar pendulum-like motions of the satellite in which the minor axis of the inertia ellipsoid lies in the orbital plane [1]. If the value of the constant h of the energy integral $H_0 = h$ satisfies the inequality $h < 3/2$, then the solution of the system with Hamiltonian (2.3) describes planar pendulum-like oscillations with amplitude ψ_0 ($\psi_0 < \pi/2$). The orbital stability of the above planar oscillations was investigated in detail in [12–15].

In what follows we suppose that $h > 3/2$. In this case the system with Hamiltonian (2.3) has the following periodic solution:

$$\psi^*(\nu, k) = \operatorname{am} \left(\frac{\sqrt{3}\nu}{k}, k \right), \quad p_\psi^*(\nu, k) = 1 + \frac{\sqrt{3}}{k} \operatorname{dn} \left(\frac{\sqrt{3}\nu}{k}, k \right), \quad k^2 = \frac{3}{2h} \quad (2.4)$$

which describes the planar pendulum-like rotation of the satellite with an average angular velocity $\omega = \pi\sqrt{3}/(kK(k))$, where $K(k)$ is a complete elliptic integral of the first kind.

It is worth noticing that the direction of rotation for the satellite and for the radius vector of its mass center can be either the same or opposite. If the directions of their rotations are the same, then $\omega > 0$ ($k > 0$) and we say that the so-called *direct* rotations take place. Otherwise, if their rotations have opposite directions, then $\omega < 0$ ($k < 0$) and we say that we deal with *retrograde* rotations.

A rigorous orbital stability analysis of the above planar rotations of the satellite was carried out in [11] and the stability diagrams were obtained (see Figs. 1 and 2). It was established that

Fig. 1. Orbital stability domains in the linear approximation for direct rotations of the satellite.

Fig. 2. Orbital stability domains in the linear approximation for retrograde rotations of the satellite.

outside of domains D_1 , D_2 and D_3 the rotations are orbitally unstable. For the values of the parameters lying within the indicated regions, conclusions on the stability for most initial conditions, formal stability or instability were obtained. Nonlinear stability analysis at the boundaries of the stability regions was not performed.

In this paper, we study the orbital stability of the planar pendulum-like rotations of the satellite at the boundaries of the stability regions D_1 , D_2 , and D_3 . In particular, we consider the stability problem on boundary curves γ_1 , γ_2 , γ_3 , γ_5 and γ_7 . The boundary curves γ_4 , γ_6 correspond to a dynamically symmetric satellite. The orbital stability in this case has been studied in [2].

3. The Hamiltonian of the problem

Following [2, 8], in the neighborhood of the unperturbed periodic motion, we perform a canonical change of the variables $\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi \rightarrow \varphi, \theta, w, p_\varphi, p_\theta, I$ by the formulas

$$\psi = \psi^*(w/\omega, k), \quad p_\psi = p_\psi^*(w/\omega, k), \quad (3.1)$$

where $k = k(I)$ is the inverse function to

$$I = \frac{2\sqrt{3}}{\pi} [E(k) - (1 - k^2)K(k)]$$

and $E(k)$ is the complete elliptic integral of the second kind. In the variables w, I , the unperturbed motion is described by relations $I = I_0$, $w = \omega\nu + w_0$.

We introduce the perturbations q_1, q_2, p_1, p_2, r according to the formulas

$$\varphi = q_1, \quad \theta = \frac{\pi}{2} + q_2, \quad p_\varphi = p_1, \quad p_\theta = p_2, \quad I = I_0 + r.$$

Then the Hamiltonian of the perturbed motion takes the form [12]

$$H = H_2 + H_4 + O_6, \quad (3.2)$$

where

$$H_2 = r + \tilde{H}_2(q_1, q_2, p_1, p_2, w), \quad (3.3)$$

$$H_4 = \frac{1}{2\omega} \frac{\partial \omega}{\partial I_0} r^2 + r\Psi(q_1, q_2, p_1, p_2, w) + \tilde{H}_4(q_1, q_2, p_1, p_2, w), \quad (3.4)$$

and

$$\begin{aligned} \tilde{H}_2 = \frac{1}{\omega} \left\{ \frac{(1-\mu)\alpha_1}{2\mu} q_1^2 + \frac{3}{2} \alpha_2 q_1 q_2 + \frac{(1-\mu)p_\psi^*}{\mu} q_1 p_2 + \frac{\alpha_1 - 3\mu}{2} q_2^2 + \right. \\ \left. + p_\psi^* q_2 p_1 + \frac{p_1^2}{2(\mu+1)} + \frac{p_2^2}{2\mu} \right\}, \end{aligned} \quad (3.5)$$

$$\Psi = \frac{1}{\omega} \left\{ \frac{(1-\mu)}{2\mu} \frac{\partial \alpha_1}{\partial I_0} q_1^2 + \frac{3}{2} \frac{\partial \alpha_2}{\partial I_0} q_1 q_2 + \frac{(1-\mu)}{\mu} \frac{\partial p_\psi^*}{\partial I_0} q_1 p_2 + \frac{1}{2} \frac{\partial \alpha_1}{\partial I_0} q_2^2 + \frac{\partial p_\psi^*}{\partial I_0} q_2 p_1 \right\}, \quad (3.6)$$

$$\begin{aligned} \tilde{H}_4 = \frac{1}{\omega} \left\{ \frac{(\mu-1)\alpha_1}{6\mu} q_1^4 - \alpha_2 q_1^3 q_2 + \frac{2(\mu-1)p_\psi^*}{3\mu} q_1^3 p_2 - \frac{(\mu-1)(\alpha_1 - 3\mu)}{2\mu} q_1^2 q_2^2 - \right. \\ \left. - \frac{(\mu-1)p_\psi^*}{\mu} q_1^2 q_2 p_1 + \frac{(\mu-1)}{2\mu} q_1^2 p_2^2 - \frac{\alpha_2}{4} q_1 q_2^3 - \frac{(\mu-1)p_\psi^*}{2\mu} q_1 q_2^2 p_2 - \right. \\ \left. - \frac{(\mu-1)}{\mu} q_1 q_2 p_1 p_2 + \alpha_3 q_2^4 + \frac{5}{6} p_\psi^* q_2^3 p_1 + \frac{1}{2} q_2^2 p_1^2 \right\}, \end{aligned} \quad (3.7)$$

$$\alpha_1 = 3\mu \cos^2 \psi^* + p_\psi^{*2}, \quad \alpha_2 = (\mu-1) \sin 2\psi^*, \quad \alpha_3 = \frac{1}{2}\mu \sin^2 \psi^* + \frac{1}{3}p_\psi^{*2}. \quad (3.8)$$

The Hamiltonian (3.2) depends on two parameters: the inertial parameter μ and the average angular velocity ω .

The problem of orbital stability of planar rotations is equivalent to the stability problem for the system with the Hamiltonian (3.2) with respect to variables q_1, q_2, p_1, p_2, r .

Let us perform isoenergy reduction and consider the motion on the zero level $H = 0$, corresponding to the unperturbed motion. This motion is described by the system of the Whittaker equations

$$\frac{dq_j}{dw} = \frac{\partial K}{\partial p_j}, \quad \frac{dp_j}{dw} = -\frac{\partial K}{\partial q_j} \quad (j = 1, 2), \quad (3.9)$$

where w plays the role of the new independent variable.

For sufficiently small q_1, q_2, p_1, p_2 , we can obtain the following series expansion of the Hamiltonian function $K(q_1, q_2, p_1, p_2, w)$:

$$K = K_2 + K_4 + O_6, \quad (3.10)$$

where

$$K_2 = \tilde{H}_2, \quad K_4 = \frac{1}{2\omega} \frac{\partial \omega}{\partial I} \tilde{H}_2^2 - \tilde{H}_2 \Psi + \tilde{H}_4. \quad (3.11)$$

Functions $\Psi, \tilde{H}_2, \tilde{H}_4$ are from (3.5)–(3.8).

Thus, the problem of the orbital stability of the planar pendulum-like rotations of the satellite is equivalent to the stability problem for the equilibrium position of the reduced system with Hamiltonian (3.11).

4. Linear system

Let us consider a linear system with the Hamiltonian (3.5). Conclusions on the stability of the linear system can be obtained by an analysis of the roots of its characteristic equation. In particular, the stability of the linear system is possible only if all roots of its characteristic equation have a modulus equal to 1, otherwise the linear system is unstable [16].

By numerical integration it was established that at the boundaries γ_1 and γ_7 (denoted in Figs. 1 and 2 by dashed curves) a second-order resonance of combinational type is realized, that is, the characteristic equation of the linear system has the form

$$(\rho^2 - 2a\rho + 1)^2 = 0, \quad a = \frac{1}{4}a_1, \quad (4.1)$$

where a_1 is the trace of the monodromy matrix of the linear system. In this case Eq. (4.1) has two multiple, complex-conjugate roots with a modulus equal to 1.

At the boundaries γ_i ($i = 2, 3, 5$) a second-order resonance of essential type is realized, that is, the characteristic equation of the linear system has the form

$$(\rho + 1)^2 (\rho^2 - 2a\rho + 1) = 0, \quad a = \frac{1}{2}a_1 + 1, \quad (4.2)$$

and thus, it has a double root equal to -1 .

Numerical calculations have shown that the monodromy matrix of the linear system cannot be reduced to a diagonal form for all values of the parameters corresponding to the boundaries of the regions D_1, D_2 and D_3 . Thus, in spite of the fact that the necessary condition of stability is fulfilled, the linear system is unstable. This circumstance, however, does not mean an orbital instability of the planar periodic motions of the satellite in a nonlinear formulation of the problem. To obtain rigorous conclusions about orbital stability (or instability) for the boundary curves, it is necessary to perform a nonlinear analysis.

5. Nonlinear stability study

In order to perform a nonlinear stability analysis of Hamiltonian systems, it is necessary to construct [17, 18] a canonical change of variables $q_j, p_j \rightarrow X_j, Y_j$ ($j = 1, 2$) that transforms the reduced system (3.10) to a form convenient for the stability study, the so-called normal form.

In the case of a resonance of essential type, the Hamiltonian (3.10) can be transformed to the following normal form [13, 19]:

$$\mathcal{H} = \frac{\delta}{2} Y_1^2 + \sigma (X_2^2 + Y_2^2) + a_{40} X_1^4 + a_{22} X_1^2 (X_2^2 + Y_2^2) + a_{04} (X_2^2 + Y_2^2)^2 + \mathcal{H}^{(5)}. \quad (5.1)$$

In the case of a combinational resonance, the normal form of the Hamiltonian (3.10) reads [15, 20]

$$\begin{aligned} \mathcal{H} = & \frac{\delta}{2} (X_1^2 + X_2^2) + \lambda (X_1 Y_2 - X_2 Y_1) + A (Y_1^2 + Y_2^2)^2 + \\ & + (Y_1^2 + Y_2^2) [B (X_1 Y_2 - X_2 Y_1) + C (X_1^2 + X_2^2)] + \mathcal{H}^{(5)}. \end{aligned} \quad (5.2)$$

In (5.1) and (5.2) $\delta = \pm 1$, the other coefficients are real constant values.

Conclusions on the stability of the equilibrium position $q_i = p_i = 0$ of the normalized canonical system can be obtained on the basis of the coefficients of the Hamiltonian normal form.

In [19] it was proven that for the canonical system with Hamiltonian (5.1) the following conditions for stability and instability take place. *If $\delta a_{40} < 0$, then the equilibrium position of the system with Hamiltonian (5.1) is unstable in the sense of Lyapunov. If $\delta a_{40} > 0$, then the equilibrium position is stable when the terms up to the fourth degree inclusive are taken into account in the Hamiltonian (5.1). If both inequalities $\delta a_{40} > 0$ and $\delta \sigma > 0$ hold, then the equilibrium position is formally stable.*

For the canonical system with Hamiltonian (5.2) in [20] the following conditions for stability and instability were obtained. *If the coefficients of the normal form (5.2) satisfy the inequality $\delta A > 0$, then the equilibrium position is formally stable. Otherwise, if $\delta A < 0$, then the Lyapunov instability takes place.*

In the general case the coefficients of the normal forms (5.1) and (5.2) can be obtained only numerically. The application of the classical methods [17, 21] for this purpose leads to rather cumbersome calculations. In [22], a normalization method which is more convenient from an algorithmic point of view was proposed. This method is based on the construction of a symplectic map generated by the phase flow of a Hamiltonian system. Using this approach, a constructive algorithm was developed in [23, 24] for constructing the normal form of the Hamiltonian in the case of a resonance of essential type and in [15] for constructing the normal form of the Hamiltonian in the case of a combinational resonance. In this paper, the normalization is carried out on the basis of the above-mentioned algorithm. We briefly describe it here.

At the first step of the algorithm a canonical linear change of variables is performed

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \mathbf{N} \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}. \quad (5.3)$$

If the parameters of the problem correspond to the values of the resonance of essential type, then the matrix $\mathbf{N}$ has the form

$$\mathbf{N} = (\delta_1 c_1 \mathbf{u}, \delta_2 c_2 \mathbf{r}, c_1 \mathbf{v}, c_2 \mathbf{s}), \quad (5.4)$$

$$\begin{aligned} \delta_1 &= \text{sign}(\mathbf{u}^T \mathbf{I} \mathbf{v}), & \delta_2 &= \text{sign}(\mathbf{r}^T \mathbf{I} \mathbf{s}), \\ c_1 &= \frac{1}{\sqrt{|\mathbf{u}^T \mathbf{I} \mathbf{v}|}}, & c_2 &= \frac{1}{\sqrt{|\mathbf{r}^T \mathbf{I} \mathbf{s}|}}, \end{aligned} \quad (5.5)$$

where $\mathbf{r}$ and $\mathbf{s}$ denote, respectively, the real and the imaginary part of the eigenvector, which corresponds to the simple complex root $\rho = e^{i2\pi\lambda}$ of Eq. (4.2), and $\mathbf{u}$ and $\mathbf{v}$ denote the eigenvector and adjointed vector, which correspond to the multiple root $\rho = -1$.

In the case of the combinational resonance the matrix $\mathbf{N}$ has the form

$$\mathbf{N} = \sqrt{2} [-\delta(d_2\mathbf{r} + d_1\mathbf{v}), -\delta(d_1\mathbf{u} - d_2\mathbf{s}), d_1\mathbf{s}, d_1\mathbf{r}], \quad (5.6)$$

$$d_1 = \frac{1}{\sqrt{|g|}}, \quad d_2 = \frac{h\delta}{|g|^{(3/2)}}, \quad g = \mathbf{r}^T \mathbf{J} \mathbf{u} + \mathbf{s}^T \mathbf{J} \mathbf{v}, \quad h = \mathbf{u}^T \mathbf{J} \mathbf{v}, \quad \delta = \text{sign } g \quad (5.7)$$

and here $\mathbf{r}$ is a real part and $\mathbf{s}$ is an imaginary part of the eigenvector of the matrix $\mathbf{X}(2\pi)$, corresponding to the multiple root $\rho = e^{i2\pi\lambda}$ of (4.1); $\mathbf{u}$ is a real and $\mathbf{v}$ is an imaginary part of the adjointed vector.

At the next step we construct a symplectic map generated by the phase flow of the system with the new Hamiltonian $\mathbf{H}^*$, which is obtained by substituting the linear change of variables given by formulas (5.3) into the Hamiltonian function (3.10). Taking into account the fact that the expansion of Hamiltonian (3.10) does not include terms of the third degree, we can write the above map in the following form [15, 22, 23]:

$$\begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ P_1^{(1)} \\ P_2^{(1)} \end{pmatrix} = \mathbf{G} \begin{pmatrix} Q_1^{(0)} - \frac{\partial F_4}{\partial P_1^{(0)}} + O_4 \\ Q_2^{(0)} - \frac{\partial F_4}{\partial P_2^{(0)}} + O_4 \\ P_1^{(0)} + \frac{\partial F_4}{\partial Q_1^{(0)}} + O_4 \\ P_2^{(0)} + \frac{\partial F_4}{\partial Q_2^{(0)}} + O_4 \end{pmatrix}. \quad (5.8)$$

In the case of resonance of essential type, the matrix $\mathbf{G}$ has the form

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & \delta_1 & 0 \\ 0 & \cos 2\pi\sigma & 0 & \sin 2\pi\sigma \\ 0 & 0 & 1 & 0 \\ 0 & -\sin 2\pi\sigma & 0 & \cos 2\pi\sigma \end{pmatrix}, \quad (5.9)$$

where $\sigma = \delta_2\lambda$. In the case of the combinational resonance, the matrix $\mathbf{G}$ reads

$$\mathbf{G} = \begin{pmatrix} \cos 2\pi\lambda & -\sin 2\pi\lambda & 0 & 0 \\ \sin 2\pi\lambda & \cos 2\pi\lambda & 0 & 0 \\ -2\pi\delta \cos 2\pi\lambda & 2\pi\delta \sin 2\pi\lambda & \cos 2\pi\lambda & -\sin 2\pi\lambda \\ -2\pi\delta \sin 2\pi\lambda & -2\pi\delta \cos 2\pi\lambda & \sin 2\pi\lambda & \cos 2\pi\lambda \end{pmatrix}. \quad (5.10)$$

In both cases λ can be obtained from the equation $\cos 2\pi\lambda = a$. Recall that the value a is calculated by formulas (4.1) and (4.2) for combinational and essential resonances, respectively. F_4 denotes the form of the fourth degree

$$F_4(Q_1^{(0)}, Q_2^{(0)}, P_1^{(0)}, P_2^{(0)}) = \sum_{i_1+i_2+j_1+j_2=4} f_{i_1i_2j_1j_2} Q_1^{(0)i_1} Q_2^{(0)i_2} P_1^{(0)j_1} P_2^{(0)j_2}, \quad (5.11)$$

where $f_{i_1 i_2 j_1 j_2} = \varphi_{i_1 i_2 j_1 j_2}(2\pi)$. The functions $\varphi_{i_1 i_2 j_1 j_2}(t)$ are defined by solving the equations

$$\frac{d\varphi_{i_1 i_2 j_1 j_2}}{dt} = -g_{i_1 i_2 j_1 j_2} \quad (i_1 + i_2 + j_1 + j_2 = 4) \quad (5.12)$$

with the initial conditions $\varphi_{i_1 i_2 j_1 j_2}(0) = 0$.

The quantities $g_{i_1 i_2 j_1 j_2}$ on the right-hand sides of (5.12) are coefficients of the forms

$$G_4(U_1, U_2, V_1, V_2, t) = \sum_{i_1+i_2+j_1+j_2=4} g_{i_1 i_2 j_1 j_2} U_1^{i_1} U_2^{i_2} V_1^{j_1} V_2^{j_2}, \quad (5.13)$$

which have been obtained by substituting

$$\begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} = \mathbf{X}^*(t) \begin{pmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{pmatrix} \quad (5.14)$$

in the form $H_4^*(Q_1, Q_2, P_1, P_2)$. The matrix $\mathbf{X}^*(t)$ is a solution of the linear differential equation

$$\frac{d\mathbf{X}^*}{dt} = \mathbf{I} \mathbf{H}_2^* \mathbf{X}^* \quad (5.15)$$

with the initial condition $\mathbf{X}^*(0) = \mathbf{E}_4$. $\mathbf{H}_2^*$ denotes the Hessian matrix of the quadratic part H_2^* of the Hamiltonian H^* .

Thus, the coefficients of the form F_4 are obtained by numerical integration of the system of 51 equations (35 equations for $\varphi_{i_1 i_2 j_1 j_2}$ and 16 equations for the elements x_{jl}^* of the matrix $\mathbf{X}^*(t)$) on the interval $[0, 2\pi]$.

The coefficients of the normal forms (5.1) and (5.2) can be written in terms of the coefficients of the form (5.11) and values δ_1, δ_2 . In particular, the coefficients a_{40} and δ of the normal form (5.1) are calculated by the formulas

$$a_{40} = -\frac{\delta_2 f_{4000}}{8\pi^3}, \quad \delta = \delta_1 \delta_2. \quad (5.16)$$

The coefficient A of the normal form (5.2) is calculated by the formula

$$A = -\frac{1}{16\pi} (3 f_{0004} + f_{0022} + 3 f_{0040}). \quad (5.17)$$

When the coefficients of the normal form are known, then by using the conditions mentioned above, conclusions on the stability or instability can be obtained.

Calculations performed in accordance with the above-mentioned algorithm have shown that, for parameter values corresponding to the boundary curves γ_i ($i = 1, 2, 3, 5$), the planar pendulum-like rotations of the satellite are orbitally unstable. The boundary curve γ_7 , on which the combinational resonance is realized, is separated by a point M ($\omega = -9.2712$, $\mu = 3.571567$) in two parts (see Fig. 3). On the segment marked with dotted line the formal orbital stability of the planar pendulum-like rotations takes place, and on the segment marked by the dashed line the planar pendulum-like rotations are unstable. To provide a stability investigation at above-mentioned point M , additional analysis, based on the 6th-degree terms in the Hamiltonian (3.10), is required. In this paper, such an analysis was not performed.

Fig. 3. Orbital stability at the boundary curve γ_7 .

6. Nonlinear stability study in the case of fast rotations

In this section we consider the special case when the satellite moves with high angular velocity, that is, we suppose $|\omega| \gg 1$. In this case $|k| \ll 1$, hence in formulas (2.4) the modulus of elliptic integrals k can be regarded as a small parameter. This circumstance allows us to perform the stability study analytically.

In [11] the linear system with the Hamiltonian K_2 has been studied by the small parameter method and the boundaries of the regions D_2 , D_3 have been found in an approximate analytical form. In particular, for $k \ll 1$ the boundary curve γ_5 is given by the following asymptotic relation:

$$\mu = \frac{\sqrt{3}}{4k} + \frac{1}{2} + O(k). \quad (6.1)$$

The asymptotic relation describing the curve γ_7 for sufficiently small values of $|k|$ reads

$$\mu = -\frac{0.736911655}{k} - 0.44620722656 + O(k). \quad (6.2)$$

Let us start from the stability study on the curve γ_5 . To this end we substitute (6.1) into (3.10), perform the change of variables $q_j = \sqrt{k} x_j$, $p_j = \frac{y_j}{\sqrt{k}}$, $j = 1, 2$ and expand the Hamiltonian (3.10) in a power series with respect to k . Then the terms K_2 and K_4 take the form

$$K_2 = K_2^{(0)} + k K_2^{(1)} + O(k^2), \quad (6.3)$$

where

$$K_2^{(0)} = -\frac{\sqrt{3}}{2} (x_1^2 - x_2^2) - (y_2 x_1 - y_1 x_2), \quad (6.4)$$

$$K_2^{(1)} = \frac{3}{16} (x_1^2 - x_2^2) \cos 2w + \frac{13}{16} (x_1^2 + x_2^2) + \frac{2}{3} (y_1^2 + y_2^2) + \frac{\sqrt{3}}{3} (3x_1 y_2 + x_2 y_1) + \frac{3}{8} x_2 x_1 \sin 2w, \quad (6.5)$$

and

$$K_4 = k^2 K_4^{(2)} + k^3 K_4^{(3)} + O(k^4), \quad (6.6)$$

where

$$K_4^{(2)} = -\frac{\sqrt{3}}{24} (5x_1^4 + x_2^4) - \frac{1}{6} (2x_1^3 y_2 + x_2^3 y_1) + \frac{\sqrt{3}}{4} x_1^2 x_2^2 + \frac{1}{2} x_1 x_2^2 y_2, \quad (6.7)$$

$$\begin{aligned} K_4^{(3)} = & \frac{1}{96} (157x_1^4 - 17x_2^4) + \frac{1}{3} y_1^2 (x_1^2 - x_2^2) + x_1^2 \left(\frac{3}{16} x_2^2 + y_2^2 \right) + \\ & + \frac{\sqrt{3}}{18} (27x_1^3 y_2 + 3x_1^2 x_2 y_1 + 6x_1 x_2^2 y_2 - 4x_2^3 y_1) - \\ & - \frac{1}{32} (x_1^4 + x_2^4) \cos 2w - \frac{1}{16} x_1 x_2 (x_1^2 + 4x_2^2) \sin 2w. \end{aligned} \quad (6.8)$$

Now we carry out a series of substitutions that allow us to normalize the quadratic part K_2 of the Hamiltonian. The first change of variables is given by the formulas

$$x_1 = \frac{(\xi_1 + \eta_2)}{3^{1/4}}, \quad x_2 = \frac{(\xi_2 + \eta_1)}{3^{1/4}}, \quad y_1 = -3^{1/4} \xi_2, \quad y_2 = -3^{1/4} \xi_1. \quad (6.9)$$

It normalizes the autonomous term $K_2^{(0)}$ of the quadratic part K_2 . After that, by means of a 2π -periodic substitution we reduce the normalized terms in K_2

$$\begin{aligned} \xi_1 &= Q_1 \cos w + P_1 \sin w, & \xi_2 &= Q_2 \cos w - P_2 \sin w, \\ \eta_1 &= -Q_1 \sin w + P_1 \cos w, & \eta_2 &= Q_2 \sin w + P_2 \cos w. \end{aligned} \quad (6.10)$$

Now by a π -periodic (in w) linear change of variables $Q_i, P_i \rightarrow u_i, v_i$ ($i = 1, 2$) we can eliminate the w from the quadratic part. This change of variables is given by the analytic (in k) generating function, whose series expansion up to terms of degree k reads

$$S(Q_1, Q_2, v_1, v_2) = Q_1 v_1 + Q_2 v_2 + k S_1 + O(k^2), \quad (6.11)$$

where

$$S_1 = \frac{\sqrt{3}}{12} \left[2(\cos 2w - 1)(Q_1 + v_2)(Q_2 + v_1) + ((Q_2 + v_1)^2 - (Q_1 + v_2)^2) \sin 2w \right]. \quad (6.12)$$

By performing one more substitution

$$\begin{aligned} u_1 &= \frac{3\sqrt[4]{3}\sqrt{7}}{14} \sqrt{k} U_1 - \frac{\sqrt[4]{7}}{14} U_2, & u_2 &= -\frac{3^{3/4}\sqrt{7}}{42} \frac{V_1}{\sqrt{k}} - \frac{3 \cdot 7^{3/4}}{14} V_2, \\ v_1 &= \frac{3 \cdot 3^{3/4}\sqrt{7}}{14} \frac{V_1}{\sqrt{k}} - \frac{7^{3/4}}{14} V_2, & v_2 &= \frac{\sqrt[4]{3}\sqrt{7}}{14} \sqrt{k} U_1 + \frac{9\sqrt[4]{7}}{14} U_2, \end{aligned} \quad (6.13)$$

we obtain the following form of the Hamiltonian quadratic part:

$$\widetilde{K}_2 = \frac{1}{2} V_1^2 + \frac{\sqrt{21}}{6} (U_2^2 + V_2^2) k + O(k^2). \quad (6.14)$$

In fact, by a linear canonical, analytic in k change of variables $U_i, V_i \rightarrow \widetilde{U}_i, \widetilde{V}_i$ ($i = 1, 2$), the Hamiltonian quadratic part (6.14) can be normalized in any orders with respect to k , so that the Hamiltonian with a normalized quadratic part reads

$$\Gamma = \frac{1}{2} \widetilde{V}_1^2 + \sigma (\widetilde{U}_2^2 + \widetilde{V}_2^2) + \Gamma_4 + O_6, \quad (6.15)$$

where σ analytically depends on k and can be calculated (for instance, by using the Deprit – Hori normalization method [17, 18]) in the form of convergent series in powers of k . The calculations have shown that

$$\sigma = k \frac{\sqrt{21}}{6} - k^2 \frac{5}{14} \sqrt{7} + O(k^3). \quad (6.16)$$

Γ_4 is a form of fourth order

$$\Gamma_4 = \sum_{i_1+i_2+j_1+j_2=4} \gamma_{i_1 i_2 j_1 j_2}(w) \tilde{U}_1^{i_1} \tilde{U}_2^{i_2} \tilde{V}_1^{j_1} \tilde{V}_2^{j_2}. \quad (6.17)$$

The coefficients $\gamma_{i_1 i_2 j_1 j_2}(w)$ π -periodically depend on w and are analytic in k . Their series expansion in powers of k can be calculated up to any finite order.

At the last step of the normalization procedure, by a nonlinear, close to identical, 2π -periodic in w canonical change of variables $\tilde{U}_i, \tilde{V}_i \rightarrow X_i, Y_i$ ($i = 1, 2$), the Hamiltonian can be brought to the form

$$\mathcal{H} = \frac{1}{2} Y_1^2 + \sigma (X_2^2 + Y_2^2) + a_{40} X_1^4 + a_{22} X_1^2 (X_2^2 + Y_2^2) + a_{04} (X_2^2 + Y_2^2)^2 + \mathcal{H}^{(5)}, \quad (6.18)$$

where the coefficient a_{40} , which is important for the stability study, reads

$$a_{40} = \frac{1}{2\pi} \int_0^{2\pi} \gamma_{4000} dw. \quad (6.19)$$

The calculations have shown that

$$a_{40} = -\frac{\sqrt{3}}{196} k^3 + O(k^4). \quad (6.20)$$

Thus, in accordance with the condition of instability mentioned in Section 5 the equilibrium position of the canonical system with the Hamiltonian (6.18) is unstable. This means that at the boundary curve γ_5 the fast planar rotations are orbitally unstable.

Similarly, the stability analysis can be performed on the boundary curve γ_7 , where the combinational resonance takes place. In this case the normalized Hamiltonian has the following form:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} (X_1^2 + X_2^2) + \lambda (X_1 Y_2 - X_2 Y_1) + A (Y_1^2 + Y_2^2)^2 + \\ & + (Y_1^2 + Y_2^2) [B (X_1 Y_2 - X_2 Y_1) + C (X_1^2 + X_2^2)] + \mathcal{H}^{(5)}. \end{aligned} \quad (6.21)$$

Calculations have shown that for sufficiently small $|k|$ the coefficient A reads

$$A = -0.0814917k + O(k^2). \quad (6.22)$$

Using the above-mentioned stability condition for the case of combinational resonance and taking into account that $k < 0$ on the curve γ_7 , we can draw the conclusion about the formal stability of fast rotations on the above boundary curve.

It should be noted that the results of the stability study obtained in the previous section by numerical calculation of the normal form are in good agreement with the results of this section obtained analytically.

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