MATHEMATICAL PROBLEMS OF NONLINEARITY

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An Extension of Herman's Theorem for Nonlinear Circle Maps with Two Breaks

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M. Herman showed that the invariant measure μ_h of a piecewise linear (PL) circle homeomorphism h with two break points and an irrational rotation number ρ_h is absolutely continuous iff the two break points belong to the same orbit. We extend Herman's result to the class P of piecewise $C^{2+\varepsilon}$ -circle maps f with an irrational rotation number ρ_f and two break points a_0, c_0 , which do not lie on the same orbit and whose total jump ratio is $\sigma_f = 1$, as follows: if μ_f denotes the invariant measure of the P -homeomorphism f , then for Lebesgue almost all values of $\mu_f([a_0, c_0])$ the measure μ_f is singular with respect to Lebesgue measure.

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1. Introduction

One of the important problems of circle dynamics is that of absolute continuity of the invariant measure for circle homeomorphisms. For smooth circle homeomorphisms this problem is well understood. Fundamental results on this were obtained by V. Arnold, M. Herman, J. Moser, J.-C. Yoccoz, Ya. Sinai and K. Khanin, Y. Kaznelson and D. Ornstein and others. Let f be an orientation-preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $F : \mathbb{R} \rightarrow \mathbb{R}$, which is continuous, strictly increasing and fulfills $F(\hat{x} + 1) = F(\hat{x}) + 1$, $\hat{x} \in \mathbb{R}$. The circle homeomorphism f is then defined by $f(x) = F(\hat{x}) \bmod 1$ with $\hat{x} \in \mathbb{R}$ a lift of $x \in S^1$. The **rotation number** ρ_f is defined by

$$\rho_f := \lim_{n \rightarrow \infty} \frac{F^n(\hat{x}) - \hat{x}}{n} \bmod 1,$$

where F^i denotes the i th iteration of the map F . It is well known that the rotation number ρ_f does not depend on the starting point $\hat{x} \in \mathbb{R}$ and is irrational if and only if f has no periodic points (see [4]). The rotation number ρ_f is invariant under topological conjugations.

Denjoy's classical theorem states [5] that a circle diffeomorphism f with an irrational rotation number $\rho = \rho_f$ and $\log Df$ of bounded variation can be conjugated to the linear rotation R_ρ with lift $\hat{R}_\rho(\hat{x}) = \hat{x} + \rho$, that is, there exists a homeomorphism $\varphi : S^1 \rightarrow S^1$ with $f = \varphi \circ R_\rho \circ \varphi^{-1}$.

It is well known that a circle homeomorphism f with an irrational rotation number ρ_f is uniquely ergodic, i.e., it has a unique invariant probability measure μ_f . A remarkable fact then is that the conjugacy φ can be defined by $\varphi(x) = \mu_f([0, x])$ (see [4]), which shows that the smoothness properties of the conjugacy φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f for a sufficiently smooth circle diffeomorphism with a typical irrational rotation number (see [2, 12, 14, 16]). The problem of smoothness of the conjugacy for smooth diffeomorphisms is by now very well understood (see, for instance, [2, 12, 14, 16, 23]).

A natural generalization of circle diffeomorphisms are piecewise smooth homeomorphisms with break points (see [14]).

The class of **P-homeomorphisms** consists of orientation-preserving circle homeomorphisms f which are differentiable except at a finite or countable number of break points, denoted by $BP(f) = \{x_b \in S^1\}$, at which the one-sided positive derivatives Df_- and Df_+ exist, but do not coincide, and for which there exist constants $0 < c_1 < c_2 < \infty$ such that

- $c_1 < Df_-(x_b) < c_2$ and $c_1 < Df_+(x_b) < c_2$;
- $c_1 < Df(x) < c_2$ for all $x \in S^1 \setminus BP(f)$;
- $\log Df$ has finite total variation v in S^1 .

Piecewise linear (for short *PL*) orientation-preserving circle homeomorphisms are the simplest examples of *P*-homeomorphisms. They occur in many different areas of mathematics such as group theory, homotopy theory or logic via the Thompson groups. A family of *PL*-homeomorphisms was first studied by M. Herman [14] as examples of circle homeomorphisms with an arbitrary irrational rotation number which admit no invariant σ -finite measure absolutely continuous with respect to Lebesgue measure.

Indeed, Herman proved thereby

Theorem 1. *The invariant measure of a PL-circle homeomorphism with two break points and an irrational rotation number is absolutely continuous with respect to Lebesgue measure if and only if these break points lie on the same orbit.*

Herman's family of maps has been studied later by several authors (see, for instance, [3, 19–21]) in the context of interval exchange transformations. Special cases are affine 2-interval exchange transformations, to which Herman's examples with break points $a_0 = 0$ and $c_0 = c$ belong.

In [8] Dzhililov et al. proved the following result. Let f be a P -homeomorphism with two break points. If f is C^2 except at the two break points and its rotation number is of bounded type, then the unique f -invariant probability measure μ_f is equivalent to Lebesgue measure if and only if the two break points of f lie on the same orbit and its total jump ratio is $\sigma_f = 1$. Notice that the condition of bounded type is essential: if ρ_f is not of bounded type, the homeomorphism f obtained by conjugating a C^2 diffeomorphism with singular invariant measure (such diffeomorphisms exist by [13]) with a P -homeomorphism with one break point is a P -homeomorphism with two break points on the same orbit and with a trivial total jump ratio, but the f -invariant probability measure μ_f is singular.

The invariant measures of P -homeomorphisms f with a finite number of break points have been studied by several authors (see, for instance, [1, 3, 6–9, 11, 15, 22]). For such a homeomorphism the character of the invariant measure strongly depends on its total jump ratio σ_f being trivial or nontrivial, i.e., $\sigma_f = 1$ or $\sigma_f \neq 1$. A recent result of Dzhililov et al. in [11] in the case $\sigma_f \neq 1$ is

Theorem 2. *Let $f \in C^{2+\varepsilon}(S^1 \setminus \{a_1, a_2, \dots, a_m\})$, $\varepsilon > 0$ be a P -homeomorphism with an irrational rotation number ρ_f and a finite number of break points $a_1, a_2, \dots, a_m$. Suppose its total jump ratio is $\sigma_f = \sigma_f(a_1) \cdot \sigma_f(a_2) \cdot \dots \cdot \sigma_f(a_m) \neq 1$. Then its invariant probability measure μ_f is singular with respect to Lebesgue measure l .*

Piecewise smooth P -homeomorphisms f with a finite number of break points and the trivial total jump ratio $\sigma_f = 1$ are more difficult to investigate. In the special case of piecewise $C^{2+\varepsilon}$ P -homeomorphisms f whose break points all lie on the same orbit, the invariant measure μ_f is absolutely continuous w.r.t. to Lebesgue measure for typical irrational rotation numbers (see [7]). Rather complicated is the case where the break points of such a homeomorphism f are not on the same orbit. In this case A. Teplinskii constructed in [22] examples of PL -homeomorphisms f with four break points and the trivial total jump ratio $\sigma_f = 1$ whose irrational rotation numbers ρ_f are of unbounded type and whose invariant measures μ_f are absolutely continuous w.r.t. Lebesgue measure l .

In the present paper we study $C^{2+\varepsilon}$ P -homeomorphisms f with an arbitrary irrational rotation number ρ_f and two break points not on the same orbit, whose total jump ratio $\sigma_f = 1$. Put

$$\Gamma := \{\rho \in [0, 1] : \rho = [k_1, k_2, \dots, k_n, \dots] \in \mathbb{R} \setminus \mathbb{Q} : \text{there exists } N = N(\rho) \in \mathbb{N}, \text{ such that } k_n \geq 3, \text{ for all } n \geq N\}. \quad (1.1)$$

Our main result for these homeomorphisms is

Theorem 3. *Let $f \in C^{2+\varepsilon}(S^1 \setminus \{b_1, b_2\})$ be a P -homeomorphism with an irrational rotation number $\rho := \rho_f \in \Gamma$ and two break points b_1, b_2 on different orbits with the trivial total jump ratio $\sigma_f = \sigma_f(b_1) \cdot \sigma_f(b_2) = 1$. Denote its invariant measure by μ_f . Then there exists a subset $M_\rho \subset [0, 1]$ of full Lebesgue measure, such that μ_f is singular w.r.t. Lebesgue measure if $\mu_f([b_1, b_2]) \in M_\rho$.*

Notice that, if $\mu_f([b_1, b_2]) = q\rho + p$, $p, q \in \mathbb{Z}$, then the two break points b_1, b_2 lie on the same orbit and the invariant measure μ_f is absolutely continuous w.r.t. Lebesgue measure (see [14]).

REMARK 1. To every piecewise smooth circle homeomorphism f with two break points b_1, b_2 , total trivial jump $\sigma_f := \sigma_f(b_1) \cdot \sigma_f(b_2) = 1$ and an irrational rotation number $\rho := \rho_f$ there corresponds a unique PL -map of Herman with two break points $a_0 = 0, c_0$, such that $\sigma_h(a_0) = \sigma_f(b_1)$ and $\mu_h([a_0, c_0]) = \mu_f([b_1, b_2])$ (for details see Section 6). Moreover, $\mu_h([a_0, c_0]) = \frac{\beta}{1+\beta}$. Hence, the condition in Theorem 3 can be written also in terms of the parameter β of this unique PL -map as $\frac{\beta}{1+\beta} \in M_\rho$.

REMARK 2. Theorem 3 can be extended to piecewise smooth homeomorphisms of the circle with finitely many break points and a trivial total jump ratio.

REMARK 3. The condition $\rho := \rho_f \in \Gamma$ in Theorem 3 is a technical one, and the theorem can be proved for all irrational rotation numbers.

2. Some definitions and well-known facts

Let f be an orientation-preserving circle homeomorphism with an irrational rotation number ρ_f and an invariant measure μ_f . Then ρ_f has a unique continued fraction expansion as $\rho_f = 1/(k_1 + 1/(k_2 + \dots)) := [k_1, k_2, \dots, k_n, \dots]$. Denote by $p_n/q_n = [k_1, k_2, \dots, k_n]$, $n \geq 1$, its n th convergent. The numbers $q_n, n \geq 1$, are called also the **first return times** of f and satisfy the recurrence relations $q_{n+1} = k_{n+1}q_n + q_{n-1}$, $n \geq 1$, where $q_0 = 1$ and $q_1 = k_1$. Fix an arbitrary $x_0 \in S^1$. Its forward orbit $O_f^+(x_0) = \{x_i = f^i(x_0), i = 0, 1, 2, \dots\}$ defines a sequence of natural partitions of the circle. Namely, denote by $I_0^{(n)}(x_0)$ the closed interval in S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$. In the clockwise orientation of the circle the point x_{q_n} is for n odd to the left of x_0 , and for n even to its right. Denote the iterates of the interval $I_0^{(n)}(x_0)$ under f by $I_i^{(n)}(x_0) := f^i(I_0^{(n)}(x_0)), i \geq 1$. We call these intervals for fixed n intervals of **rank** n . If

$$\Delta_n := \mu_f(I_0^{(n)}(x_0)) = \mu_f(I_i^{(n)}(x_0)) \quad (2.1)$$

denotes the measure of the intervals of rank n , then $\Delta_n = |q_n \rho_f - p_n|$.

It is well known that the set $\xi_n(x_0)$ of intervals with mutually disjoint interiors, defined as

$$\xi_n(x_0) = \{I_0^{(n-1)}(x_0), I_1^{(n-1)}(x_0), \dots, I_{q_n-1}^{(n-1)}(x_0)\} \cup \{I_0^{(n)}(x_0), I_1^{(n)}(x_0), \dots, I_{q_n-1-1}^{(n)}(x_0)\},$$

determines a partition of the circle for any n . The partition $\xi_n(x_0)$ is called the n th **dynamical partition** of S^1 determined by the point x_0 and the map f , (where we suppressed the dependence on the map f for simplicity).

Proceeding from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$, all the intervals $I_j^{(n)}(x_0)$, $0 \leq j \leq q_n - 1$, are preserved, whereas each of the intervals $I_i^{(n-1)}(x_0)$, $0 \leq i \leq q_n - 1$, is partitioned into $k_{n+1} + 1$ subintervals belonging to $\xi_{n+1}(x_0)$ such that

$$I_i^{(n-1)}(x_0) = I_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} I_{i+q_n-1+s q_n}^{(n)}(x_0). \quad (2.2)$$

Obviously, one has $\xi_1(x_0) \leq \xi_2(x_0) \leq \dots \leq \xi_n(x_0) \leq \dots$

Definition 1. Let $K > 1$ be a constant. We call two intervals I_1 and I_2 of S^1 **K -comparable** if the inequality $K^{-1}\ell(I_2) \leq \ell(I_1) \leq K\ell(I_2)$ holds.

Next we introduce another partition $\xi_n^*(x_0)$. Define

$$J_i^{(n)} := I_i^{(n)}(x_0) \cup I_i^{(n-1)}(x_0), \quad 0 \leq i \leq q_{n-1} - 1.$$

The system of intervals

$$\xi_n^*(x_0) := \{J_0^{(n)}, J_1^{(n)}, \dots, J_{q_{n-1}-1}^{(n)}\} \cup \{I_{q_{n-1}}^{(n-1)}, I_{q_{n-1}+1}^{(n-1)}, \dots, I_{q_n-1}^{(n-1)}\}$$

determines also a partition of the circle.

Definition 2. An interval $I = [\tau, t] \subset S^1$ is said to be q_n -small and its endpoints q_n -close if the intervals $f^i(I)$, $0 \leq i \leq q_n - 1$, are, except for the endpoints, pairwise disjoint.

Let f be a P -homeomorphism with an irrational rotation number ρ_f and $|BP(f)| < \infty$. The following facts about P -homeomorphisms are standard:

- if the interval $I = (x, y) \subset S^1$ is q_n -small and $f^s(x), f^s(y) \notin BP(f)$ for all $0 \leq s < q_n$, then for any $k \in [0, q_n)$ Finzi's inequality

$$e^{-v} \leq \frac{Df^k(x)}{Df^k(y)} \leq e^v \quad (2.3)$$

holds, where v is the total variation of $\log Df$ on S^1 ;

- for any y_0 with $y_s := f^s(y_0) \notin BP(f)$ for all $0 \leq s < q_n$, the inequality

$$e^{-v} \leq \prod_{s=0}^{q_n-1} Df(y_s) \leq e^v \quad (2.4)$$

holds. This inequality is called **Denjoy's inequality**;

- for an arbitrary element $I^{(n)}$ of the dynamical partition $\xi_n(x_0)$

$$\ell(I^{(n)}) \leq \text{const } \lambda^n \quad (2.5)$$

holds, where $\lambda = (1 + e^{-v})^{-\frac{1}{2}} < 1$;

- the homeomorphism f is topologically conjugate to the linear rotation R_ρ i.e., there exists a homeomorphism φ such that $\varphi \circ f = R_\rho \circ \varphi$;
- let μ_f be the unique invariant probability measure μ_f of f . Then (see [14, p. 71])

$$\int_{S^1} \log Df(x) d\mu_f(x) = 0. \quad (2.6)$$

3. Herman's family of PL -homeomorphisms with two break points

In [14, Section 7 of Chapter VI] M. Herman introduced a family of PL -homeomorphisms with two break points, for which he studied their invariant measures and the regularity of the

Fig. 1

maps conjugating them to linear rotations: given two real numbers $\lambda > 1$ and $\beta > 0$, he defines for $\hat{x} \in [0, 1]$ the piecewise linear map $H_{\beta, \lambda}: [0, 1] \rightarrow [0, 1]$ as

$$H_{\beta, \lambda}(\hat{x}) = \begin{cases} \lambda \hat{x}, & \text{if } 0 \leq \hat{x} \leq c, \\ \lambda^{-\beta}(\hat{x} - 1) + 1, & \text{if } c \leq \hat{x} \leq 1, \end{cases}$$

such that $\lambda c = \lambda^{-\beta}(c - 1) + 1$.

Then Herman considered for $0 \leq \theta \leq 1$ the one-parameter family of *PL*-maps $H_{\beta, \lambda, \theta}$ of the unit interval with

$$H_{\beta, \lambda, \theta}(\hat{x}) = H_{\beta, \lambda}(\hat{x}) + \theta \pmod{1},$$

and the induced piecewise linear homeomorphisms of the circle

$$h_{\beta, \lambda, \theta}(x) = H_{\beta, \lambda, \theta}(\hat{x}) \pmod{1}. \quad (3.1)$$

Obviously, $a_0 = 0$ and $c_0 = c$ are break points of all these $h_{\beta, \lambda, \theta}$. Denote their rotation number for fixed $\lambda > 1$ and $\beta > 0$ by ρ_θ . Continuity and monotonicity of ρ_θ as a function of θ imply that for an arbitrary irrational number $\alpha \in [0, 1]$ there exists a unique $\theta = \theta(\alpha) \in [0, 1]$ with $\rho_\theta = \alpha$. Herman then proved in [14]

Theorem 4. *The following properties are equivalent:*

- (i) $h_{\beta, \lambda, \theta}$ is conjugate to the linear rotation R_α through an absolutely continuous homeomorphism;
- (ii) $h_{\beta, \lambda, \theta}$ is conjugate to R_α through a Lipschitz homeomorphism;
- (iii) $h_{\beta, \lambda, \theta}$ can be conjugated to R_α by a piecewise C^∞ homeomorphism, which is not *PL*;
- (iv) $\frac{\beta}{\beta + 1} \in \mathbb{Z} \alpha \pmod{1}$;
- (v) the break points a_0 and c_0 belong to the same orbit under $h_{\beta, \lambda, \theta}$.

It can be easily checked (see [14]) that up to two points

$$\frac{\log Dh_{\beta, \lambda, \theta}(x)}{(1 + \beta) \log \lambda} = \chi_{[a_0, c_0]}(x) - \frac{\beta}{1 + \beta}, \quad (3.2)$$

where $\chi_{[a_0, c_0]}$ is the characteristic function of the interval $[a_0, c_0]$. Obviously,

$$\sigma = \sigma(a_0) = \frac{Dh_-(0)}{Dh_+(0)} = \lambda^{-1-\beta}$$

for $h = h_{\beta,\lambda,\theta}$, and hence $\log \sigma = -(1 + \beta) \log \lambda$. Then we can rewrite (3.2) as

$$\frac{\log Dh_{\beta,\lambda,\theta}(x)}{\log \sigma} = \frac{\beta}{1 + \beta} - \chi_{[a_0, c_0]}(x), \quad (3.3)$$

and hence also for any $n \geq 1$

$$\frac{\log Dh_{\beta,\lambda,\theta}^n(x)}{\log \sigma} = \frac{n\beta}{1 + \beta} - \sum_{k=0}^{n-1} \chi_{[a_0, c_0]}(h_{\beta,\lambda,\theta}^k(x)). \quad (3.4)$$

Therefore, the following useful lemma holds for Herman's homeomorphism $h_{\beta,\lambda,\theta}$

Lemma 1. *For every $n \geq 1$*

$$\frac{1}{\log \sigma} \log Dh_{\beta,\lambda,\theta}^n(x) = n \frac{\beta}{1 + \beta} \pmod{1}. \quad (3.5)$$

Furthermore, one has

Lemma 2. *Let μ_θ be the invariant measure of Herman's map $h_{\beta,\lambda,\theta}$ with break points a_0, c_0 and an irrational rotation number ρ_θ . Then*

$$\mu_\theta([a_0, c_0]) = \frac{\beta}{1 + \beta}. \quad (3.6)$$

Proof. By Eq. (2.6) one has $\int_{S^1} \frac{\log Dh_{\beta,\lambda,\theta}(x)}{\log \sigma} d\mu_\theta(x) = 0$. Inserting for $\frac{\log Dh_{\beta,\lambda,\theta}(x)}{\log \sigma}$ the right-hand side of (3.3), we get

$$\int_{S^1} \left\{ \frac{\beta}{1 + \beta} - \chi_{[a_0, c_0]}(x) \right\} d\mu_\theta(x) = \frac{\beta}{1 + \beta} - \mu_\theta([a_0, c_0]) = 0,$$

and hence the lemma is proved. ■

REMARK 4. In (3.6) the right-hand side does not depend on the parameter θ .

The uniform distribution of sequences is one of the classical problems of ergodic theory (see, for instance, [17]). Indeed, one has

Theorem 5 (see [17]). *For $[a, b] \subset \mathbb{R}$ let $u_n: [a, b] \rightarrow \mathbb{R}$, $n = 1, 2, \dots$, be a sequence of continuously differentiable real-valued functions. Suppose, for arbitrary $m, n \in \mathbb{N}$, $n \neq m$, the function $Du_n(x) - Du_m(x)$ is monotone with respect to x and that furthermore $|Du_n(x) - Du_m(x)| \geq K > 0$ for some constant K not depending on x , m and n . Then the sequence $u_n(x)$, $n = 1, 2, \dots$ is uniformly distributed $\pmod{1}$ for almost all x in $[a, b]$.*

Since $u_n(x) = q_n \cdot x$ fulfills the assumptions of Theorem 5, it follows that the sequence $\frac{q_n \cdot \beta}{1 + \beta} \pmod{1}$ is uniformly distributed for Lebesgue almost all $\frac{\beta}{1 + \beta}$. Clearly, this sequence is not uniformly distributed for all $\frac{\beta}{1 + \beta}$, since for $\frac{\beta}{1 + \beta} = m\rho_{f_{\beta,\lambda,\theta}} \pmod{1}$ for some integer m ,

$\lim_{n \rightarrow \infty} \left\| \frac{q_n \cdot \beta}{1 + \beta} \right\| = 0$, where $\|x\|$ denotes the distance of x to the nearest integer. In the case of rotation numbers of bounded type one has indeed the following result.

Theorem 6 (see [17, 18]). *Let α be an irrational number of bounded type with partial quotients $\frac{p_n}{q_n}$. Then*

$$\lim_{n \rightarrow \infty} \|q_n x\| = 0$$

if and only if $x \in \mathbb{Z} \alpha \bmod 1$.

4. On the location of break points

In this section we recall four lemmas from [10] concerning the break points for an arbitrary P -homeomorphism f with an irrational rotation number ρ_f and two break points $a_0 = a_0^*$ and $c_0 = c_0^*$ not on the same orbit. They describe the locations of break points of the map f^{q_n} and the values of the derivative Df^{q_n} on S^1 , where $\frac{p_n}{q_n}$ are the partial quotients of ρ_f . Obviously, the map f^{q_n} has $2q_n$ break points denoted by $BP(f^{q_n}) := BP(f^{q_n}; a_0^*) \cup BP(f^{q_n}; c_0^*)$ with $BP(f^{q_n}; a_0^*) := \{a_0^*, a_{-1}^*, \dots, a_{-q_n+1}^*\}$, respectively, $BP(f^{q_n}; c_0^*) := \{c_0^*, c_{-1}^*, \dots, c_{-q_n+1}^*\}$, where $a_{-i}^* = f^{-i}(a_0^*)$, respectively, $c_{-i}^* = f^{-i}(c_0^*)$, $0 \leq i \leq q_n - 1$.

It is clear that the break points of the map f^{q_n} define a partition $\mathcal{P}_n(f)$ of the circle S^1 into $2q_n$ intervals with pairwise non-intersecting interior.

Let $\xi_n(a_0^*)$ be the n th dynamical partition determined by the break point $a_0 = a_0^*$ with respect to the map f . Then one has for the second break point c_0^* either $c_0^* \in I_{i_0}^{(n)}(a_0)$ for some $0 \leq i_0 < q_{n-1}$, or $c_0^* \in I_{j_0}^{(n-1)}(a_0) = f^{j_0}((a_0, a_{-q_n}]) \cup f^{j_0}((a_{-q_n}, a_{q_{n-1}}))$ for some $0 \leq j_0 < q_n$, i.e., $c_0^* \in f^{j_0}((a_0, a_{-q_n}])$ or $c_0^* \in f^{j_0}((a_{-q_n}, a_{q_{n-1}}))$. The two last cases have to be treated separately. The following three lemmas describe the location of the break points of f^{q_n} in intervals of different n dynamical partitions (for the proofs see [10]).

Lemma 3 ([10]). *Assume $c_0^* \in I_{i_0}^{(n)}(a_0^*)$ for some i_0 with $0 \leq i_0 < q_{n-1}$. Then the break points $\{a_{-i}^*, c_{-i}^*, 0 \leq i \leq q_n - 1\}$ of f^{q_n} belong to the following elements of the dynamical partition $\xi_n(a_0^*)$ (see also Fig. 2):*

- $a_0^* \in I_0^{(n)}(a_0^*)$;
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*) \in I_s^{(n)}(a_0^*)$, $0 \leq s \leq i_0$;
- $a_{-q_n+s}^* = f^s(a_{-q_n}^*) \in f^s((a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*)$, $1 \leq s \leq i_0$;
- $a_{-q_n+s}^*, c_{-q_n-i_0+s}^* = f^s(c_{-q_n-i_0}^*) \in f^s((a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*)$, $i_0 + 1 \leq s \leq q_n - 1$.

Lemma 4 ([10]). *Assume $c_0^* \in f^{i_0}((a_0^*, a_{-q_n}])$ for some $0 \leq i_0 < q_n$. Then the break points of f^{q_n} belong to the following elements of the dynamical partition $\xi_n(c_{-i_0}^*)$ of the break point $c_{-i_0}^*$ (see Fig. 3):*

- $c_{-i_0}^*, a_0^* \in I_0^{(n)}(c_{-i_0}^*)$
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*)$, $a_{-q_n+s}^* = f^s(a_{-q_n}^*) \in f^s([c_{-i_0}^*, a_{-q_n}]) \subset I_s^{(n-1)}(c_{-i_0}^*)$, $1 \leq s \leq i_0$;
- $c_{-q_n-i_0+s}^* = f^s(c_{-q_n-i_0}^*)$, $a_{-q_n+s}^* = f^s(a_{-q_n}^*) \in f^s([c_{-i_0}^*, c_{-q_n}]) \subset I_s^{(n-1)}(c_{-i_0}^*)$, $i_0 + 1 \leq s \leq q_n - 1$.

Fig. 2

Fig. 3

Lemma 5 ([10]). If $c_0^* \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}])$ for some i_0 with $0 \leq i_0 < q_n$, the break points of f^{q_n} are located in the following elements of the dynamical partition $\xi_n(a_{-q_n+1}^*)$ of the break point $a_{-q_n+1}^*$ (see also Fig. 4):

- $a_{-q_n+1+s}^* \in I_s^{(n-1)}(a_{-q_n+1}^*), \quad c_{-i_0+1+s}^* \in I_s^{(n-1)}(a_{-q_n+1}^*), \quad 0 \leq s \leq i_0 - 1;$
- $a_{-q_n+i_0+1+s}^* \in I_{i_0+s}^{(n-1)}(a_{-q_n+1}^*), \quad c_{-q_n+1+s}^* \in I_{i_0+s}^{(n-1)}(a_{-q_n+1}^*), \quad 0 \leq s \leq q_n - i_0 - 1.$

Fig. 4

Next we recall the case of a P -homeomorphism f with an irrational rotation number ρ_f and two break points $a_0^* := a_0$, $a_{i_0}^* := f^{i_0}(a_0)$, $i_0 > 0$, on the same orbit. Put $n_{i_0} := \min\{n : q_n \geq i_0\}$. Assume that $n > n_{i_0}$. If the total jump ratio is $\sigma_f = 1$, the map f^{q_n} has $2i_0$ break points

$$a_{-q_n+1}^* := a_{-q_n+1}, \quad a_{-q_n+2}^* := a_{-q_n+2}, \quad \dots, \quad a_{-q_n+i_0}^* := a_{-q_n+i_0}$$

and

$$a_1^* := a_1, a_2^* := a_2, \dots, a_{i_0}^* := a_{i_0}.$$

If $\sigma_f \neq 1$, the map f^{q_n} has $q_n + i_0$ break points

$$a_{-q_n+1}^* := a_{-q_n+1}, \quad a_{-q_n+2}^* := a_{-q_n+2}, \dots, a_0^* := a_0, \dots, a_{i_0}^* := a_{i_0}.$$

Then one has

Lemma 6 ([10]). Assume f is a P -homeomorphism with an irrational rotation number ρ_f and two break points $a_0^* := a_0$, $a_{i_0}^* := f^{i_0}(a_0)$, $i_0 > 0$, on the same orbit. Choose $n > n_{i_0}$.

1) If $\sigma_f = 1$, then one finds for the break points $a_{-q_n+s+1}^*$, a_{s+1}^* of f^{q_n}

- $a_{-q_n+s+1}^*, a_{s+1}^* \in f^s([a_1^*, a_{-q_n+1}^*]) \subset I_{s+1}^{(n-1)}(a_0^*) \in \xi_n(a_0^*)$, $0 \leq s \leq i_0 - 1$;

(see Fig. 5);

2) if $\sigma_f \neq 1$, we have

- $a_0^* \in I_0^{(n-1)}(a_0^*)$;
- $a_{-q_n+1+s}^*, a_{1+s}^* \in f^s([a_{i_0+1}^*, a_{-q_n+i_0+1}^*]) \subset I_{i_0+1+s}^{(n-1)}(a_0^*)$, $0 \leq s \leq q_n - i_0 - 2$.
- $a_{s+1}^* \in f^s([a_1^*, a_{-q_n+1}^*]) \subset I_{1+s}^{(n-1)}(a_0^*)$, $i_0 \leq s \leq q_n - i_0 - 1$.

Fig. 5

Lemmas 3 to 6 show the location of the break points of f^{q_n} on elements of different n th dynamical partitions determined by the map f , respectively, their order along the circle. Indeed, these lemmas hold true also for any pure rotation R_ρ with ρ irrational and any two points $a_0, c_0 \in S^1$, whose preimages under $R_\rho^{q_n}$ correspond to the break points of the P -homeomorphism f^{q_n} .

Lemmas 3–5 imply the following corollary.

Corollary 1. *Let h be a PL -homeomorphism with two break points a_0 and c_0 . Then each interval of the partition ξ_n^* contains exactly one $z \in BP(h^{q_n}; a_0)$ and one $z \in BP(h^{q_n}; c_0)$.*

5. Denjoy equality for piecewise linear circle homeomorphisms with two break points

In this section we recall the explicit expressions derived in [10] for the derivatives Dh^{q_n} of the piecewise linear homeomorphism h with two break points a_0, c_0 in terms of the jump ratio $\sigma_h(a^{(0)})$ and the μ_h -measures of certain intervals of the partition $\mathcal{P}_n(h)$ of S^1 determined by the break points of h^{q_n} .

They follow from Lemmas 3–5 applied to a PL -circle homeomorphism h with an irrational rotation number ρ_h and two break points $a_0^* = a_0 = 0$, $c_0^* = c_0$ not on the same orbit and with the total jump ratio $\sigma_h = 1$.

In the case of Lemma 3 and Lemma 5 the break points $PB(h^{q_n}; a_0^*)$ of h^{q_n} , associated with $a_0^* = 0$, and $PB(h^{q_n}; c_0^*)$, associated with c_0^* , alternate in their order along the circle S^1 .

Let n be odd. Obviously, the break points of h^{q_n} define a system of disjoint subintervals of the circle, given in the case of the assumption in Lemma 3 by (see Fig. 2)

$$[c_{-i_0+s}^*, a_{-q_n+s}^*], \quad 1 \leq s \leq i_0, \quad (5.1)$$

respectively,

$$[c_{-i_0-q_n+s}^*, a_{-q_n+s}^*], \quad i_0 + 1 \leq s \leq q_n. \quad (5.2)$$

We combine these subintervals to the subsets

$$A_n(i_0) := \bigcup_{s=1}^{i_0} [c_{-i_0+s}^*, a_{-q_n+s}^*], \quad B_n(i_0) := \bigcup_{s=i_0+1}^{q_n} [c_{-i_0-q_n+s}^*, a_{-q_n+s}^*]. \quad (5.3)$$

In the case of the assumption in Lemma 5 the subintervals are given by (see Fig. 4)

$$[a_{-q_n+s}^*, c_{-i_0+s}^*], \quad 1 \leq s \leq i_0, \quad (5.4)$$

respectively,

$$[a_{-q_n+s}^*, c_{-i_0-q_n+s}^*], \quad i_0 + 1 \leq s \leq q_n, \quad (5.5)$$

which we combine to the subsets

$$A_n(i_0) := \bigcup_{s=1}^{i_0} [a_{-q_n+s}^*, c_{-i_0+s}^*], \quad B_n(i_0) := \bigcup_{s=i_0+1}^{q_n} [a_{-q_n+s}^*, c_{-i_0-q_n+s}^*]. \quad (5.6)$$

For n even, the orientation of the above intervals has to be reversed. Therefore, in the case of Lemma 3 we have the following system of disjoint intervals:

$$[a_{-q_n+s}^*, c_{-i_0+s}^*], \quad 1 \leq s \leq i_0, \quad (5.7)$$

respectively,

$$[a_{-q_n+s}^*, c_{-i_0-q_n+s}^*], \quad i_0 + 1 \leq s \leq q_n. \quad (5.8)$$

In the case of Lemma 5 one finds

$$[c_{-i_0+s}^*, a_{-q_n+s}^*], \quad 1 \leq s \leq i_0, \quad (5.9)$$

respectively,

$$[c_{-i_0-q_n+s}^*, a_{-q_n+s}^*], \quad i_0 + 1 \leq s \leq q_n. \quad (5.10)$$

In the case of Lemma 3 and n even, respectively, in the case of Lemma 5 and n odd, the subsets $A_n(i_0)$ and $B_n(i_0)$ can be defined as before. The above constructions show that the boundaries of every interval in the subsets $A_n(i_0)$ and $B_n(i_0)$ consist of break points from $PB(h^{q_n}; a_0^*)$, respectively, $PB(h^{q_n}; c_0^*)$. In the following we abbreviate the jump ratio of h at the break point a_0^* by

$$\sigma := \sigma_h(a_0^*) = \frac{Dh_-(0)}{Dh_+(0)}.$$

Next, we recall the first result on the values of $(Dh^{q_n}(x))$ from [10].

Theorem 7. *Let h be a PL-circle homeomorphism with an irrational rotation number ρ_h and two break points $a_0^* = 0$ and c_0^* , whose total jump ratio is $\sigma_h = 1$, and which lie on different orbits. Assume c_0^* fulfills the assumptions of Lemma 3 with $0 \leq i_0 < q_{n-1}$, respectively, Lemma 5 for some i_0 with $0 \leq i_0 < q_n$. Then in the case of Lemma 3*

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n(i_0) \cup B_n(i_0)) - 1}, & \text{if } x \in A_n(i_0) \cup B_n(i_0), \\ \sigma^{\mu_h(A_n(i_0) \cup B_n(i_0))}, & \text{if } x \in S^1 \setminus (A_n(i_0) \cup B_n(i_0)); \end{cases} \quad (5.11)$$

respectively, in the case of Lemma 5,

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(A_n(i_0) \cup B_n(i_0)) - 1}, & \text{if } x \in A_n(i_0) \cup B_n(i_0), \\ \sigma^{\mu_h(A_n(i_0) \cup B_n(i_0))}, & \text{if } x \in S^1 \setminus (A_n(i_0) \cup B_n(i_0)). \end{cases} \quad (5.12)$$

Theorem 7 shows that Dh^{q_n} is constant on every element of the partition $\mathcal{P}_n(h)$ and takes only two values under the assumptions of Lemmas 3 and 5. Moreover, the values of Dh^{q_n} are determined by the jump ratio $\sigma = \sigma_h(a_0^*)$ and the μ_h -measure of $A_n(i_0) \cup B_n(i_0)$.

In the case of the assumption on c_0^* in Lemma 4 we can define again a system of disjoint subintervals determined by the elements in $\mathcal{P}_n(h)$. Let n be odd. Then these subintervals are as follows (see Fig. 3):

$$[c_{-i_0+s}^*, a_{-q_n+s}^*], \quad 1 \leq s \leq i_0, \quad (5.13)$$

respectively,

$$[a_{-q_n+s}^*, c_{-i_0-q_n+s}^*], \quad i_0 + 1 \leq s \leq q_n. \quad (5.14)$$

For n even the orientation of the above intervals has to be reversed. To determine in the case of Lemma 4 the values of Df^{q_n} , we define

$$A_n(i_0) := \bigcup_{s=1}^{i_0} [c_{-i_0+s}^*, a_{-q_n+s}^*], \quad B_n(i_0) := \bigcup_{s=i_0+1}^{q_n} [a_{-q_n+s}^*, c_{-i_0-q_n+s}^*]. \quad (5.15)$$

Then the following theorem holds.

Theorem 8 ([10]). *Let h be a PL-circle homeomorphism with two break points a_0^* and c_0^* with $\sigma_h = 1$, which lie on different orbits. Assume c_0^* fulfills the assumption of Lemma 4 for some i_0 with $0 \leq i_0 < q_n$. Then for all $n \geq 1$*

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n(i_0)) - \mu_h(B_n(i_0)) - 1} & \text{if } x \in A_n(i_0), \\ \sigma^{\mu_h(A_n(i_0)) - \mu_h(B_n(i_0)) + 1} & \text{if } x \in B_n(i_0), \\ \sigma^{\mu_h(A_n(i_0)) - \mu_h(B_n(i_0))} & \text{if } x \notin A_n(i_0) \cup B_n(i_0). \end{cases} \quad (5.16)$$

It remains to discuss the case of a PL-homeomorphism h with an irrational rotation number ρ_h and two break points $a_0^* = 0$ and $a_{i_0}^* = h^{i_0}(a_0^*)$, $i_0 > 0$, on the same orbit. In this case the break points of h^{q_n} alternate in their order along the circle S^1 . Denote by $U_n(a_s^*)$, $1 \leq s \leq i_0$, the closed intervals with endpoints a_s^* and $a_{-q_n+s}^*$. Obviously, these subintervals are disjoint. Lemma 6 implies that $U_n(a_s^*) \subset I_s^{(n-1)}(a_0^*)$, $1 \leq s \leq i_0$. Next, we define for every $n \geq 1$

$$U_n = \bigcup_{s=1}^{i_0} U_n(a_s^*). \quad (5.17)$$

Then one has

Theorem 9 ([10]). *Let h be a PL-circle homeomorphism with two break points $a_0^* = 0$ and $a_{i_0}^* = h^{i_0}(a_0^*)$, $i_0 > 0$, with $\sigma_h = 1$, which lie on the same orbit. Put $n_{i_0} := \min\{n : q_n \geq i_0\}$. For $n > n_{i_0}$ one finds*

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(U_n)} & \text{if } x \in U_n, \\ \sigma^{\mu_h(U_n) - 1} & \text{if } x \in S^1 \setminus U_n. \end{cases} \quad (5.18)$$

Theorems 7 and 8 lead to the following

Theorem 10. *Let h be a PL-circle homeomorphism with an irrational rotation number ρ_h and two break points $a_0^* = 0$ and c_0^* , which lie on different orbits. Assume c_0^* fulfills the assumptions in Lemma 3 for some i_0 with $0 \leq i_0 < q_{n-1}$, respectively, in Lemma 4 or Lemma 5 for some i_0 with $0 \leq i_0 < q_n$. Then in the case of Lemma 3 or Lemma 4*

$$\frac{1}{\log \sigma} \log Dh^{q_n}(x) = (-1)^n (\mu_h(A_n(i_0)) + \mu_h(B_n(i_0))) = q_n \cdot \frac{\beta}{1 + \beta} \pmod{1}, \quad (5.19)$$

respectively, in the case of Lemma 5,

$$\frac{1}{\log \sigma} \log Dh^{q_n}(x) = (-1)^{n+1} (\mu_h(B_n(i_0)) - \mu_h(A_n(i_0))) = q_n \cdot \frac{\beta}{1 + \beta} \pmod{1}. \quad (5.20)$$

6. Proof of Theorem 3

Let $f \in C^{2+\epsilon}(S^1 \setminus \{b_1, b_2\})$ be a P -homeomorphism of the circle with an irrational rotation number ρ_f and two break points b_1 and b_2 not on the same orbit, whose total jump ratio is $\sigma_f = \sigma_f(b_1) \cdot \sigma_f(b_2) = 1$. Denote by μ_f its unique invariant probability measure. Define the parameters β and λ through

$$\frac{\beta}{1+\beta} := \mu_f([b_1, b_2]), \quad \lambda^{-1-\beta} := \sigma_f(b_1). \quad (6.1)$$

Let $h = h_{\beta, \lambda, \theta}$ be Herman's PL -homeomorphism of S^1 with break points $a_0 = 0$ and $c_0 = c$ such that $\lambda c = \lambda^{-\beta}(c-1) + 1$. Since the rotation number ρ_f is irrational, we can find a unique $\bar{\theta}$ such that the rotation number $\rho_{\bar{\theta}}$ of $h_{\bar{\theta}} = h_{\beta, \lambda, \bar{\theta}}$ coincides with ρ_f . Denote by $\mu_{\bar{\theta}}$ the invariant measure of $h_{\bar{\theta}}$. By Lemma 2 $\mu_{\bar{\theta}}([a_0, c_0]) = \frac{\beta}{1+\beta}$. Since $\rho_f = \rho_{\bar{\theta}}$, the homeomorphisms f and $h_{\bar{\theta}}$ are topologically conjugate via some homeomorphism φ . We can choose φ such that $a_0 = \varphi(b_1)$ and $c_0 = \varphi(b_2)$, because $\mu_f([b_1, b_2]) = \mu_{\bar{\theta}}([a_0, c_0])$. Then one has also $(\varphi^* \mu_{\bar{\theta}})([b_1, b_2]) = \mu_f([b_1, b_2]) = \mu_{\bar{\theta}}([a_0, c_0])$, since the invariant probability measure of f is unique. Hence, we have proved the following fact, which will play a key role in our proof of the main theorem.

Theorem 11. *Consider the P -homeomorphism $f : S^1 \rightarrow S^1$ with an irrational rotation number ρ_f and two break points b_1, b_2 with jump ratios $\sigma := \sigma_f(b_1)$ and $\sigma_f(b_2) = \sigma^{-1}$. Let $h_{\bar{\theta}} := h_{\beta, \lambda, \bar{\theta}}$ be Herman's PL -homeomorphism with two break points $a_0 = 0$, c_0 , parameters β, λ as defined by (6.1) and an irrational rotation number $\rho_{\bar{\theta}} = \rho_f$. Then*

- (I) $\sigma_{\bar{\theta}}(a_0) = \sigma$ and $\mu_{\bar{\theta}}([a_0, c_0]) = \mu_f([b_1, b_2])$;
- (II) *the maps f and $h_{\bar{\theta}}$ are topologically conjugate by some circle homeomorphism φ with $\varphi(b_1) = a_0$ and $\varphi(b_2) = c_0$.*

Since the rotation number ρ_f is irrational, the invariant probability measure μ_f has no discrete ergodic component. Indeed, one knows that every such P -homeomorphism is ergodic also w.r.t. Lebesgue measure l (see [14]). Suppose μ_f has an absolutely continuous component $\mu_f^{a.c.}$ with support A and $\mu_f^{a.c.}(A) > 0$. Then also $l(A) > 0$. If $p(x)$ is the density of $\mu_f^{a.c.}$, then on A obviously $p(x) \geq 0$ and on $S^1 \setminus A$ one has $p(x) = 0$. Since $p(x)$ satisfies the functional equation $p(f(x)) = \frac{1}{Df(x)}p(x)$, $x \in S^1$ and $Df(x) \geq \text{const} > 0$, the subset $A_+ = \{x : p(x) > 0\}$ is f -invariant. Ergodicity of f with respect to Lebesgue measure l implies that either $\mu_f^{a.c.}(A) = 1$ or $\mu_f^{a.c.}(A) = 0$. Hence, the invariant measure μ_f is either pure absolutely continuous or pure singular on S^1 .

The idea for the proof of Theorem 3 is to construct for the homeomorphism f a sequence of measurable subsets $G_{n_m} \subset S^1$ such that $l(G_{n_m}) > \text{const} > 0$ and

$$\liminf_{m \rightarrow \infty} l\{x \in G_{n_m} : |Df^{q_{n_m}}(x) - 1| > \delta\} > 0$$

for some $\delta > 0$. On the other hand, one knows that under the assumption of absolute continuity of the invariant measure μ_f w.r.t. Lebesgue measure l , $Df^{q_n}(x)$ tends, as $n \rightarrow \infty$, to 1 in probability with respect to the probability measure l .

Hence, let us start with the following proposition shown in [14].

Proposition 1. *Let f be a P -homeomorphism of the circle with an irrational rotation number ρ_f . If its invariant probability measure μ_f is equivalent to Lebesgue measure l , then for all $\delta > 0$ the sequence $l(\{x: |Df^{q_n}(x) - 1| > \delta\})$ tends to zero as $n \rightarrow \infty$.*

Important for the construction of the above mentioned sets G_{n_m} will be

Lemma 7. *For arbitrary $\delta \in (0, 1)$ and $n \geq 1$ consider three points $z_1, z_2, z_3 \in S^1$, $z_2 \in BP(f^{q_n})$, with $z_1 \prec z_2 \prec z_3 \prec z_1$ such that the intervals $[z_1, z_2]$ and $[z_2, z_3]$ are q_n -small. Assume the P -homeomorphism $f^{q_n} \in C^{2+\varepsilon}(S^1 \setminus \{z_2\})$ has a jump ratio $\sigma_{f^{q_n}}(z_2) = \Lambda$ at the break point z_2 . For v the total variation of $\log Df$ on S^1 and $t_l \in (z_1, z_2)$ and $t_r \in (z_2, z_3)$ with*

$$\frac{l([t_l, z_2])}{l([z_1, z_2])} = \frac{l([t_r, z_3])}{l([z_2, z_3])} = \delta \quad (6.2)$$

one has either

$$\log Df^{q_n}(x) \leq -\frac{\log \Lambda}{2} + Ke^v \delta, \quad (6.3)$$

for all $x \in [t_l, z_2)$, or

$$\log Df^{q_n}(y) \geq \frac{\log \Lambda}{2} - Ke^v \delta \quad (6.4)$$

for all $y \in (z_2, t_r]$, when $\Lambda > 1$.

In the case $\Lambda < 1$ one has either

$$\log Df^{q_n}(x) \geq -\frac{\log \Lambda}{2} - Ke^v \delta, \quad (6.5)$$

for all $x \in [t_l, z_2)$, or

$$\log Df^{q_n}(y) \leq \frac{\log \Lambda}{2} + Ke^v \delta \quad (6.6)$$

for all $y \in (z_2, t_r]$.

Proof. Assume $\log \Lambda = \log \frac{Df_-^{q_n}(z_2)}{Df_+^{q_n}(z_2)} > 0$, the case $\log \Lambda < 0$ can be treated analogously. Then $\log Df_-^{q_n}(z_2) = \log \Lambda + \log Df_+^{q_n}(z_2)$, and hence,

$$\log Df_+^{q_n}(z_2) \leq -\frac{\log \Lambda}{2} \text{ if and only if } \log Df_-^{q_n}(z_2) \leq \frac{\log \Lambda}{2}, \quad (6.7)$$

respectively,

$$\log Df_+^{q_n}(z_2) \geq -\frac{\log \Lambda}{2} \text{ if and only if } \log Df_-^{q_n}(z_2) \geq \frac{\log \Lambda}{2}. \quad (6.8)$$

Hence, either

$$\log Df_+^{q_n}(z_2) \leq -\frac{\log \Lambda}{2} \quad (6.9)$$

or

$$\log Df_-^{q_n}(z_2) \geq \frac{\log \Lambda}{2}. \quad (6.10)$$

Then for arbitrary $x \in [t_l, z_2]$ one finds

$$\begin{aligned} \left| \log \frac{Df^{q_n}(z_2)}{Df^{q_n}(x)} \right| &\leq \sum_{j=0}^{q_n-1} |\log Df_-(f^j(z_2)) - \log Df_-(f^j(x))| \leq K \sum_{j=0}^{q_n-1} l([f^j(x), f^j(z_2)]) \leq \\ &\leq K \sum_{j=0}^{q_n-1} l([f^j(t_l), f^j(z_2)]) \leq K \sum_{j=0}^{q_n-1} \frac{l([f^j(t_l), f^j(z_2)])}{l([f^j(z_1), f^j(z_2)])} l([f^j(z_1), f^j(z_2)]) = \\ &= K \sum_{j=0}^{q_n-1} \frac{Df^j(\zeta)}{Df^j(\vartheta)} \frac{l([t_l, z_2])}{l([z_1, z_2])} l([f^j(z_1), f^j(z_2)]) \end{aligned}$$

for certain $\zeta \in [t_l, z_2]$, $\vartheta \in [z_1, z_2]$ and an universal constant $K > 0$ depending only on f . According to inequality (2.3),

$$e^{-v} \leq \frac{Df^j(\zeta)}{Df^j(\vartheta)} \leq e^v$$

and therefore

$$K \sum_{j=0}^{q_n-1} \frac{Df^j(\zeta)}{Df^j(\vartheta)} \frac{l([t_l, z_2])}{l([z_1, z_2])} l([f^j(z_1), f^j(z_2)]) \leq K e^v \delta,$$

since $\frac{l([t_l, z_2])}{l([z_1, z_2])} = \delta$. We have also used the fact that the interval $[z_1, z_2]$ is q_n -small and hence the intervals $[f^j(z_1), f^j(z_2)]$, $0 \leq j \leq q_n - 1$ are disjoint. This leads finally to the bound

$$\left| \log \frac{Df^{q_n}(z_2)}{Df^{q_n}(x)} \right| \leq K e^v \delta. \quad (6.11)$$

In the same way it can be shown that

$$\left| \log \frac{Df^{q_n}(z_2)}{Df^{q_n}(y)} \right| \leq K e^v \delta \quad (6.12)$$

for all $y \in [t_r, z_2]$. Inserting the bounds (6.9)–(6.12), we get the bounds (6.3)–(6.6) in Lemma 7. ■

An important role in the proof of Theorem 3 is played by certain neighborhoods of the break points of the P -homeomorphisms f^{q_n} , which we define next. For this, recall the partition $\mathcal{P}_n(f)$ of the circle defined by the breakpoints $BP(f^{q_n})$ of the map f^{q_n} in Section 4. For $z \in BP(f^{q_n})$ we denote by $V_n^l(z)$, respectively, $V_n^r(z)$ the interval in $\mathcal{P}_n(f)$, whose right, respectively, left boundary point is the break point z . Given some $\delta \in (0, 1)$, using Lemma 7 we can then construct left and right subintervals $V_n^l(z; \delta) \subset V_n^l(z)$, respectively, $V_n^r(z; \delta) \subset V_n^r(z)$, both with the break point z as a boundary point, such that

$$\frac{l(V_n^l(z; \delta))}{l(V_n^l(z))} = \delta, \quad \frac{l(V_n^r(z; \delta))}{l(V_n^r(z))} = \delta.$$

Definition 3. The subintervals $V_n^l(z; \delta)$, $V_n^r(z; \delta)$, respectively, the interval $V_n(z; \delta) = V_n^l(z; \delta) \cup V_n^r(z; \delta)$ are called *the left normalized δ -neighborhood*, *the right normalized δ -neighborhood*, respectively, *the normalized δ -neighborhood of the break point z* .

Assume now there exists for $\rho = \rho_f = \rho_{\bar{\theta}}$, $\rho \in \Gamma$ a set M_ρ such that the invariant measure μ_f is absolutely continuous w.r.t. Lebesgue measure l if $\mu_f([b_1, b_2]) \in M_\rho$.

We will show that this assumption contradicts Proposition 1.

For this, we derive first some properties of the distribution of $\log Dh_{\bar{\theta}}^{q_n}(x)$ with respect to the partition $\mathcal{P}_n(h_{\bar{\theta}})$ of Herman's PL -homeomorphisms $h_{\bar{\theta}}$ defined in Theorem 11.

For this, recall the sets $A_n(i_0), B_n(i_0)$ defined in (5.3), (5.6), respectively, (5.15). Later we will need also the closely related sets $A_n^*(i_0), B_n^*(i_0)$, defined according to Lemmas 3–5 as follows:

- if the break point c_0 of Dh^{q_n} fulfills the assumption in Lemmas 3 or 5, we set (see Fig. 2 and Fig. 4)

$$A_n^*(i_0) := \bigcup_{s=1}^{i_0} [a_{-q_n+s}^*, a_{q_{n-1}+s}], \quad (6.13)$$

and

$$B_n^*(i_0) := \bigcup_{s=i_0+1}^{q_n} [a_{-q_n+s}^*, a_{q_{n-1}+s}]; \quad (6.14)$$

- if the break point c_0 of Dh^{q_n} , on the other hand, fulfills the assumption in Lemma 4, we set (see Fig. 3)

$$A_n^*(i_0) := \bigcup_{s=1}^{i_0} [a_{-q_n+s}^*, c_{-i_0+q_{n-1}+s}], \quad (6.15)$$

and

$$B_n^*(i_0) := \bigcup_{s=1}^{q_n-i_0-1} [c_{-q_n+s}^*, c_{q_{n-1}+s}]. \quad (6.16)$$

Consider first the case when the break point c_0 of Dh^{q_n} fulfills the assumption of Lemma 3 or Lemma 4. We begin our discussion with the case of Lemma 3. By Theorem 10

$$\frac{1}{\log \sigma} \log Dh^{q_n}(x) = (-1)^n (\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0))) = q_n \cdot \frac{\beta}{1+\beta} \pmod{1} \quad (6.17)$$

for all $x \in A_n(i_0) \cup B_n(i_0)$ as defined in (5.3). Therefore,

$$\frac{1}{\log \sigma} \log Dh^{q_n}(x) \pmod{1} = \begin{cases} 1 - (\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0))) & \text{if } n \text{ is odd,} \\ \mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) & \text{if } n \text{ is even.} \end{cases} \quad (6.18)$$

Next, we estimate $\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0))$. By Lemma 3 (see Fig. 2) we have obviously

$$A_n(i_0) = \bigcup_{s=1}^{i_0} [c_{-i_0+s}^*, a_{-q_n+s}^*] \subset \bigcup_{s=1}^{i_0} [a_{q_n+s}, a_{-q_n+s}^*] \quad (6.19)$$

and

$$B_n(i_0) = \bigcup_{s=i_0+1}^{q_n} [c_{-i_0-q_n+s}^*, a_{-q_n+s}^*] \subset \bigcup_{s=i_0+1}^{q_n} [a_s, a_{-q_n+s}^*]. \quad (6.20)$$

It is easy to see that

$$\mu_{\bar{\theta}}(A_n(i_0)) \leq 2i_0 \cdot \Delta_n, \quad \mu_{\bar{\theta}}(B_n(i_0)) \leq (q_n - i_0) \cdot \Delta_n, \quad (6.21)$$

Using these bounds and the recurrence relations

$$q_n = k_n q_{n-1} + q_{n-2}, \quad \Delta_n = k_n \Delta_{n+1} + \Delta_{n+2}, \quad n \geq 1,$$

we obtain

$$\begin{aligned} \mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) &\leq (q_n + i_0) \cdot \Delta_n \leq \frac{(q_n + q_{n-1})\Delta_n}{q_n \Delta_{n-1}} = \\ &= \left(1 + \frac{q_{n-1}}{q_n}\right) \cdot \frac{\Delta_n}{\Delta_{n-1}} < \left(1 + \frac{1}{k_n}\right) \cdot \frac{1}{k_{n+1}}. \end{aligned}$$

Since $k_n \geq 3$ for $n \geq N_\rho$, we hence get

$$\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) < \frac{4}{9} \quad (6.22)$$

for $n \geq N_\rho$. Summarizing Eqs. (6.17), (6.18) and the last bound, we conclude that in the case of Lemma 3 and $n > N_\rho$

$$0 \leq \frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1} < \frac{4}{9}, \quad \text{for } n \text{ even}, \quad (6.23)$$

and

$$\frac{5}{9} < \frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1} \leq 1, \quad \text{for } n \text{ odd}. \quad (6.24)$$

Consider now the case of Lemma 4.

Using the second assertion of Theorem 10, it follows that

$$\frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1} \in \{|\mu_{\bar{\theta}}(B_n(i_0)) - \mu_{\bar{\theta}}(A_n(i_0))|, 1 - |\mu_{\bar{\theta}}(B_n(i_0)) - \mu_{\bar{\theta}}(A_n(i_0))|\} \quad (6.25)$$

for all $x \in A_n(i_0) \cup B_n(i_0)$ defined in (5.15). Lemma 4 and (5.15) imply for $n \geq N_\rho$

$$\begin{aligned} |\mu_{\bar{\theta}}(A_n(i_0)) - \mu_{\bar{\theta}}(B_n(i_0))| &\leq \mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) < \\ &< i_0 \cdot \Delta_n + (q_n - i_0) \cdot \Delta_n = q_n \cdot \Delta_n < \frac{q_n \cdot \Delta_n}{q_n \cdot \Delta_{n-1}} = \frac{\Delta_n}{\Delta_{n-1}} < \frac{1}{3}. \end{aligned}$$

This and (6.25) lead, in case the break point c_0 of Dh^{q_n} fulfills the assumption of Lemma 4, to

$$\frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1} \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

This together with the bounds (6.23) and (6.24) show that, in case the break point c_0 of Dh^{q_n} fulfills either the assumption of Lemma 3 or of Lemma 4, $\frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1}$ takes values only in $[0, 1] \setminus \left[\frac{4}{9}, \frac{5}{9}\right]$, i.e., for $n \geq N_\rho$ and $x \in A_n(i_0) \cup B_n(i_0)$

$$\frac{1}{\log \sigma} \log Dh_{\bar{\theta}}^{q_n}(x) \pmod{1} \in \left[0, \frac{4}{9}\right] \cup \left[\frac{5}{9}, 1\right]. \quad (6.26)$$

From Theorem 10, however, we know that the sequence

$$\left\{ \frac{1}{\log \sigma} \log Dh_{\theta}^{q_n}(x) = q_n \cdot \frac{\beta}{1+\beta} \pmod{1} \right\}$$

is uniformly distributed on $[0, 1]$ and hence its values are dense in $[0, 1]$. From (6.26) we therefore conclude that the cases where the break point c_0 of Dh^{q_n} fulfills either the assumption of Lemma 3 or of Lemma 4 can happen only for a finite number of q_n 's. This means, on the other hand, that the values of $\frac{1}{\log \sigma} \log Dh_{\theta}^{q_n}$ are dense on $[0, 1]$ only in case the break point c_0 of $Dh_{\theta}^{q_n}$ fulfills the assumption of Lemma 5.

Hence, we have to discuss next this case. By Theorem 10 we have for $x \in A_n(i_0) \cup B_n(i_0)$, defined in (5.6),

$$\frac{1}{\log \sigma} \log Dh^{q_n}(x) = (-1)^n (\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0))) \pmod{1}. \quad (6.27)$$

But this means that

$$Dh^{q_n}(x) \pmod{1} = \begin{cases} \mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) & \text{if } n \text{ is even,} \\ 1 - (\mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0))) & \text{if } n \text{ is odd.} \end{cases} \quad (6.28)$$

We can assume that for some subsequence $\{q_{n_m}\}$

$$\lim_{m \rightarrow \infty} Dh^{q_{2n_m}}(x) = \lim_{m \rightarrow \infty} (\mu_h(B_{2n_m}(i_0)) + \mu_h(A_{n_{2n_m}}(i_0))) = \omega \quad (6.29)$$

with $\omega \in [0, 1]$. The case

$$\lim_{m \rightarrow \infty} Dh^{q_{2n_m+1}}(x) = \lim_{m \rightarrow \infty} (1 - (\mu_h(B_{2n_m+1}(i_0)) + \mu_h(A_{n_{2n_m+1}}(i_0)))) = \omega$$

with $\omega \in [0, 1]$ can be treated analogously.

We have to consider two possible cases, namely, when $0 \leq i_0 < \frac{q_n}{2}$ and when $\frac{q_n}{2} \leq i_0 < q_n$. Consider the case $0 \leq i_0 < \frac{q_n}{2}$.

We start with several bounds and estimate first the ratio $\frac{\mu_{\bar{\theta}}(A_n(i_0))}{\mu_{\bar{\theta}}(B_n(i_0))}$. It is clear that

$$\frac{\mu_{\bar{\theta}}(A_n(i_0))}{\mu_{\bar{\theta}}(B_n(i_0))} = \frac{i_0 \mu_{\bar{\theta}}([a_{-q_n+1}^*, c_{-i_0+1}^*])}{(q_n - i_0) \mu_{\bar{\theta}}([a_{-q_n+1}^*, c_{-i_0+1}^*]) + (q_n - i_0) \mu_{\bar{\theta}}([c_1, c_{-q_n+1}^*])} \leq \frac{i_0}{q_n - i_0} \leq \frac{\frac{q_n}{2}}{q_n - \frac{q_n}{2}} = 1,$$

where we have used $0 \leq i_0 < \frac{q_n}{2}$. Consequently,

$$\mu_{\bar{\theta}}(A_n(i_0)) \leq \mu_{\bar{\theta}}(B_n(i_0)). \quad (6.30)$$

Using this bound, we obtain $1 \geq \mu_{\bar{\theta}}(A_n(i_0)) + \mu_{\bar{\theta}}(B_n(i_0)) \geq 2\mu_{\bar{\theta}}(A_n(i_0))$, which implies

$$\mu_{\bar{\theta}}(A_n(i_0)) \leq \frac{1}{2}. \quad (6.31)$$

From the definition of $B_n(i_0)$ and $B_n^*(i_0)$ in (5.6) and (6.14) it follows that

$$\mu_{\bar{\theta}}(B_n^*(i_0)) = (q_n - i_0)\Delta_{n-1} - \mu_{\bar{\theta}}(B_n(i_0)). \quad (6.32)$$

Next, we estimate $(q_n - i_0)\Delta_{n-1}$. It is obvious that

$$q_n\Delta_{n-1} + q_{n-1}\Delta_n = 1. \quad (6.33)$$

It is clear that

$$\frac{q_n\Delta_{n-1}}{q_{n-1}\Delta_n} = \frac{k_n q_{n-1} + q_{n-2}}{q_{n-1}} \cdot \frac{k_{n+1}\Delta_n + \Delta_{n+2}}{\Delta_n} = \left(k_n + \frac{q_{n-2}}{q_{n-1}}\right) \left(k_{n+1} + \frac{\Delta_{n+1}}{\Delta_n}\right).$$

This and (6.33) imply

$$q_n\Delta_{n-1} > 1 - \frac{1}{k_n k_{n+1} + 1} = \frac{k_n k_{n+1}}{k_n k_{n+1} + 1}. \quad (6.34)$$

Next, we compare $q_n\Delta_{n-1}$ and $(q_n - i_0)\Delta_{n-1}$. Since $0 \leq i_0 < \frac{q_n}{2}$, we have

$$\frac{(q_n - i_0)\Delta_{n-1}}{q_n\Delta_{n-1}} \geq \frac{1}{2}.$$

Together with (6.34) this implies that

$$(q_n - i_0)\Delta_{n-1} \geq \frac{1}{2}q_n\Delta_{n-1} \geq \frac{1}{2} \cdot \frac{k_n k_{n+1}}{k_n k_{n+1} + 1}.$$

Since $k_n \geq 3$ for $n \geq N_\rho$, we therefore have shown

$$(q_n - i_0)\Delta_{n-1} \geq \frac{1}{2} \cdot \frac{9}{10} = \frac{9}{20}. \quad (6.35)$$

Suppose now $n_m, m = 1, 2, \dots$ is some sequence of even numbers such that

$$\lim_{m \rightarrow \infty} (\mu_{\bar{\theta}}(B_{n_m}(i_0)) + \mu_{\bar{\theta}}(A_{n_m}(i_0))) = \omega_1 \quad (6.36)$$

with $\omega_1 \in [0, 1]$. Then, for any $\varepsilon > 0$ and sufficiently large n_m ,

$$\omega_1 - \mu_{\bar{\theta}}(A_{n_m}(i_0)) - \varepsilon \leq \mu_{\bar{\theta}}(B_{n_m}(i_0)) \leq \omega_1 - \mu_{\bar{\theta}}(A_{n_m}(i_0)) + \varepsilon.$$

This together with (6.31) implies

$$\omega_1 - \frac{1}{2} - \varepsilon \leq \mu_{\bar{\theta}}(B_n(i_0)) < \omega_1 + \varepsilon.$$

This together with (6.32) implies for sufficiently large n_m

$$\mu_{\bar{\theta}}(B_{n_m}^*(i_0)) \geq \frac{9}{20} - \omega_1 - \varepsilon. \quad (6.37)$$

The last two relations show that $\mu_{\bar{\theta}}(B_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(B_{n_m}^*(i_0))$ are comparable with some constant $C > 1$.

In the next step of the proof of Theorem 3 we consider two copies of the unit circle. Suppose the homeomorphism f acts on the first circle, and Herman's homeomorphism $h_{\bar{\theta}}$, on the second one. It is obvious that

$$\mu_f(\varphi^{-1}(B_{n_m}(i_0))) = \mu_{\bar{\theta}}(B_{n_m}(i_0)), \quad \mu_f(\varphi^{-1}(B_{n_m}^*(i_0))) = \mu_{\bar{\theta}}(B_{n_m}^*(i_0)).$$

These relations together with (6.37) imply that

$$c_1 \leq \mu_f(\varphi^{-1}(A_{n_m}(i_0))) \leq c_2, \quad c_1 \leq \mu_f(\varphi^{-1}(A_{n_m}^*(i_0))) \leq c_2.$$

The last relations, the arrangement of the break points of f^{q_n} and absolute continuity of the invariant measure μ_f w.r.t. Lebesgue measure l imply for sufficiently large n_m the following bounds:

$$c_3 \leq l(\varphi^{-1}(A_{n_m}(i_0))) \leq c_4, \quad c_3 \leq l(\varphi^{-1}(A_{n_m}^*(i_0))) \leq c_4 \quad (6.38)$$

for some constants $c_3, c_4 \in (0, 1)$.

Consider next the set of break points

$$BP_1(f^{q_{n_m}}) := \{c_{-q_n+1}^*, c_{-q_n+2}^*, \dots, c_{-i_0}^*\} \subset BP(f^{q_{n_m}}, c_{-i_0}^*)$$

and for $z \in BP_1(f^{q_{n_m}})$ the left and right normalized δ -neighborhoods $V_{n_m}^l(z, \delta)$, respectively, $V_{n_m}^r(z, \delta)$. Put

$$V_{n_m}^l(\delta) := \bigcup_{z \in BP_1(f^{q_{n_m}})} V_n^l(z, \delta), \quad V_{n_m}^r(\delta) := \bigcup_{z \in BP_1(f^{q_{n_m}})} V_n^r(z, \delta).$$

Obviously,

$$V_{n_m}^l(\delta) \subset \varphi^{-1}(B_{n_m}(i_0)), \quad V_{n_m}^r(\delta) \subset \varphi^{-1}(B_{n_m}^*(i_0)).$$

We conclude that the length l of each of these intervals covering the normalized δ -neighborhoods by definition is δ^{-1} times the length of the latter ones. Using the definitions of $B_{n_m}(i_0)$, $B_{n_m}^*(i_0)$ (see also Fig. 4) and the normalized one-sided δ -neighborhoods $V_n(z, \delta)$, we obtain

$$l(\varphi^{-1}(B_{n_m}(i_0) \cap V_{n_m}^l(\delta))) = \delta \cdot l(\varphi^{-1}(B_{n_m}(i_0))), \quad (6.39)$$

$$l(\varphi^{-1}(B_{n_m}^*(i_0) \cap V_{n_m}^r(\delta))) = \delta \cdot l(\varphi^{-1}(B_{n_m}^*(i_0))) \quad (6.40)$$

and consequently

$$l((\varphi^{-1}(B_{n_m}(i_0)) \cup \varphi^{-1}(B_{n_m}^*(i_0))) \cap (V_{n_m}^l(\delta) \cup V_{n_m}^r(\delta))) = \delta \cdot l(\varphi^{-1}(B_{n_m}(i_0) \cup B_{n_m}^*(i_0))). \quad (6.41)$$

From this we can now derive bounds on the values of $Df^{q_{n_m}}$. Put $\delta = \left| \frac{\log \Lambda}{aK e^v} \right|$ with Λ as defined in Lemma 7 and $a > \left| \frac{\log \Lambda}{K e^v} \right|$. From relations (6.3) and (6.4) of Lemma 7 it follows that **either** on the left **or** on the right normalized δ -neighborhood of every break point z of $f^{q_{n_m}}$

$$|\log Df^{q_{n_m}}(x)| \geq \left| \frac{(a-2) \log \Lambda}{2a} \right|. \quad (6.42)$$

Define

$$G_{n_m}(\delta) = \bigcup \{z \in BP_1(f^{q_{n_m}}) : |\log Df^{q_n}(x)| \geq \left| \frac{(a-2) \log \Lambda}{2a} \right| \text{ for all } x \in V_{n_m}^s(z, \delta), s \in \{l, r\}\}.$$

It is clear that

$$l(G_{n_m}(\delta)) \geq \min\{\delta \cdot l(B_{n_m}(i_0)), \delta \cdot l(B_{n_m}^*(i_0))\} \geq c_5 \cdot \delta$$

with some constant $c_5 > 0$. Finally, we obtain

$$|\log Df^{q_{n_m}}(x)| \geq \left| \frac{(a-2) \log \Lambda}{2a} \right|$$

for all $x \in G_{n_m}(\delta)$. But this contradicts convergence of $Df^{q_n(x)}$ to 1 in probability with respect to normalized Lebesgue measure.

In the second case when $\frac{q_n}{2} \leq i_0 < q_n$ we will show only that either $A_n(i_0)$ and $A_n^*(i_0)$ or $B_n(i_0)$ and $B_n^*(i_0)$ are comparable. The construction of the sets G_{n_m} in these two cases proceeds in complete analogy to the case when $0 \leq i_0 < \frac{q_n}{2}$.

In the case $\frac{q_n}{2} \leq i_0 < q_n$ assume first

$$\mu_{\bar{\theta}}(A_n(i_0)) \geq \mu_{\bar{\theta}}(B_n(i_0)). \quad (6.43)$$

We will then show that $\mu_{\bar{\theta}}(A_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(A_{n_m}^*(i_0))$ are comparable for some subsequence $\{n_m\}$. Using the definition of $A_n(i_0)$ and $A_n^*(i_0)$ and Lemma 5, we get

$$\mu_{\bar{\theta}}(A_n^*(i_0)) = i_0 \Delta_{n-1} - \mu_{\bar{\theta}}(A_n(i_0)). \quad (6.44)$$

Assume again we are given a sequence n_m , $m = 1, 2, \dots$ of even numbers such that

$$\lim_{m \rightarrow \infty} (\mu_{\bar{\theta}}(B_{n_m}(i_0)) + \mu_{\bar{\theta}}(A_{n_m}(i_0))) = \omega_2 \quad (6.45)$$

with $\omega_2 \in [0, 1]$. Then for any $\epsilon > 0$ and for sufficiently large n_m

$$\omega_2 - \epsilon < \mu_{\bar{\theta}}(A_{n_m}(i_0)) + \mu_{\bar{\theta}}(B_{n_m}(i_0)) < \omega_2 + \epsilon. \quad (6.46)$$

This together with (6.43) implies

$$\frac{\omega_2 - \epsilon}{2} < \mu_{\bar{\theta}}(A_{n_m}(i_0)) < \omega_2 + \epsilon. \quad (6.47)$$

Hence, together with (6.35) and (6.43) we get

$$\begin{aligned} \mu_{\bar{\theta}}(A_{n_m}^*(i_0)) &= i_0 \Delta_{n_m-1} - \mu_{\bar{\theta}}(A_{n_m}(i_0)) \geq \\ &\geq \frac{1}{2} q_{n_m} \Delta_{n_m-1} - \omega_2 - \epsilon \geq \frac{1}{2} \cdot \frac{9}{20} - \omega_2 - \epsilon = \frac{9}{40} - \omega_2 - \epsilon. \end{aligned} \quad (6.48)$$

The bounds (6.47) and (6.48) show that, choosing ω_2 less than $\frac{9}{40}$, one finds for sufficiently large n and $\epsilon > 0$ small

$$\mu_{\bar{\theta}}(A_{n_m}(i_0)) > c_1 \quad \text{and} \quad \mu_{\bar{\theta}}(A_{n_m}^*(i_0)) > c_2 \quad (6.49)$$

for some constants $c_1, c_2 > 0$. Hence, under the assumptions (6.43) $\mu_{\bar{\theta}}(A_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(A_{n_m}^*(i_0))$ are comparable with some constant $K > 1$.

Assume next in the case $\frac{q_n}{2} \leq i_0 < q_n$

$$\mu_{\bar{\theta}}(A_n(i_0)) < \mu_{\bar{\theta}}(B_n(i_0)). \quad (6.50)$$

This assumption and (6.46) imply:

$$\frac{\omega_2 - \epsilon}{2} < \mu_{\bar{\theta}}(B_{n_m}(i_0)) < \omega_2 + \epsilon. \quad (6.51)$$

Using the definition of $A_n(i_0)$, $B_n(i_0)$, Eq. (6.44) and Lemma 5 (see Fig. 4), we obtain

$$\omega_2 - \epsilon < \mu_{\bar{\theta}}(A_{n_m}(i_0)) + \mu_{\bar{\theta}}(B_{n_m}(i_0)) = q_n \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) + (q_{n_m} - i_0) \Delta_{n_m} < \omega_2 + \epsilon. \quad (6.52)$$

Put $i_0 := i_0(n_m) = l_0(n_m)q_{n_m}$, $0 \leq l_0(n_m) \leq 1$. Then we have

$$(q_{n_m} - i_0) \Delta_{n_m} = (1 - l_0)q_{n_m} \cdot \Delta_{n_m}. \quad (6.53)$$

It easy to show that

$$\frac{9}{40} < q_{n_m} \Delta_{n_m} < \frac{1}{3},$$

and hence

$$\frac{9}{40}(1 - l_0) < (q_{n_m} - i_0) \Delta_{n_m} < \frac{1}{3}(1 - l_0). \quad (6.54)$$

Assume now

$$\frac{1}{3}(1 - l_0) < \frac{\omega_2 - \epsilon}{2}, \quad (6.55)$$

then by (6.52) and (6.54) we obtain

$$q_{n_m} \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) > \frac{\omega_2 - \epsilon}{2},$$

which together with (6.52) and $q_{n_m} - i_0 < i_0$ implies

$$\begin{aligned} \frac{\omega_2 - \epsilon}{2} &< q_{n_m} \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) = \\ &= i_0 \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) + (q_{n_m} - i_0) \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) \leq \\ &\leq 2i_0 \mu_{\bar{\theta}}([a_{-q_{n_m}+1}^*, c_{-i_0+1}^*]) = 2\mu_{\bar{\theta}}(A_{n_m}(i_0)) \end{aligned}$$

or

$$\frac{\omega_2 - \epsilon}{4} < \mu_{\bar{\theta}}(A_{n_m}(i_0)). \quad (6.56)$$

Using this and (6.44), we can see that

$$\mu_{\bar{\theta}}(A_{n_m}^*(i_0)) \geq \frac{9}{20} - \frac{\omega_2 - \epsilon}{4} = \frac{9}{20} - \frac{\omega_2}{4} + \frac{\epsilon}{4},$$

which together with (6.56) shows that, under assumptions (6.55), $\mu_{\bar{\theta}}(A_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(A_{n_m}^*(i_0))$ are comparable for sufficiently large n_m and sufficiently small $\epsilon > 0$.

It remains to discuss the case where

$$\frac{1}{3}(1 - l_0) \geq \frac{\omega_2 - \epsilon}{2}. \quad (6.57)$$

In this case we obtain, using $q_n \Delta_{n-1} \in \left(\frac{9}{10}, 1\right)$,

$$(q_{n_m} - i_0) \Delta_{n_m} = (1 - l_0) q_{n_m} \Delta_{n_m} \geq \frac{3}{2}(\omega_2 - \epsilon) q_{n_m} \Delta_{n_m} \geq \frac{27}{20}(\omega_2 - \epsilon).$$

Together with (6.51) we then get

$$\mu_{\bar{\theta}}(B_{n_m}^*(i_0)) = (q_{n_m} - i_0) \Delta_{n_m-1} - \mu_{\bar{\theta}}(B_{n_m}(i_0)) \geq \frac{27}{20}(\omega_2 - \epsilon) - (\omega_2 + \epsilon) \geq \frac{7}{20}\omega_2 - \frac{47}{20}\epsilon,$$

which together with (6.51) shows that, under the assumptions (6.50) and (6.57) for sufficiently large n_m , $\mu_{\bar{\theta}}(B_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(B_{n_m}^*(i_0))$ are comparable if ω_2 is small and ϵ is sufficiently small.

Finally, we conclude that, under the assumptions of Theorem 3, either $\mu_{\bar{\theta}}(B_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(B_{n_m}^*(i_0))$ are K-comparable or $\mu_{\bar{\theta}}(A_{n_m}(i_0))$ and $\mu_{\bar{\theta}}(A_{n_m}^*(i_0))$ are K-comparable, with some constant $K > 1$, depending on ω .

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References

- [1] Adouani, A. and Marzougui, H., Singular Measures for Class P -Circle Homeomorphisms with Several Break Points, *Ergodic Theory Dynam. Systems*, 2014, vol. 34, no. 2, pp. 423–456.
- [2] Arnol'd, V. I., Small Denominators: 1. Mapping the Circle onto Itself, *Izv. Akad. Nauk SSSR. Ser. Mat.*, 1961, vol. 1, no. 1, pp. 21–86 (Russian).
- [3] Coelho, Z., Lopes, A., and da Rocha, L. F., Absolutely Continuous Invariant Measures for a Class of Affine Interval Exchange Maps, *Proc. Amer. Math. Soc.*, 1995, vol. 123, no. 11, pp. 3533–3542.
- [4] Cornfeld, I. P., Fomin, S. V., and Sinai, Ya. G., *Ergodic Theory*, Grundlehren Math. Wiss., vol. 245, New York: Springer, 1982.
- [5] Denjoy, A., Sur les courbes définies par les équations différentielles à la surface du tore, *J. Math. Pures Appl. (9)*, 1932, vol. 11, pp. 333–375.
- [6] Dzhaliilov, A. A. and Khanin, K. M., On an Invariant Measure for Homeomorphisms of a Circle with a Point of Break, *Funct. Anal. Appl.*, 1998, vol. 32, no. 3, pp. 153–161; see also: *Funktsional. Anal. i Prilozhen.*, 1998, vol. 32, no. 3, pp. 11–21.
- [7] Dzhaliilov, A. A., Piecewise Smoothness of Conjugate Homeomorphisms of a Circle with Corners, *Theoret. and Math. Phys.*, 1999, vol. 120, no. 2, pp. 179–192; see also: *Teoret. Mat. Fiz.*, 1999, vol. 120, no. 2, pp. 961–972.
- [8] Dzhaliilov, A. A. and Liousse, I., Circle Homeomorphisms with Two Break Points, *Nonlinearity*, 2006, vol. 19, no. 8, pp. 1951–1968.
- [9] Dzhaliilov, A., Liousse, I., and Mayer, D., Singular Measures of Piecewise Smooth Circle Homeomorphisms with Two Break Points, *Discrete Contin. Dyn. Syst.*, 2009, vol. 24, no. 2, pp. 381–403.
- [10] Dzhaliilov, A., Jalilov, A., and Mayer, D., A Remark on Denjoy's Inequality for PL Circle Homeomorphisms with Two Break Points, *J. Math. Anal. Appl.*, 2018, vol. 458, no. 1, pp. 508–520.

- [11] Dzhaliyov, A. A., Maier, D., and Safarov, U. A., Piecewise-Smooth Circle Homeomorphisms with Several Break Points, *Izv. Math.*, 2012, vol. 76, no. 1, pp. 94–112; see also: *Izv. Ross. Akad. Nauk Ser. Mat.*, 2012, vol. 76, no. 1, pp. 101–120.
- [12] Katznelson, Y. and Ornstein, D., The Absolute Continuity of the Conjugation of Certain Diffeomorphisms of the Circle, *Ergodic Theory Dynam. Systems*, 1989, vol. 9, no. 4, pp. 681–690.
- [13] Hawkins, J. and Schmidt, K., On C^2 -Diffeomorphisms of the Circle Which Are of Type III₁, *Invent. Math.*, 1982, vol. 66, no. 3, pp. 511–518.
- [14] Herman, M., Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Études Sci. Publ. Math.*, 1979, vol. 49, pp. 5–233.
- [15] Khanin, K. M. and Kocić, S., Hausdorff Dimension of Invariant Measure of Circle Diffeomorphisms with Breaks, *Ergod. Theory Dyn. Syst.*, 2017, vol. 63, pp. 1–9.
- [16] Sinai, Ya. G. and Khanin, K. M., Smoothness of Conjugacies of Diffeomorphisms of the Circle with Rotations, *Russian Math. Surveys*, 1989, vol. 44, no. 1, pp. 69–99; see also: *Uspekhi Mat. Nauk*, 1989, vol. 44, no. 1(265), pp. 57–82, 247.
- [17] Kuipers, L. and Niederreiter, H., Uniform Distribution of Sequences, New York: Wiley, 1974.
- [18] Larcher, G., A Convergence Problem Connected with Continued Fractions, *Proc. Amer. Math. Soc.*, 1988, vol. 103, no. 3, pp. 718–722.
- [19] Liousse, I., PL Homeomorphisms of the Circle Which Are Piecewise C^1 Conjugate to Irrational Rotations, *Bull. Braz. Math. Soc. (N. S.)*, 2004, vol. 35, no. 2, pp. 269–280.
- [20] Liousse, I., Nombre de rotation, mesures invariantes et ratio set des homéomorphismes affines par morceaux du cercle, *Ann. Inst. Fourier (Grenoble)*, 2005, vol. 55, no. 2, pp. 431–482.
- [21] Nakada, H., Piecewise Linear Homeomorphisms of Type III and the Ergodicity of Cylinder Flows, *Keio Math. Sem. Rep.*, 1982, No. 7, pp. 29–40.
- [22] Teplinsky, A., A Circle Diffeomorphism with Breaks That Is Smoothly Linearizable, *Ergod. Theory Dyn. Syst.*, 2018, vol. 38, no. 1, pp. 371–383.
- [23] Yoccoz, J.-Ch., Il n'y a pas de contre-exemple de Denjoy analytique, *C. R. Acad. Sci. Paris Sér. 1 Math.*, 1984, vol. 298, no. 7, pp. 141–144.