MATHEMATICAL PROBLEMS OF NONLINEARITY

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Determination of the Homotopy Type of a Morse – Smale Diffeomorphism on a 2-torus by Heteroclinic Intersection

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According to the Nielsen – Thurston classification, the set of homotopy classes of orientation-preserving homeomorphisms of orientable surfaces is split into four disjoint subsets. Each subset consists of homotopy classes of homeomorphisms of one of the following types: T_1) periodic homeomorphism; T_2) reducible non-periodic homeomorphism of algebraically finite order; T_3) a reducible homeomorphism that is not a homeomorphism of algebraically finite order; T_4) pseudo-Anosov homeomorphism. It is known that the homotopic types of homeomorphisms of torus are T_1 , T_2 , T_4 only. Moreover, all representatives of the class T_4 have chaotic dynamics, while in each homotopy class of types T_1 and T_2 there are regular diffeomorphisms, in particular, Morse – Smale diffeomorphisms with a finite number of heteroclinic orbits. The author has found a criterion that allows one to uniquely determine the homotopy type of a Morse – Smale diffeomorphism with a finite number of heteroclinic orbits on a two-dimensional torus. For this, all heteroclinic domains of such a diffeomorphism are divided into trivial (contained in the disk) and non-trivial. It is proved that if the heteroclinic points of a Morse – Smale diffeomorphism are contained only in the trivial domains then such diffeomorphism has the homotopic type T_1 . The orbit space of non-trivial heteroclinic domains consists of a finite number of two-dimensional tori, where the saddle separatrices participating in heteroclinic intersections are projected as transversally intersecting knots. That whether the Morse – Smale diffeomorphisms belong to types T_1 or T_2 is uniquely determined by the total intersection index of such knots.

Keywords: Morse – Smale diffeomorphisms, Nielsen – Thurston theory, heteroclinic intersections, homotopy class of a map

1. Introduction and formulation of the results

Let X be a topological space. Two continuous maps $f_0, f_1: X \rightarrow X$ are called *homotopic* (denoted by $f_0 \simeq f_1$), if there is a continuous map $H: X \times [0, 1] \rightarrow X$ such that $H(x, 0) =$

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$= f_0(x)$ and $H(x, 1) = f_1(x)$. A map H is called a *homotopy* between f_0 and f_1 . For every fixed $t \in [0, 1]$ we set $H(x, t) = H_t(x)$. Denote by $[f]$ the *homotopy class* of the map f , that is $f' \in [f] \iff f' \simeq f$.

The set of homotopy classes of homeomorphisms of the space X forms the group $\Gamma(X)$ with the group operation $[f_1] \circ [f_2] = [f_1 \circ f_2]$, which is called *mapping class group*. Thus,

$$\Gamma(X): \stackrel{\text{def}}{=} \text{Homeo}(X)/\simeq,$$

where $\text{Homeo}(X)$ is the set of homeomorphisms of the space X onto itself.

A homotopy $H_t: X \rightarrow X$, $t \in [0, 1]$ is called an *isotopy* if, for any t the map H_t is a homeomorphism of the space X .

According to the Nielsen–Thurston classification [1] there are four types of homotopy classes of surface homeomorphisms. Each subset contains only homotopy classes of homeomorphisms of one of the following types:

- T_1) periodic homeomorphism;
- T_2) reducible non-periodic homeomorphism of algebraically finite order;
- T_3) reducible homeomorphism that is not a homeomorphism of algebraically finite order;
- T_4) pseudo-Anosov homeomorphism.

Let M^2 be a connected compact (possibly with boundary) orientable surface of genus g . Recall that a homeomorphism $h: M^2 \rightarrow M^2$ is called a *periodic homeomorphism*, if there exists $m \in \mathbb{N}$, such that $h^m = id$, where id is the identity transformation. The smallest such number m is called the *period* of the periodic homeomorphism.

A homeomorphism $h: M_2 \rightarrow M_2$ is called a *reducible* by a system C of disjoint simple closed curves C_i , $i = 1, \dots, l$, non-homotopic to zero and pairwise non-homotopic to each other if the system of curves C is invariant under h .

A reducible non-periodic homeomorphism $h: M^2 \rightarrow M^2$ is called a *homeomorphism of algebraically finite order*, if there exists an h -invariant neighbourhood $\mathbb{C}$ of curves of the set C , consisting of the union two-dimensional annuli and such that for each connected component M_j^2 , $j = 1, \dots, n$ of the set $M^2 \setminus \text{int } \mathbb{C}$ there is a number $m_j \in \mathbb{N}$ such that $h^{m_j}|_{M_j^2}: M_j^2 \rightarrow M_j^2$ is a periodic homeomorphism.

Since the Nielsen–Thurston theory was originally described for surfaces of genus $g \geq 2$, it is worth mentioning the book [2], in which this theory is also generalized to a torus and a fact is presented that the homotopic types of homeomorphisms of torus are T_1, T_2, T_4 only (Theorem 13.1, p. 369).

Recall that the diffeomorphism $f: M^n \rightarrow M^n$, given on a smooth, closed, connected n -manifold ($n \geq 1$) M^n is called *Morse–Smale diffeomorphism*, if

- 1) the non-wandering set Ω_f consists of a finite number of hyperbolic orbits;
- 2) the manifolds W_p^s, W_q^u intersect transversally for any non-wandering points p, q .

In the set of periodic orbits of such a diffeomorphism, one can introduce a complete order relation, which is an extension of S. Smale's partial order [3]. Namely, let $\mathcal{O}_i, \mathcal{O}_j$ be the periodic orbits of the Morse–Smale diffeomorphism f . One can say that the orbits $\mathcal{O}_i, \mathcal{O}_j$ are in the relation $\prec (\mathcal{O}_i \prec \mathcal{O}_j)$, if $W_{\mathcal{O}_i}^s \cap W_{\mathcal{O}_j}^u \neq \emptyset$. A sequence of different periodic orbits $\mathcal{O}_i = \mathcal{O}_{i_0}, \mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_k} = \mathcal{O}_j$ ($k \geq 1$), such that $\mathcal{O}_{i_0} \prec \mathcal{O}_{i_1} \prec \dots \prec \mathcal{O}_{i_k}$ is called a *chain of the length k , connecting periodic orbits $\mathcal{O}_i$ and $\mathcal{O}_j$* . The chain connecting the periodic orbits of saddle points will be called *saddle chain*. Due to the finiteness of the non-wandering set, for any diffeomorphism $f \in MS(M^n)$ there is a well-defined number equal to the length of the maximal saddle chain, which is denoted by $beh(f)$.

Denote by $MS_1(\mathbb{T}^2)$ the set of orientation-preserving Morse–Smale diffeomorphisms f defined on the two-dimensional torus $\mathbb{T}^2$ and having $\text{beh}(f) \leq 1$. According to [4, 5] the Morse–Smale diffeomorphism with $\text{beh}(f) \leq 1$ exists in both the first T_1 and the second T_2 Nielsen–Thurston type. We state $F = f^d: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, where $d \in \mathbb{N}$ is the smallest number such that $\forall x \in \Omega_f: f^d(x) = x$ and the map $f|_{W_x^u}$ preserves orientation. The dynamics of any diffeomorphism $F \in MS_1(\mathbb{T}^2)$ can be represented as follows. The set Ω_F of periodic orbits of the map F can be splatted into subsets Ω_F^i , $i \in \{\omega, s, u, \alpha\}$:

- Ω_F^ω — the set of all sink orbits;
- Ω_F^s — the set of saddle orbits, whose unstable sets do not contain heteroclinic points;
- Ω_F^u — the set of the remaining saddle orbits of the system;
- Ω_F^α — the set of source orbits.

Let

$$\mathcal{A}_F = \Omega_F^\omega \cup W_{\Omega_F^s}^u, \quad \mathcal{R}_F = \Omega_F^\alpha \cup W_{\Omega_F^u}^s, \quad V_F = \mathbb{T}^2 \setminus (\mathcal{A}_F \cup \mathcal{R}_F).$$

By virtue of the paper [6], the sets $\mathcal{A}_F$ and $\mathcal{R}_F$ are an attractor and a repeller of the diffeomorphism F , respectively. All heteroclinic points of the diffeomorphism F belong to the set V_F , which consists of a finite number of connected components V_i , $i = 1, \dots, m$. Each component V_i is homeomorphic to an open two-dimensional annuli (see, for example [7]) and has a fundamental domain $K_i \cong [0, 1] \times \mathbb{S}^1$, bounded by the circles $b_i \cong \{0\} \times \mathbb{S}^1$, $F(b_i) \cong \{1\} \times \mathbb{S}^1$, so that

$$V_i = \bigcup_{n \in \mathbb{Z}} F^n(K_i).$$

The component V_i of the set V_F will be called *trivial*, if the curve b_i is homotopic to zero on the torus, and *non-trivial* otherwise. The component V_i of the set V_F will be called a *heteroclinic domain*, if the component V_i contains at least one heteroclinic orbit.

The following lemma is proved in section 2.

Lemma 1. *If all heteroclinic domains of a diffeomorphism F are trivial, then the diffeomorphism F is isotopic to the identity map and, therefore, the homotopy class $[f]$ is of the type T_1 .*

In what follows, assume that the diffeomorphism F has non-trivial heteroclinic domains. Let us renumber the connected components of the set V_F : V_i , $i = 1, \dots, m$ in such a way that the first m_0 components are non-trivial heteroclinic domains, and the rest are trivial.

Consider the covering of the torus $q: \mathbb{R}^2 \rightarrow \mathbb{T}^2$, representing the torus as a quotient group of the plane $\mathbb{R}^2$ by the group of integer shifts. Let

$$a = q(Ox), \quad b = q(Oy).$$

Then the curves a , b are generators on the torus with a positive traversal direction inherited from the coordinate axes, and their homotopy classes are $[a] = \langle 1, 0 \rangle$, $[b] = \langle 0, 1 \rangle$. Then the homotopy class $[c]$ of any simple closed curve $c \in \mathbb{T}^2$ is determined by two numbers $[c] = \langle \mu, \nu \rangle$, $\mu \in \mathbb{Z}$, $\nu \in \mathbb{N}$, where μ , ν denote the number of revolutions around the generators a , b . Moreover, the numbers μ , ν are coprime, that is, $(\mu, \nu) = 1$.

Without loss of generality (see, for example, [8]), we assume that all curves b_i , $i \in \{1, \dots, m_0\}$ are of the form $\{y_i\} \times b$, and the generator a intersects the annuli K_i along the segment a_i , connecting the point $Y_i = y_i \times \{0\}$ with the point $F(Y_i)$. We state $\widehat{V}_i = V_i/F = K_i/F$ is the quotient

space by the action of the diffeomorphism F . Denote by $p_i: V_i \rightarrow \widehat{V}_i$ the natural projection, which is also a covering map for the space $\widehat{V}_i$. Then (see, for example, [7]) $\widehat{V}_i$ is homeomorphic to the two-dimensional torus and the curves $\widehat{a}_i = p_i(a_i)$, $\widehat{b}_i = p_i(b_i)$ are generators on $\widehat{V}_i$ such that $[\widehat{a}_i] = \langle 1, 0 \rangle$ and $[\widehat{b}_i] = \langle 0, 1 \rangle$ (see, Fig. 1), while the orientations on a_i and b_i are induced by the orientations on a and b .

Fig. 1. The projection q defines generators a, b on the torus $\mathbb{T}^2$ as images of the axes Ox and Oy . The projection p_i defines the generators $\widehat{a}_i, \widehat{b}_i$ on the torus $\widehat{V}_i$ as the images of the segments a_i, b_i . The connection between the basis of the torus $\mathbb{T}^2$ and of the torus $\widehat{V}_i$ is established in a natural way using the map p_i for all $i \in 1, \dots, m_0$

Denote by L_i^s (L_i^u) the union of the stable (the unstable) separatrices belonging to the domain V_i . Let $\widehat{L}_i^s = p_i(L_i^s)$ ($\widehat{L}_i^u = p_i(L_i^u)$). Denote by r_i^s and r_i^u the number of curves in the sets $\widehat{L}_i^s$ and $\widehat{L}_i^u$ respectively. Since the curves of the set $\widehat{L}_i^s$ ($\widehat{L}_i^u$) are pairwise disjoint and $F = f^d$, then any curve $\widehat{l}_i^s \subset \widehat{L}_i^s$ ($\widehat{l}_i^u \subset \widehat{L}_i^u$) is of homotopy type $\langle \delta_i, d_i^s \rangle$ ($\langle \delta_i, d_i^u \rangle$), where $\delta_i = \pm 1$ and d is a multiple of d_i^s (d_i^u). Since the separatrices are oriented in a natural way from the repeller to the attractor, the number δ_i always matches for $\widehat{l}_i^s$ and $\widehat{l}_i^u$. For any two curves $\widehat{l}_i^s \subset \widehat{L}_i^s$, $\widehat{l}_i^u \subset \widehat{L}_i^u$ at each point $x \in \widehat{l}_i^s \cap \widehat{l}_i^u$ we define $\vec{v}_x^s$ ($\vec{v}_x^u$) as a tangent vector to $\widehat{l}_i^s$ ($\widehat{l}_i^u$) at the point x , directed according to the orientation $\widehat{l}_i^s$ ($\widehat{l}_i^u$). Then we define an *index* ξ_x of the point x equal to 1 (−1), if the orientation that an ordered pair of vectors $(\vec{v}_x^u, \vec{v}_x^s)$ defines on the manifold $\widehat{V}_i$, coincides (does not coincide) with the orientation of the manifold $\widehat{V}_i$. Then we will call the number

$$\xi_i = \sum_{x \in \widehat{l}_i^s \cap \widehat{l}_i^u} \xi_x$$

by the *index of heteroclinic intersection of curves* $\widehat{l}_i^s$ and $\widehat{l}_i^u$. The index of the heteroclinic intersection of curves coincides for any two curves $\widehat{l}_i^s$ and $\widehat{l}_i^u$ from the sets $\widehat{L}_i^s$ and $\widehat{L}_i^u$ respectively. We state

$$\xi_F = \sum_{i=1}^{m_0} \xi_i$$

and will call the number ξ_F by the *heteroclinic intersection index of the diffeomorphism* $F \in MS_1(\mathbb{T}^2)$. The following result shows that the heteroclinic intersection index uniquely determines the type of the homotopy class $[f]$ of the diffeomorphism $f \in MS_1(\mathbb{T}^2)$.

Theorem 1. *Let $f \in MS_1(\mathbb{T}^2)$ and $F = f^d$. The homotopy class $[f]$ is of type T_1 , if and only if $\xi_F = 0$. The homotopy class $[f]$ is of type T_2 , if and only if $\xi_F \neq 0$.*

2. Proof of the Lemma 1

Let $f \in MS_1(\mathbb{T}^2)$, $F = f^d: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ and all heteroclinic domains V_i , $i \in \{1, \dots, m'\}$ of the diffeomorphism F are trivial. Then the curves b_i , $i \in \{1, \dots, m'\}$ bound the disks $D_i \subset \mathbb{T}^2$ such that $F(D_i) \subset \text{int}(D_i)$ or $F^{-1}(D_i) \subset \text{int}(D_i)$. Since the domain V_i contains heteroclinic points, then D_i contains at least one saddle point. By virtue of [9, Lemma 6.3, p. 1115]¹ there is an isotopy $H_t: \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $t \in [0, 1]$, such that $H_0 = F$, $H_1 = F_1$, where $F_1: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ Morse–Smale diffeomorphism coinciding with F on the set $\mathbb{T}^2 \setminus (D_1 \cup \dots \cup D_{m'})$ and such that each disk D_i , $i \in \{1, \dots, m'\}$ contains a single fixed sink or source of the diffeomorphism F_1 (see Fig. 2). By construction, the diffeomorphism F_1 is a gradient-like diffeomorphism, all non-wandering points of which are fixed and have a positive orientation type. According to [10, Theorem 4.2], the diffeomorphism F is isotopic to the identity map. Since $F = f^d$, then $f_*^d = id$. Thus, the matrix f_* is periodic and, therefore, the diffeomorphism $[f]$ has the homotopy type T_1 .

Fig. 2. An example of the application of the lemma 1. On the left is the phase portrait of the diffeomorphism F on the torus $\mathbb{T}^2$ with marked curves b_i , $i \in \{1, \dots, 4\}$ and the disk D_3 containing the repeller connected component $\mathcal{R}_f$. On the right is a diffeomorphism F_1 on the torus $\mathbb{T}^2$, which coincides with the diffeomorphism F everywhere except for the disk D_3 , on which, using the isotopy, we trivialize the connected component of the repeller to the only sink $\tilde{\alpha}$

3. Proof of the Theorem 1

Let $f \in MS_1(\mathbb{T}^2)$ and $F = f^d$. Without loss of generality, we will assume that $V_i = S^1 \times \mathbb{R}$, $F|_{V_i}(s, r) = (s, r + 1)$ and $K_i = S^1 \times [0, 1]$. We state $U_i^+ = S^1 \times [0, +\infty)$.

Consider the set E_F of such diffeomorphisms $G: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that G coincides with F in some neighborhood U_G^+ of the attractor $\mathcal{A}_F$, containing $\bigcup_{i=1}^{m_0} K_i$, in some neighborhood U_G^- of the repeller $\mathcal{R}_F$ and $\Omega_F = \Omega_G$. For any diffeomorphism $G \in E_F$ we denote by $L_{G,i}^s$ and $L_{G,i}^u$ the set of stable and unstable saddle separatrices of the diffeomorphism G , that belong to the set V_i . Then each connected component of the orbit spaces $L_{G,i}^s/G$ and $L_{G,i}^u/G$ is homeomorphic to a circle. Since K_i is a common fundamental domain of diffeomorphisms $G|_{V_i}$ and $F|_{V_i}$, then $L_{G,i}^s/G$ is

¹Lemma 6.3 from [9] asserts that any gradient-like diffeomorphism f on the surface M^2 , with an attractor A belonging to the disk D and such that $f(D) \subset D$, $A = \bigcap_{n \in \mathbb{N}} f^n(D)$, is connected by a stable arc (diffeotopy admitting only flip and saddle knot bifurcations) with some gradient-like diffeomorphism g , coinciding with f on $M^2 \setminus D$ and having a single non-wandering point in the disk D , which is a fixed sink.

homeomorphic to $p_i(L_{G,i}^s \cap K_i)$ and $L_{G,i}^u/G$ is homeomorphic to $p_i(L_{G,i}^u \cap K_i)$. Since $G|_{U_i^+} = F|_{U_i^+}$, then $L_{G,i}^s \cap U_i^+ = \bigcup_{n \geq 0} F^n(L_{G,i}^s \cap K_i)$, $L_{G,i}^u \cap U_i^+ = \bigcup_{n \geq 0} F^n(L_{G,i}^u \cap K_i)$ and, therefore, $p_i(L_{G,i}^s \cap K_i) = p_i(L_{G,i}^s \cap U_i^+)$, $p_i(L_{G,i}^u \cap K_i) = p_i(L_{G,i}^u \cap U_i^+)$. Let

$$\widehat{L}_{G,i}^s = p_i(L_{G,i}^s \cap U_i^+), \quad \widehat{L}_{G,i}^u = p_i(L_{G,i}^u \cap U_i^+).$$

Let us prove an auxiliary lemma.

Lemma 2. *Let $\widehat{h}: \widehat{V}_i \rightarrow \widehat{V}_i$ is isotopic to the identity diffeomorphism. Then there exists a diffeotopy $\zeta_t \subset E_F$ such that $\zeta_0 = F$, $\zeta_1 = F_1$ and $\widehat{L}_{F_1,i}^u = \widehat{h}(\widehat{L}_{F,i}^u)$, $\widehat{L}_{F_1,i}^s = \widehat{L}_{F,i}^s$.*

Proof. Take an open cover $D = \{\widehat{D}_1, \dots, \widehat{D}_q\}$ of the manifold $\widehat{V}_i$ by two-dimensional disks so that $p_i^{-1}(\widehat{D}_j) \cap S^1 \times \{r_j, r_j + 1\} = \emptyset$ for some $r_j \in \mathbb{R}$. According to [11, Fragmentation lemma] there exist diffeomorphisms $\widehat{w}_1, \dots, \widehat{w}_q: \widehat{V}_i \rightarrow \widehat{V}_i$ smoothly isotopic to the identity and with the following properties:

i) for each $j \in \{1, \dots, q\}$ there is a smooth isotopy $\{\widehat{w}_{j,t}\}$, identical outside $\widehat{D}_j$ and joining the identity map and $\widehat{w}_j$;

ii) $\widehat{h} = \widehat{w}_1 \dots \widehat{w}_q$.

Let $U_{G,i}^+ = V_i \cap U_G^+$, $U_{G,i}^- = V_i \cap U_G^-$ and denote by r_G^-, r_G^+ real positive numbers such that $S^1 \times (-\infty, -r_G^-) \subset U_{G,i}^-$ and $S^1 \times (-r_G^+, +\infty) \subset U_{G,i}^+$.

Take q disks $D_1, \dots, D_q$ one in each of the sets $p_i^{-1}(\widehat{D}_1), \dots, p_i^{-1}(\widehat{D}_q)$ so that $D_j \subset \widetilde{K}_j$, with $\widetilde{K}_j = S^1 \times (-r_j - 1, -r_j)$, $r_G^- < r_1$ and $r_j + 1 < r_{j+1}$ for $j \in \{1, \dots, q-1\}$. Let $w_{j,t}: V \rightarrow V$ be a diffeomorphism that coincides with $(p_i|_{\widetilde{K}_j})^{-1} \widehat{w}_{j,t} p_i$ on the set $\widetilde{K}_j$ and coincides with the identity map outside the set $\widetilde{K}_j$. Let

$$\zeta_t = w_{1,t} \dots w_{q,t} F.$$

By the construction $\zeta_t \subset E_f$ and $\zeta_0 = F$. Let $F_1 = \zeta_1$. Let us show that $\widehat{L}_{F_1,i}^u = \widehat{h}(\widehat{L}_{F,i}^u)$, $\widehat{L}_{F_1,i}^s = \widehat{L}_{F,i}^s$.

Since the manifolds $L_{F_1,i}^s$ are obtained from the local manifolds $L_{F_1,i}^s \cap U_{F_1,i}^+$ by iteration due to the map F_1^{-1} and $L_{F_1,i}^s \cap U_{F_1,i}^+ = L_{F,i}^s \cap U_{F_1,i}^+$, then $L_{F_1,i}^s \cap U_i^+ = L_{F,i}^s \cap U_i^+$ and therefore $\widehat{L}_{F_1,i}^s = \widehat{L}_{F,i}^s$.

We state $\widetilde{U}_i^- = S^1 \times (-\infty, -r_q - 1)$ and $F_j = w_j \dots w_q F$, $j \in \{1, \dots, q\}$. By the construction $F_j \in E_F$. Then, to prove the lemma, it suffices to show that $\widehat{L}_{F_j,i}^u = \widehat{w}_j(\widehat{L}_{F_{j+1},i}^u)$, where $F_{q+1} = F$.

Take natural numbers k_j^-, k_j^+ such that $\widetilde{K}_j^- = F_j^{-k_j^-}(\widetilde{K}_j) \subset \widetilde{U}_i^-$, $\widetilde{K}_j^+ = F_j^{k_j^+}(\widetilde{K}_j) \subset U_i^+$. We state $k_j = k_j^- + k_j^+$. Since the manifolds $L_{F_j,i}^u$ are obtained from the local manifolds $L_{F_{j+1},i}^u \cap U_{F_j,i}^-$ by iteration due to the map F_j and $L_{F_j,i}^u \cap \widetilde{U}_i^- = L_{F_{j+1},i}^u \cap \widetilde{U}_i^-$, then (see Fig. 3)

$$\begin{aligned} \widehat{L}_{F_{j+1},i}^u &= p_i F_{j+1}^{k_j} (L_{F_{j+1},i}^u \cap \widetilde{K}_j^-), \\ \widehat{L}_{F_j,i}^u &= p_i F_j^{k_j} (L_{F_{j+1},i}^u \cap \widetilde{K}_j^-). \end{aligned}$$

Fig. 3. The figure (a) shows an illustration of the action of the formula step by step. $x \xrightarrow{F_{j+1}^-} y \xrightarrow{F_j^{k_j^-}} z \xrightarrow{F_j^{-k_j^+}} z' \xrightarrow{F_{j+1}^{k_j^+}} F_j^{k_j^+}(z')$. The figure (b) shows the form of separates $L_{F_1,i}^s$, $L_{F_1,i}^u$ for the diffeomorphism F_1

Then

$$\begin{aligned} \widehat{L}_{F_j,i}^u &= p_i F_{j+1}^{k_j^+} w_j F_{j+1}^{k_j^-} (L_{F_{j+1},i}^u \cap \widetilde{K}_j^-) = \\ &= p_i F_j^{k_j^+} w_j F_j^{-k_j^+} F_{j+1}^{k_j^-} (L_{F_{j+1},i}^u \cap \widetilde{K}_j^-) = p_i F_j^{k_j^+} w_j F_j^{-k_j^+} (p_i|_{\widetilde{K}_j^+})^{-1} p_i F_{j+1}^{k_j^-} (L_{F_{j+1},i}^u \cap \widetilde{K}_j^-) = \\ &= p_i w_j (p_i|_{\widetilde{K}_j})^{-1} (\widehat{L}_{F_{j+1},i}^u) = \widehat{w}_j (\widehat{L}_{F_{j+1},i}^u). \end{aligned}$$

□

Let's return to the proof of the theorem. By assumption, the diffeomorphism F has non-trivial heteroclinic domains V_i , $i \in \{1, \dots, m_0\}$ and curves from the sets $\widehat{L}_{F,i}^s$ and $\widehat{L}_{F,i}^u$ have homotopy types $\langle \delta_i, d_i^s \rangle$ and $\langle \delta_i, d_i^u \rangle$ respectively on the torus $\widehat{V}_i$. Then, by virtue of [8, Chapter D, Exercise 7] the equality $\xi_i = \delta_i(d_i^u - d_i^s)$ is holds.

We define the function $\nu: [0, 1] \rightarrow [0, 1]$ by the formula

$$\nu(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{1 + \exp\left(\frac{\frac{1}{2}-t}{t^2(t-1)^2}\right)}, & 0 < t < 1, \\ 1, & t = 1. \end{cases}$$

Define on the annuli K_i the map $R_i: K_i \rightarrow K_i$, by the formula

$$R_i(e^{i\phi}, r) = (e^{i(\phi+2\pi\xi_i\nu(r))}, r).$$

The projection $\widehat{R}_i = p_i R_i p_i^{-1}$ of the map R_i acts on the fundamental group of the torus $\widehat{V}_i$ by the matrix $\widehat{R}_{i*}$. Due to the consistency of the bases a, b and $\widehat{a}_i, \widehat{b}_i$, the equality $\widehat{R}_{i*} = R_{i*}$ holds. Then, the diffeomorphism R_i acts on the fundamental group of the torus $\mathbb{T}^2$ by the matrix $R_{i*} = \begin{pmatrix} 1 & \xi_i \\ 0 & 1 \end{pmatrix}$. Let $\widetilde{F} = R_{m_0}^{-1} \circ \dots \circ R_1^{-1} \circ F$. Then $\widetilde{F} \in E_F$ and for any $i \in \{1, \dots, m_0\}$ the equality holds

$$\begin{pmatrix} \delta_i & d_i^u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\xi_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_i & d_i^s \\ 0 & 1 \end{pmatrix}.$$

Since the global stable manifold of a saddle point is obtained from a local one by iteration due to the inverse map and in a neighborhood of the attractor $\mathcal{A}_F$ the manifolds L_i^s and $L_{\widetilde{F},i}^s$ are coincide, then $L_{\widetilde{F},i}^s \cap K_i = L_i^s \cap K_i$. Since the global unstable manifold of a saddle point is obtained from the local by iterating by the direct map and in the neighborhood of the repeller $\mathcal{R}_F$ the manifolds L_i^u and $L_{\widetilde{F},i}^u$ are coincide, then $L_{\widetilde{F},i}^u \cap K_i = R_i(L_i^u \cap K_i)$.

Thus, the knots $\widehat{L}_{\widetilde{F},i}^s, \widehat{L}_{\widetilde{F},i}^u$ have the same homotopy type on the torus $\widehat{V}_i$. By virtue of [8], there exists a diffeomorphism $\widehat{h}: \widehat{V}_i \rightarrow \widehat{V}_i$ isotopic to the identity such that $\widehat{h}(\widehat{L}_{\widetilde{F},i}^u) \cap \widehat{L}_{\widetilde{F},i}^s = \emptyset$. According to the lemma 2, the diffeomorphism $\widetilde{F}$ is diffeotopic to some diffeomorphism $\widetilde{F}_1 \in E_F$ such that $\widehat{L}_{\widetilde{F}_1,i}^u \cap \widehat{L}_{\widetilde{F}_1,i}^s = \emptyset$. By construction, the diffeomorphism $\widetilde{F}_1$ belongs to the class $MS(\mathbb{T}^2)$ and has only trivial heteroclinic domains (see Fig. 4).

Fig. 4. On the left is the phase portrait of the diffeomorphism $\widetilde{F}$ on the torus $\mathbb{T}^2$ with a non-trivial heteroclinic domain, $\widetilde{F}$ is reduced by some isotopy to the diffeomorphism $\widetilde{F}_1$, the phase portrait of which is shown on the right

By virtue of the lemma 1, the diffeomorphism $\widetilde{F}_1$ is isotopic to the identity diffeomorphism. Then the diffeomorphism F is isotopic to the diffeomorphism $R_1 \circ \dots \circ R_{m_0}$. Because

$$R_{i*} = \begin{pmatrix} 1 & \xi_i \\ 0 & 1 \end{pmatrix},$$

then

$$F_* = \begin{pmatrix} 1 & \xi_F \\ 0 & 1 \end{pmatrix}.$$

Then F belongs to the homotopy type T_1 if and only if $\xi_F = 0$. Since $F = f^d$, then the matrix f_* is periodic if and only if $\xi_F = 0$. Thus, the diffeomorphism $[f]$ has homotopy type T_1 if and only if $\xi_F = 0$.

Since the Morse – Smale diffeomorphisms are representatives of only homotopy classes from T_1 and T_2 , then $[f] \in T_2$ if and only if $\xi_F \neq 0$.

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