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On the Dynamics of a Rigid Body in the Hess Case at High-Frequency Vibrations of a Suspension Point

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The motion of a heavy rigid body with a mass geometry corresponding to the Hess case is considered. The suspension point of the body is assumed to perform high-frequency periodic vibrations of small amplitude in the three-dimensional space. It is proved that for any law of vibrations of this type, the approximate autonomous equations of the body motion admit an invariant relation (the first integral at the zero level), which coincides with a similar relation that exists in the Hess case of the motion of a body with a fixed point. In the approximate equations of motion written in Hamiltonian form, the cyclic coordinate is introduced and the corresponding reduction is performed. For the laws of vibration of the suspension point corresponding to the integrable cases (when there is another cyclic coordinate in the system), a detailed study of the model one-degree-of-freedom system is given. For the nonintegrable cases, an analogy with the approximate problem of the motion of a Lagrange top with a vibrating suspension point is drawn, and the results obtained earlier for the top are used. Some properties of the body motion at the nonzero level of the above invariant relation are also discussed.

Keywords: Hess case, high-frequency vibrations, integrable case, reduced system, Lagrange top

1. Introduction

The existence of a particular integral (an invariant relation at the zero level) of the Euler–Poisson equations of motion of a heavy rigid body around a fixed point was discovered by Hess in 1890 [1] and, somewhat later, by Appelrot [2]. Zhukovskii [3] gave a geometric analysis of motion and proposed a model of a rigid body with Hess mass geometry. An analytical study was carried out in the paper [4], where the system of equations of motion of the body in the Hess case was reduced to (one) Riccati equation; this equation was obtained in a different way in the monograph [5].

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By now, a number of cases of the existence of the Hess integral or its generalization have been discovered in various problems of solid mechanics and related problems. Particular Hess integrals have been found in the problem of the motion of a gyrostat around a fixed point [7]; the problem of the motion of a heavy rigid body suspended on a weightless rod [8]; for a body moving along a smooth horizontal plane [9]; in the Kirchhoff equations of the motion of a rigid body in an ideal fluid (the Chaplygin case) [10, 11], in the Poincaré–Zhukovskii equations of motion of a rigid body with cavities filled with fluid [6]. Two cases of the existence of the Hess integral in the interaction of two force fields are described in [12].

The general conditions for the existence of a Hess type integral in the problem of the motion of a rigid body around a fixed point in a generalized potential field, as well as in the problem of the motion of a rigid body on a smooth plane, are obtained in [13]; here, conditions are given for the existence of a similar integral in the problem of the motion of a gyroscope in gimbals. The paper [14] considers the Suslov problem on the motion of a rigid body around a fixed point in the presence of a nonholonomic constraint; conditions are obtained under which the equations of motion of the body admit the Hess type first integral; in this case, the problem is shown to be isomorphic to the classical Hess case in the Euler–Poisson equations.

The investigation of the Hess case of motion of a heavy rigid body with a fixed point can be reduced to the study of a vector field on a two-dimensional torus. In the series of papers [15–18], some topological properties of the phase flow of the problem are studied in the presence of perturbation (in the class of the Euler–Poisson system). In particular, limit cycles are considered near a critical circle case. The neighborhood of a saddle point in the presence of perturbations splitting the separatrices and chaotic behavior of the system are also discussed.

In [19], it is established that limit cycles on the torus (for the classical Hess problem) exist only at a zero value of the rotation number. For the generalized Hess case on a pencil of Poisson brackets, the effect of quantization of the rotation number, which can take arbitrary integer values depending on the parameter, is discovered. The presence of a cubic perturbation in a specific Hamiltonian system is shown to lead to a general situation (the Cantor ladder).

The purpose of this paper is to study the dynamics of a rigid body with the Hess mass geometry in a gravity field under the assumption that the suspension point of the body performs high-frequency vibrations of small amplitude in the three-dimensional space. Previously, the approximate autonomous equations of motion of a body with an arbitrary mass geometry in the presence of vibrations of the suspension point of this type were obtained in [20, 21]. These equations have the form of the modified Euler–Poisson equations or Hamiltonian equations, in which the vector of a vibration moment or the corresponding vibration potential is additionally taken into account.

In our paper, we will prove that the vibration moment acting on a body with the Hess mass geometry is always orthogonal to the radius vector of its center of mass relative to the suspension point. From this fact, it is easy to establish that the approximate system of equations of the body motion admits an invariant relation which coincides with the Hess integral for a body with a fixed point. This means that in the approximate problem of the dynamics of the body with a vibrating suspension in the Hess case (as well as for a body with a fixed point) there is symmetry with respect to rotations, and a cyclic coordinate (at the zero level of the integral) can be introduced, which leads to a reduced system with two degrees of freedom.

If there is another cyclic coordinate in this system (the case of vibration symmetry with respect to the vertical), the reduced system can be integrated. In this paper, the corresponding model system with one degree of freedom is investigated in detail. A geometric interpretation of

the motion of the body axis containing its center of mass is given. In the absence of the second cyclic coordinate, the reduced system is apparently not integrable.

Note that the Hamiltonian of the reduced system with two degrees of freedom coincides with the Hamiltonian in the approximate problem of the dynamics of a Lagrange top with a vibrating suspension point, with a zero angular velocity of proper rotation of the top. This analogy is known for bodies in the Lagrange and Hess cases with fixed suspension points. The problem of the motion of the Lagrange top with a vibrating suspension was investigated in various formulations in [22–27]. Taking into account the analogy of two problems, some results obtained earlier in the dynamics of the top are used in the Hess case study.

Some features of the particular motions (stationary rotations around the vertical) of the body in the Hess case in the presence of vibration symmetry relative to the vertical are also discussed under the assumption that the above-mentioned invariant relation does not hold.

2. Problem statement. Equations of motion

Consider the motion of a heavy rigid body, assuming that one of its points O (referred below as the suspension point) performs a specified motion in the three-dimensional space. Let the radius vector $\mathbf{O}_*\mathbf{O}$ have projections $u(t), v(t), w(t)$ in a fixed coordinate system O_*XYZ , with the axis O_*Z directed vertically upwards. We consider these functions to be periodic (with a frequency Ω) and twice continuously differentiable, with zero time average values.

Let us introduce a coordinate system $OXYZ$ obtained from the system O_*XYZ by translational motion, and the body-fixed system $Oxyz$ formed by the body's principal inertia axes for the point O . The unit vectors of the axes OX, OY, OZ and Ox, Oy, Oz are denoted, respectively, by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Let m be the body's mass, A, B, C its principal moments of inertia for the point O , with $A > B > C$. The coordinates of the center of mass G of the body in the body-fixed system are denoted by x_G, y_G, z_G .

Let the maximum length a of the radius vector $\mathbf{O}_*\mathbf{O}$ be small compared with the length $\ell = OG$, and the oscillation frequency Ω be large compared with the characteristic frequency $\sqrt{g/\ell}$, where g is the acceleration of gravity. We introduce a small parameter ε and the dimensionless frequency ω_* according to the formulae

$$\varepsilon^2 = a/\ell, \quad g/(\ell\Omega^2) = \varepsilon^4\omega_*^2, \quad \omega_* = \sqrt{g\ell}/a\Omega \sim 1.$$

In [20, 21], an approximate autonomous system of differential equations describing the body motion relative to the point O was obtained by perturbation theory methods. It represents the modified system of Euler–Poisson equations and has the form

$$d\mathbf{M}/dt = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{m}_g + \mathbf{m}_v, \quad (2.1)$$

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \quad (2.2)$$

In Eqs. (2.1), (2.2) and further, the symbol d/dt and the dot denote the time derivative in the coordinate systems $OXYZ$ and $Oxyz$, respectively, $\mathbf{M} = \mathbf{I}\boldsymbol{\omega}$ is the angular momentum of the body, $\mathbf{I} = \text{diag}(A, B, C)$; $\boldsymbol{\omega}$ is the angular velocity vector of the body, with the projections $\tilde{p}, \tilde{q}, \tilde{r}$ onto the body-fixed axes; $\mathbf{m}_g = mg\boldsymbol{\gamma} \times \mathbf{OG}$ is the moment of gravity, and $\mathbf{m}_v$ is the vibration moment defined by the formula [20, 21]

$$\mathbf{m}_v = \langle m\mathbf{V}_O \times (\mathbf{OG} \times \boldsymbol{\delta}) \rangle. \quad (2.3)$$

Hereinafter, the angular brackets denote the time averages of the functions contained in them.

The vector $\mathbf{V}_O$ in (2.3) is the suspension point velocity. In the coordinate systems O_*XYZ and $Oxyz$ its components will be denoted by V_X, V_Y, V_Z and v_x, v_y, v_z , respectively, where

$$V_X = du/dt, \quad V_Y = dv/dt, \quad V_Z = dw/dt; \quad v_x = \mathbf{V}_O \cdot \mathbf{e}_1, \quad v_y = \mathbf{V}_O \cdot \mathbf{e}_2, \quad v_z = \mathbf{V}_O \cdot \mathbf{e}_3.$$

The components of the vector $\boldsymbol{\delta}$ in the coordinate system $Oxyz$ are calculated by the formulae

$$\delta_x = \frac{m}{A}(z_G v_y - y_G v_z), \quad \delta_y = \frac{m}{B}(x_G v_z - z_G v_x), \quad \delta_z = \frac{m}{C}(y_G v_x - x_G v_y). \quad (2.4)$$

The motion of the body relative to the point O can also be described using the approximate autonomous canonical Hamiltonian equations. For this purpose, the vibration potential Π_v is added to the potential energy, which is defined by the formula

$$\Pi_v = \frac{1}{2} \langle A\delta_x^2 + B\delta_y^2 + C\delta_z^2 \rangle. \quad (2.5)$$

Further, we will study body motions within the framework of the approximate system written in the form of Eqs. (2.1), (2.2) or in the form of the relevant Hamiltonian equations. An estimate of the accuracy of solutions of these approximate equations (as compared with the solutions of the complete nonautonomous systems) and the corresponding time interval are given in [20, 21].

The subject of this paper is the study (within the framework of the above systems) of the dynamics of the body, the mass geometry of which satisfies the Hess case:

$$y_G = 0, \quad (B - C)Ax_G^2 = (A - B)Cz_G^2. \quad (2.6)$$

Without loss of generality, we will further assume that $x_G > 0, z_G > 0$. This can always be achieved by redirecting the axes of the body-fixed system.

Conditions (2.6) imply that the center of mass of the body lies in the principal inertia plane Oxz , and the radius vector $\mathbf{OG}$ of the center of mass relative to the suspension point is perpendicular to one of two circular sections of the gyration ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \text{const},$$

which contains the axis Oy of the middle moment of inertia [3].

It is well known that in the Hess case the system of equations of motion of a body with a fixed point admits an invariant relation (the first integral at the zero level) of the form

$$\mathbf{M} \cdot \mathbf{OG} = 0, \quad (2.7)$$

which allows one to perform a reduction of the system of equations of motion and to obtain a solution to the reduced system in terms of quadratures.

In this paper, we prove (Section 3) that for arbitrary vibrations of the suspension point (within the framework of the assumptions made), the approximate system of equations of the body motion with the Hess mass geometry also admits an invariant relation of the form (2.7). In Section 4, an integrable case is investigated, the properties of the corresponding model system with one degree of freedom are studied in detail, and a geometric interpretation of the motion of the body axis is given. Section 5 discusses the properties of certain particular body motions (families of equilibria and stationary rotations) in nonintegrable cases.

3. The existence of an invariant relation

3.1. Transformation of the vibration moment and vibration potential

If the body mass geometry satisfies conditions (2.6), then the formulae for the vibration moment and the vibration potential can be transformed to a form more convenient for further study.

Consider first the vibration moment. We expand the double vector product from the formula (2.3)

$$\mathbf{m}_v = m \langle \mathbf{OG}(\mathbf{V}_O \cdot \boldsymbol{\delta}) - \boldsymbol{\delta}(\mathbf{V}_O \cdot \mathbf{OG}) \rangle \quad (3.1)$$

and transform the scalar product $\mathbf{V}_O \cdot \boldsymbol{\delta}$ taking into account conditions (2.4) and (2.6):

$$\begin{aligned} \mathbf{V}_O \cdot \boldsymbol{\delta} &= \frac{m}{A} z_G v_x v_y + \frac{m}{B} v_y (x_G v_z - z_G v_x) - \frac{m}{C} x_G v_y v_z = \\ &= m v_y \left[\frac{B-A}{AB} z_G v_x + \frac{C-B}{BC} x_G v_z \right] = \\ &= m v_y \frac{C-B}{BC} \frac{x_G}{z_G} (v_x x_G + v_z z_G) = m v_y \frac{C-B}{BC} \frac{x_G}{z_G} (\mathbf{V}_O \cdot \mathbf{OG}). \end{aligned}$$

After that, relation (3.1) can be rewritten as

$$\mathbf{m}_v = m \langle (\mathbf{V}_O \cdot \mathbf{OG}) \boldsymbol{\delta}_1 \rangle, \quad \boldsymbol{\delta}_1 = m v_y \frac{C-B}{BC} \frac{x_G}{z_G} \mathbf{OG} - \boldsymbol{\delta}. \quad (3.2)$$

Further, note that the components of the vector $\boldsymbol{\delta}_1$ with regard to Eqs. (2.4) and (2.6) can be represented in the form

$$\delta_{1x} = -\frac{m z_G v_y}{B}, \quad \delta_{1y} = -\frac{m}{B} (x_G v_z - z_G v_x), \quad \delta_{1z} = \frac{m x_G v_y}{B},$$

and therefore, $\boldsymbol{\delta}_1 = (m/B) \mathbf{OG} \times \mathbf{V}_O$. Thus, equality (3.2) is reduced to

$$\mathbf{m}_v = \frac{m^2}{B} \langle (\mathbf{OG} \cdot \mathbf{V}_O) \mathbf{OG} \times \mathbf{V}_O \rangle. \quad (3.3)$$

Hence, it follows that the vector $\mathbf{m}_v$ of the vibration moment (as well as the vector $\mathbf{m}_g$) is orthogonal to the radius vector $\mathbf{OG}$.

We now consider Eq. (2.5) for the vibration potential. Rewrite it in the form

$$\Pi_v = \frac{m^2}{2A} \langle (z_G v_y)^2 \rangle + \frac{m^2}{2B} \langle (x_G v_z - z_G v_x)^2 \rangle + \frac{m^2}{2C} \langle (x_G v_y)^2 \rangle$$

taking into account the formula (2.4) and the condition $y_G = 0$, We exclude the term v_y^2 from this expression using the relation $v_x^2 + v_y^2 + v_z^2 = V_O^2(t)$ where $V_O^2(t)$ is the square of the suspension point velocity (and $\langle V_O^2(t) \rangle = \text{const}$). Discarding the nonessential additive constants, we represent the transformed formula for Π_v as follows:

$$\Pi_v = \frac{m^2}{2} \left\langle - \left(\frac{z_G^2}{A} + \frac{x_G^2}{C} \right) (v_x^2 + v_z^2) + \frac{1}{B} (x_G v_z - z_G v_x)^2 \right\rangle.$$

We can regroup further summands in angle brackets as

$$\Pi_v = \frac{m^2}{2} \left\langle \left(\frac{B-A}{AB} z_G^2 + \frac{x_G^2}{C} \right) v_x^2 + \left(\frac{C-B}{BC} x_G^2 - \frac{z_G^2}{A} \right) v_z^2 - 2 \frac{x_G z_G}{B} v_x v_z \right\rangle.$$

The second condition in (2.6) implies that the relations in the two parentheses are equal to x_G^2/B and z_G^2/B , and the expression in the angle brackets is represented by a complete square. We get in the end

$$\Pi_v = -\frac{m^2}{2B} \left\langle (x_G v_x + z_G v_z)^2 \right\rangle = -\frac{m^2}{2B} \langle (\mathbf{OG} \cdot \mathbf{V}_O)^2 \rangle. \quad (3.4)$$

3.2. Invariant relation for Eq. (2.1)

We now prove that equality (2.7) is an invariant relation for Eq. (2.1). We differentiate the expression $\mathbf{M} \cdot \mathbf{OG}$ with respect to time, taking into account Eq. (2.1) and the orthogonality of the vectors $\mathbf{m}_g$ and $\mathbf{m}_v$ to the radius vector $\mathbf{OG}$:

$$\begin{aligned} \frac{d}{dt}(\mathbf{M} \cdot \mathbf{OG}) &= \frac{d\mathbf{M}}{dt} \cdot \mathbf{OG} + \mathbf{M} \cdot \frac{d\mathbf{OG}}{dt} = \\ &= (\mathbf{m}_g + \mathbf{m}_v) \cdot \mathbf{OG} + \mathbf{M} \cdot (\boldsymbol{\omega} \times \mathbf{OG}) = \mathbf{M} \cdot (\boldsymbol{\omega} \times \mathbf{OG}). \end{aligned}$$

The product $\mathbf{M} \cdot (\boldsymbol{\omega} \times \mathbf{OG})$ can be written in terms of the components of the multiplied vectors, with regard to conditions (2.6):

$$\begin{aligned} \mathbf{M} \cdot (\boldsymbol{\omega} \times \mathbf{OG}) &= [(A-B)z_G \tilde{r} + (B-C)x_G \tilde{p}] \tilde{q} = \\ &= \frac{(B-C)x_G}{Cz_G} \tilde{q} (A \tilde{p} x_G + C \tilde{r} z_G) = \frac{(B-C)x_G}{Cz_G} \tilde{q} (\mathbf{M} \cdot \mathbf{OG}). \end{aligned}$$

The result is

$$\frac{d}{dt}(\mathbf{M} \cdot \mathbf{OG}) = \frac{(B-C)x_G}{Cz_G} \tilde{q} (\mathbf{M} \cdot \mathbf{OG}). \quad (3.5)$$

It follows from this that relation (2.7) is an invariant of Eq. (2.1). Thus, we have proved the following theorem:

Theorem. *If the body mass geometry corresponds to the Hess case, then Eq. (2.1) admits the invariant relation (2.7) for any vibration features of the body suspension point determined by the vibration moment (3.3).*

3.3. Reducing the order of the system of equations

The existence of an invariant relation in the problem under study (as in the case of a body with a fixed point) is due to symmetry with respect to the rotation of the body around the axis OG , which is perpendicular to the aforementioned circular section of the gyration ellipsoid. This circumstance allows one to introduce a cyclic coordinate (at the zero level of the impulse) and to reduce the order of the system. For a body with a fixed point in the Hess case, this procedure is described, for example, in [6].

Let us turn to the system $Ox'y'z'$ of nonprincipal inertia axes, with unit vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, by rotating the principal inertia plane Oxz around the Oy axis at an angle

$$\alpha = \arctan \frac{x_G}{z_G} = \arctan \sqrt{\frac{(A-B)C}{(B-C)A}}.$$

As a result, the Oz' axis passes through the center of mass G , and $x'_G = y'_G = 0$, $z'_G = OG = \ell$.

We will define the orientation of the body-fixed system $Ox'y'z'$ in the $OXYZ$ -system by the Euler angles ψ, θ, φ introduced in the usual way.

3.3.1. Transformed Euler Equations

Let us project Eq. (2.1) onto the axis of the coordinate system $Ox'y'z'$:

$$\begin{aligned} I_1 \dot{p} + I_{13} \dot{r} + (I_3 - I_2)qr + I_{13}pq &= mg\ell\gamma_2 + m_{vx'}, \\ I_2 \dot{q} + (I_1 - I_3)rp - I_{13}(p^2 - r^2) &= -mg\ell\gamma_1 + m_{vy'}, \\ I_3 \dot{r} + I_{13} \dot{p} + (I_2 - I_1)pq - I_{13}rq &= m_{vz'}. \end{aligned} \quad (3.6)$$

In Eqs. (3.6), the projections $\gamma_1 = \sin\theta \sin\varphi$, $\gamma_2 = \sin\theta \cos\varphi$, $\gamma_3 = \cos\theta$ of the vectors $\boldsymbol{\gamma}$ as well as the projections $m_{vx'}, m_{vy'}, m_{vz'}$ of the vibration moment vector onto the axis Ox', Oy', Oz' are used. The introduced moments of inertia are calculated using the formulae $I_2 = B$ and

$$I_1 = A \cos^2 \alpha + C \sin^2 \alpha, \quad I_3 = A \sin^2 \alpha + C \cos^2 \alpha, \quad I_{13} = (A - C) \sin \alpha \cos \alpha. \quad (3.7)$$

Note that $I_{13} > 0$, due to the assumptions made.

Given the second condition in (2.6), we find that

$$\Delta = I_1 I_3 - I_{13}^2 = I_2 I_3. \quad (3.8)$$

Then, taking into account the triangle inequality for axial moments of inertia, we have

$$I_{13}^2 = I_3(I'_1 - I_2) < I_3^2. \quad (3.9)$$

Further, we note that the expression (3.3) for the vibration moment is written in vector form and, for a fixed axis $Oy = Oy'$, does not depend on the choice of the two other axes of the body-fixed system. Setting the components of the multiplied vectors in the coordinate system $Ox'y'z'$, we obtain

$$\mathbf{m}_v = \frac{m^2 \ell^2}{B} (-\langle v_{y'} v_{z'} \rangle, \langle v_{x'} v_{z'} \rangle, 0), \quad (3.10)$$

where

$$v_{x'} = \mathbf{V}_O \cdot \mathbf{e}'_1, \quad v_{y'} = \mathbf{V}_O \cdot \mathbf{e}'_2, \quad v_{z'} = \mathbf{V}_O \cdot \mathbf{e}'_3.$$

The third equation of the system (3.6) can be rewritten with regard to equality (3.8) in the form

$$I_{13} \dot{p} + I_3 \dot{r} = \frac{I_{13}^2}{I_3} q(I_{13}p + I_3r), \quad (3.11)$$

whence we obtain the invariant relation (2.7), which reduces to the condition $M_{z'} = 0$ and has the form

$$I_{13}p + I_3r = 0. \quad (3.12)$$

Using this equality, we exclude the variable r from the first two equations of the system (3.6). Then, taking into account relation (3.9), we rewrite these equations in the form

$$\dot{p} + \frac{I_{13}}{I_3}pq = \frac{1}{I_2}(mg\ell\gamma_2 + m_{vx'}), \quad \dot{q} - \frac{I_{13}}{I_3}p^2 = \frac{1}{I_2}(-mg\ell\gamma_1 + m_{vy'}), \quad (3.13)$$

where the quantities $m_{vx'}$ and $m_{vy'}$ are defined in the formula (3.10).

3.3.2. Hamiltonian of the system

We now obtain the Hamiltonian of the system. The expression (2.5) for the vibration potential can be rewritten with regard to the condition $\mathbf{OG} = \ell\mathbf{e}'_3$ in the form

$$\Pi_v = -\frac{m^2\ell^2}{2B}\langle(\mathbf{VO} \cdot \mathbf{e}'_3)^2\rangle = -\frac{m^2\ell^2}{2B}\langle(V_X \sin \psi \sin \theta - V_Y \cos \psi \sin \theta + V_Z \cos \theta)^2\rangle. \quad (3.14)$$

From the expression (3.14) it follows that the vibration potential depends on six quantities that determine the type and properties of vibration, the average values of the squares of the components (in the coordinate system $OXYZ$) of the velocity vector $\mathbf{VO}$ and their combined products:

$$\begin{aligned} a_X &= \langle V_X^2 \rangle, & a_Y &= \langle V_Y^2 \rangle, & a_Z &= \langle V_Z^2 \rangle, \\ a_{XY} &= \langle V_X V_Y \rangle, & a_{XZ} &= \langle V_X V_Z \rangle, & a_{YZ} &= \langle V_Y V_Z \rangle. \end{aligned} \quad (3.15)$$

These constant quantities are parameters of the problem. By turning the axes O_*X and O_*Y one can always achieve the condition $a_{XY} = 0$.

With these notations, the expression (3.14) can be rewritten as

$$\Pi_v = -\frac{m^2\ell^2}{2B}[(a_1 \sin^2 \psi + a_2 \cos^2 \psi) \sin^2 \theta + (a_{XZ} \sin \psi - a_{YZ} \cos \psi) \sin 2\theta]. \quad (3.16)$$

Additive constants are omitted here and two more notations are introduced as follows:

$$a_1 = a_X - a_Z, \quad a_2 = a_Y - a_Z. \quad (3.17)$$

Using the nonprincipal inertia axes, the kinetic energy of the body in the coordinate system $OXYZ$ can be written in the form

$$T = \frac{1}{2}(I_1 p^2 + I_2 q^2 + I_3 r^2) + I_{13}pr. \quad (3.18)$$

The quantities p , q , r satisfy the kinematic relations

$$p = \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi, \quad q = \dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi, \quad r = \dot{\psi} \cos \theta + \dot{\varphi}. \quad (3.19)$$

We introduce the generalized momenta related to the angles ψ , θ , φ by

$$\begin{aligned} p_\psi &= \frac{\partial T}{\partial \dot{\psi}} = L_1 \sin \theta \sin \varphi + L_2 \sin \theta \cos \varphi + L_3 \cos \theta, \\ p_\theta &= \frac{\partial T}{\partial \dot{\theta}} = L_1 \cos \varphi - L_2 \sin \varphi, \quad p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = L_3. \end{aligned} \quad (3.20)$$

Here

$$L_1 = \frac{\partial T}{\partial p} = I_1 p + I_{13}r, \quad L_2 = \frac{\partial T}{\partial q} = I_2 q, \quad L_3 = \frac{\partial T}{\partial r} = I_3 r + I_{13}p. \quad (3.21)$$

Solve Eqs. (3.21) for p , q , r and rewrite the expression (3.18) in the form

$$T = \frac{I_3}{2\Delta} L_1^2 + \frac{1}{2I_2} L_2^2 + \frac{I_1}{2\Delta} L_3^2 - \frac{I_{13}}{\Delta} L_1 L_3. \quad (3.22)$$

According to relation (3.8), the coefficients in the terms with L_1^2 and L_2^2 are the same.

We then find the quantities L_k ($k = 1, 2, 3$) using the formulae (3.20). After some transformations, we have

$$\tilde{T} = \frac{1}{2I_2} \left[p_\theta^2 + \frac{(p_\psi - p_\varphi \cos \theta)^2}{\sin^2 \theta} \right] + \frac{I_1}{2\Delta} p_\varphi^2 - \frac{I_{13}}{\Delta} \frac{(p_\psi - p_\varphi \cos \theta) \sin \varphi - p_\theta \sin \theta \cos \varphi}{\sin \theta} p_\varphi. \quad (3.23)$$

The Hamiltonian of the system is calculated by the formula

$$\tilde{H} = \tilde{T} + \Pi_g + \Pi_v, \quad (3.24)$$

where $\Pi_g = mg\ell \cos \theta$, and the expression for Π_v is given in Eq. (3.16).

The invariant relation corresponding to the system with the Hamiltonian (3.24) has the form $p_\varphi = 0$. Given the third equation in (3.21) and equalities (3.19), we rewrite this relation in the form

$$(I_3 \cos \theta + I_{13} \sin \theta \sin \varphi) \dot{\psi} + I_{13} \dot{\theta} \cos \varphi + I_3 \dot{\varphi} = 0. \quad (3.25)$$

At the zero level of the impulse p_φ , the φ coordinate is cyclic, and the Hamiltonian reduced at this level is determined by the expression

$$H = \frac{1}{2I_2} \left(p_\theta^2 + \frac{p_\psi^2}{\sin^2 \theta} \right) + mg\ell \cos \theta - \frac{m^2 \ell^2}{2I_2} [(a_1 \sin^2 \psi + a_2 \cos^2 \psi) \sin^2 \theta + (a_{XZ} \sin \psi - a_{YZ} \cos \psi) \sin 2\theta]. \quad (3.26)$$

We rewrite the Hamiltonian (3.26) by introducing the dimensionless momenta p_1 and p_2 , and the dimensionless time τ according to the formulae

$$p_\psi = I_2 \omega p_1, \quad p_\theta = I_2 \omega p_2, \quad \tau = \omega_0 t, \quad \omega_0 = (mg\ell/I_2)^{1/2}.$$

As a result, we obtain

$$\Gamma = \frac{1}{2} \left(p_2^2 + \frac{p_1^2}{\sin^2 \theta} \right) + \cos \theta - \frac{m\ell}{2I_2 g} [(a_1 \sin^2 \psi + a_2 \cos^2 \psi) \sin^2 \theta + (a_{XZ} \sin \psi - a_{YZ} \cos \psi) \sin 2\theta]. \quad (3.27)$$

The reduced Hamiltonian (3.27) (or (3.26)) describes the motion of a two-degree-of-freedom system, which will also be called reduced. To the equations of motion of this system we add the (dimensionless) equation for the angle φ , which follows from relation (3.25):

$$\varphi' = -(a \sin \theta \sin \varphi + \cos \theta) \psi' - a \theta' \cos \varphi. \quad (3.28)$$

Here we have introduced the notation $a = I_{13}/I_3$, with $0 < a < 1$ due to inequality (3.9); the prime means differentiation with respect to τ .

Note that Hamiltonian Γ coincides with a similar Hamiltonian describing (in an approximate autonomous problem) the motion of a Lagrange top with a vibrating suspension point [26], at zero constant of the cyclic integral corresponding to the proper rotation angle. For bodies with the Lagrange and Hess mass geometry, when they move around their fixed points in the gravity field, a similar analogy is well known (see, for example, [6]), and the relevant reduced system can be integrated by quadratures.

For arbitrary values of the vibration parameters, the reduced system with Hamiltonian (3.27) is apparently not integrable. Integration of the system can, however, be carried out if there is a symmetry of the vibrational field with respect to the vertical. This case will be discussed in the next section.

4. Integrable case

Let the vibration parameters of the problem at hand satisfy the relations as follows:

$$a_1 = a_2 \quad (a_X = a_Y), \quad a_{XY} = a_{XZ} = a_{YZ} = 0. \quad (4.1)$$

This symmetric case is realized, for example, if the suspension point of the body moves according to the law (ℓ_1 and ℓ_2 are constants)

$$u(t) = \varepsilon \ell_1 \cos \Omega t, \quad v(t) = \varepsilon \ell_1 \cos 2\Omega t, \quad w(t) = \varepsilon \ell_2 \sin \Omega t.$$

If conditions (4.1) hold, then in the system with Hamiltonian (3.27) there is another cyclic coordinate, ψ (as in the case of a body with a fixed point), and $p_1 = c = \text{const}$. The Hamiltonian of a corresponding one-degree-of-freedom system (hereinafter referred to as the model Hamiltonian) has the form

$$\Gamma_1 = \frac{1}{2} \left(p_2^2 + \frac{c^2}{\sin^2 \theta} \right) + \cos \theta - \frac{1}{2} \sigma \cos^2 \theta, \quad \sigma = \frac{m\ell}{I_2 g} (a_Z - a_X). \quad (4.2)$$

The dimensionless parameter σ characterizes the ratio between the intensities of the vertical and horizontal components of vibration, and can take values of any sign. Further, we assume that $\sigma \neq 0$.

Consider the energy integral $\Gamma_1 = h = \text{const}$ of the model system. We write it, given that $p_2 = \theta'$, in the form

$$\theta'^2 = 2h - 2 \cos \theta + \sigma \cos^2 \theta - \frac{c^2}{\sin^2 \theta}. \quad (4.3)$$

Equation (4.3) can also be represented in algebraic form by introducing the variable $u = \cos \theta$ ($|u| \leq 1$):

$$u'^2 = f(u), \quad f(u) = (1 - u^2)(2h - 2u + \sigma u^2) - c^2. \quad (4.4)$$

The regions of possible motions of the system are determined by the condition $f(u) \geq 0$. In these regions we obtain the quadrature

$$\int_{u_0}^u \frac{du}{\sqrt{f(u)}} = \pm(\tau - \tau_0). \quad (4.5)$$

For $\sigma \neq 0$, the function $f(u)$ is a fourth-degree polynomial in u , so solutions of Eq. (4.5) are written in terms of elliptic functions. If $u = \tilde{u}(\tau)$ is a solution of Eq. (4.5) and $\theta = \tilde{\theta}(\tau)$ is

the corresponding change in the nutation angle, then the change in the angles of precession is determined by quadrature from the equation

$$\psi' = \frac{c}{\sin^2 \theta}, \quad (4.6)$$

which completes the integration of the reduced system with the Hamiltonian (3.27) under conditions (4.1). After that, it remains to integrate the nonautonomous first-order differential equation (3.28).

Further, we also need Eqs. (3.13), rewritten with regard to conditions (4.1). We calculate the components of the vibration moment vector (3.10) under these conditions. Then we move to dimensionless time τ , make dimensionless the projections p and q of the angular velocity with the help of the factor ω_0 and, retaining the same notation for these quantities, we represent Eqs. (3.13) in the form

$$p' + apq = (1 - \sigma\gamma_3)\gamma_2, \quad q' - ap^2 = -(1 - \sigma\gamma_3)\gamma_1. \quad (4.7)$$

To Eqs. (4.7) we add the Poisson equations for the values $\gamma_1, \gamma_2, \gamma_3$. After excluding the variable r using Eq. (3.12), the Poisson equations take the following dimensionless form:

$$\gamma_1' = -a\gamma_2p - \gamma_3q, \quad \gamma_2' = (a\gamma_1 + \gamma_3)p, \quad \gamma_3' = \gamma_1q - \gamma_2p. \quad (4.8)$$

The system of equations (4.7) and (4.8) has three first integrals, namely, energy and area integrals, and a geometric integral, of the form

$$\frac{1}{2}(p^2 + q^2) + \gamma_3 - \frac{\sigma}{2}\gamma_3^2 = \text{const}, \quad (4.9)$$

$$p\gamma_1 + q\gamma_2 = \text{const}, \quad (4.10)$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \quad (4.11)$$

We now consider in more detail the body motion in the integrable case under study.

4.1. The case $c = 0$

First let $c = 0$. Relation (4.6) yields that $\psi = \text{const}$ in this case, and the trajectory of the apex (the end of the vector $\mathbf{e}_3'$) on the Poisson sphere lies on one of the circles passing through the upper and lower poles of the sphere.

The axis Oz' of the body performs pendulum-type motions described by the canonical equations with a model Hamiltonian

$$\Gamma_{10} = \frac{1}{2}p_2^2 + \cos \theta - \frac{1}{2}\sigma \cos^2 \theta. \quad (4.12)$$

When θ is replaced by $\pi - \theta$, this Hamiltonian coincides with the Hamiltonian function describing the pendulum-type motions of the Lagrange top with a vibrating suspension point, considered in [27]. We briefly describe the motion of this model system.

4.1.1. Motion of the model system

The particular solutions $p_2 = 0$ and $\theta = 0$ or $\theta = \pi$ of the system with the Hamiltonian (4.12) correspond to the relative body equilibria for which the radius vector $\mathbf{OG}$ is in a vertical position (upper or lower poles of the Poisson sphere). The upper equilibrium is stable

when the condition $\sigma > 1$ is fulfilled (or, in the original notation, when $a_Z - Bg/(m\ell) > a_X$) and is unstable otherwise. The lower equilibrium is stable for $\sigma \geq -1$ ($a_Z + Bg/(m\ell) > a_X$), and unstable otherwise.

In addition, for $|\sigma| \geq 1$, the system with the Hamiltonian (4.12) has also two symmetric equilibria $p_2 = 0$, $\theta = \theta_* = \pm \arccos(1/\sigma)$ corresponding to the positions of the radius vector $\mathbf{OG}$ that are inclined to the vertical; they are stable if $\sigma \leq -1$ ($a_X \geq a_Z + Bg/(m\ell)$), and unstable if $\sigma \geq 1$ ($a_X \leq a_Z - Bg/(m\ell)$).

The equilibria considered correspond to the energy levels $h_0 = 1 - \sigma/2$, $h_\pi = -1 - \sigma/2$ and $h_* = 1/(2\sigma)$ of the system.

The regions of other possible motions of the system are given by the square inequality $2h - 2u + \sigma u^2 > 0$, which follows from the condition of positivity of the function $f(u)$ from Eq. (4.4) at $c = 0$. We introduce the notation for the roots of the above square trinomial as

$$u_1 = \frac{1 - \sqrt{1 - 2h\sigma}}{\sigma}, \quad u_2 = \frac{1 + \sqrt{1 - 2h\sigma}}{\sigma}.$$

The phase portraits of the system with Hamiltonian (4.12) are shown in Fig. 1a, 1b, and 1c for the cases $\sigma < -1$, $-1 < \sigma < 1$, and $\sigma > 1$, respectively. The corresponding regions with different types of the system motion are shown in Fig. 2 in the parameter plane σ, h .

In region 0 in Fig. 2 motions of the model system are not possible. In region 1, given by the conditions $\sigma < -1$ and $h_* < h < h_\pi$, oscillations in the vicinity of the lateral equilibrium $\theta = \theta_*$ (Fig. 1a) occur in the system, with the frequency ω_1 written below.

Fig. 1. Phase portraits of the model system at $c = 0$.

Fig. 2. Regions with different types of the motions of the model system at $c = 0$.

When $h_\pi < h < h_0$ (region 2), we have oscillations, with the frequency ω_2 , in the vicinity of the lower equilibrium $\theta = \pi$. For $0 < |\sigma| < 1$ they are all oscillations of this type (Fig. 1b), for $\sigma > 1$ they are part of such oscillations (with energy level $h < h_0$, see Fig. 1c), and for $\sigma < -1$ they are oscillations corresponding to the closed curves covering three singular points of the system in Fig. 1a.

For the values $\sigma > 1$ and $h_0 < h < h_*$ (region 3), two types of motions occur in the system, with the same frequency ω_3 ; they are one of the oscillations in the vicinity of the lower position $\theta = \pi$ (with energy level $h > h_0$) and one of the oscillations in the vicinity of the upper position $\theta = 0$ (Fig. 1c).

Finally, when $\sigma < 1$, $h > h_0$ or $\sigma > 1$, $h > h_*$, rotations occur in the system (see Figs. 1a–1c); the rotation frequencies are equal to ω_4 and ω_5 in regions 4 and 5 of the part of the parameter plane at hand, lying, respectively, to the left (with $\sigma \neq 0$) and to the right of the dotted curve.

The frequencies introduced are calculated by the formulae as follows [28, 30]:

$$\omega_i = \frac{\pi \xi_i}{K(k_i)} \quad (i = 1, \dots, 4), \quad \omega_5 = \frac{\pi \xi_5}{2K(k_5)}, \quad (4.13)$$

where $K(k_i)$ is the complete elliptic integral of the first kind, and

$$\begin{aligned} \xi_1 &= \frac{1}{2} \sqrt{|\sigma|(1-u_1)(1+u_2)}, \quad \xi_2 = \frac{\sqrt{2}}{2} \sqrt{|\sigma(u_2-u_1)|}, \quad \xi_3 = \xi_1, \quad \xi_5 = \sqrt{\sigma MN}, \\ k_1^2 &= \frac{2(u_2-u_1)}{(1-u_1)(1+u_2)}, \quad k_3^2 = \frac{(1-u_2)(1+u_1)}{(1-u_1)(1+u_2)}, \quad k_5^2 = \frac{4-(M-N)^2}{4MN}, \\ M^2 &= (1-m)^2 + n^2, \quad N^2 = (1+m)^2 + n^2, \quad m = 1/\sigma, \quad n^2 = (2h\sigma - 1)/\beta^2. \end{aligned}$$

The quantities ξ_4 and k_4 are obtained from ξ_1 and k_1 by replacing $u_1 \leftrightarrow u_2$, and $k_2 = k_4^{-1}$.

The oscillations and rotations considered are expressed in terms of elliptic functions. Exceptions are the cases $h = h_0$, $h = h_\pi$ and $h = h_*$, corresponding to the equilibrium points and one of the oscillations or asymptotic motions; because of the presence of a multiple root of the function $f(u)$, these motions are defined via elementary functions. We do not write the explicit relations $\theta = \tilde{\theta}(\tau)$ on the trajectories of the model system; they can be obtained using the formulae from [28, 30].

4.1.2. On body motions at $c = 0$

The motion of the body relative to the axis Oz' with $\psi = \text{const}$ is described, according to Eq. (3.28), as follows:

$$\varphi' = -a\theta' \cos \varphi. \quad (4.14)$$

In particular, for the equilibrium values $\theta = \theta_0 = \text{const}$ of the model system with the Hamiltonian (4.12), Eq. (4.14) yields $\varphi' = 0$. We obtain the families of equilibria of the body in the coordinate system $OXYZ$, when angle φ takes arbitrary values. Note that, in view of the analogy mentioned above, these families of equilibria correspond to the stationary rotations of the Lagrange top around the vertical (inverted and hanging top) and inclined axes studied in [25].

Let's solve the stability problem of the found body equilibria. Put in the system (4.7), (4.8)

$$p = x_1, \quad q = x_2, \quad \gamma_1 = \gamma_{10} + x_3, \quad \gamma_2 = \gamma_{20} + x_4, \quad \gamma_3 = \cos \theta_0 + x_5 \quad (4.15)$$

under the condition $\gamma_{10}^2 + \gamma_{20}^2 + \gamma_{30}^2 = 1$.

We calculate the characteristic equations of the linearized equations of perturbed motion. For the cases $\gamma_{30} = \pm 1$, $\gamma_{10} = \gamma_{20} = 0$ they have the form

$$\lambda(\lambda^2 + \sigma \mp 1)^2 = 0,$$

where the upper and lower signs correspond to the cases $\gamma_{30} = 1$ and $\gamma_{30} = -1$.

The necessary stability condition of these equilibria (the condition of the absence of roots with positive real parts) is determined by the inequality $\sigma \mp 1 \geq 0$.

The characteristic equation for the case $\gamma_{30} = 1/\sigma$ is

$$\lambda^3 \left(\lambda^2 + \frac{1 - \sigma^2}{\sigma} \right) = 0.$$

Taking into account the existence region, we find the necessary stability condition as $\sigma \leq -1$.

To obtain the sufficient stability conditions, we take the first integrals of the system (4.7), (4.8) and use the Lyapunov stability theorem [29]. For body equilibria corresponding to the values $\theta_0 = 0$ and $\theta_0 = \pi$, we introduce perturbations in relations (4.9) and (4.11) using formulae (4.15), and rewrite these relations in the form

$$V_1 = \frac{1}{2}(x_1^2 + x_2^2) + (1 \mp \sigma)x_5 - \frac{\sigma}{2}x_5^2, \quad (4.16)$$

$$V_2 = x_3^2 + x_4^2 \pm 2x_5 + x_5^2, \quad (4.17)$$

where, as above, the upper and lower signs correspond to $\theta_0 = 0$ and $\theta_0 = \pi$.

The Lyapunov function is chosen as the sum of squares of the first integrals

$$V = V_1^2 + V_2^2. \quad (4.18)$$

It is sign-definite with respect to the variables $x_1, \dots, x_5$ if the function V_1 considered on the set of values of x_j that make the function V_2 vanish is sign-definite.

Solve the equation $V_2 = 0$ for x_5 as

$$x_5 = \mp \frac{x_3^2 + x_4^2}{2} + \dots$$

and substitute this expression in relation (4.16):

$$V_1|_{V_2=0} = \frac{1}{2}(x_1^2 + x_2^2) + \frac{\sigma \mp 1}{2}(x_3^2 + x_4^2) + \dots$$

Ellipses in the last two formulae mean sets of terms higher than the second degree in x_3 and x_4 .

If the condition $\sigma \mp 1 > 0$ is fulfilled, then the function V_1 obtained in such a way is positive-definite with respect to x_1, x_2, x_3, x_4 , and the function V is positive-definite in all five variables. Therefore, this inequality is a sufficient condition for the stability of the considered equilibrium positions of the body with respect to perturbations of the quantities $p, q, \gamma_1, \gamma_2, \gamma_3$.

To study the stability of the body equilibria corresponding to the value $\theta = \theta_*$, let us represent the integrals (4.9) and (4.10), with regard to relations (3.19), in the form

$$\frac{1}{2}(\psi'^2 \sin^2 \theta + \theta'^2) + \cos \theta - \frac{\sigma}{2} \cos^2 \theta = \text{const}, \quad (4.19)$$

$$\psi' \sin^2 \theta = \text{const}. \quad (4.20)$$

We introduce the perturbations by the formulae $\theta = \theta_* + x_1$, $\theta' = x_2$, $\psi' = x_3$ and rewrite these integrals as

$$V_1 = \frac{\sigma^2 - 1}{2\sigma^2}(-\sigma x_1^2 + x_3^2) + \frac{1}{2}x_2^2, \quad V_2 = x_3(\sin^2 \theta_* + \sin 2\theta_* x_1 + \dots).$$

Here, the ellipsis means the set of the terms of the second and higher degrees in x_1 .

The Lyapunov function is again chosen in the form (4.18). Considering the relation $V_2 = 0$, we find that $x_3 = 0$. Taking into account this condition as well as the existence region of the equilibrium, we conclude that the function V_1 is positively defined with respect to the variables x_1, x_2 if the inequality $\sigma < -1$ holds. This condition ensures the sign-definiteness of the function V with respect to x_1, x_2, x_3 and represents the sufficient stability condition in relation to these perturbations for the body equilibrium under study.

Thus, the necessary and sufficient stability conditions (with respect to the corresponding sets of variables) of the body equilibria at hand, up to an equal sign, coincide with the stability conditions for the corresponding equilibria of the model system.

Consider now the body motions different from the equilibrium positions. The body axis Oz' performs the pendulum-type motions (oscillations or rotations) described above. From relation (4.14) it follows that under the condition $\cos \varphi \equiv 0$ ($\varphi = \pm\pi/2$, $\varphi = \pm 3\pi/2$, etc.) there are no body rotations around the axis Oz' , i.e., the body moves as a physical pendulum.

If $\cos \varphi \neq 0$, then Eq. (4.14) can be integrated, and the result is represented as

$$\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right) = C e^{-a\theta} \quad (C = \text{const}).$$

When the solution $\theta = \tilde{\theta}(\tau)$ of the model system is substituted into this equation, we obtain the relation $\varphi(\tau)$.

If the solution $\theta = \tilde{\theta}(\tau)$ describes oscillations of the body axis, then the angle φ (along with θ) varies in a limited range periodically, and its period is equal to the period of oscillation of the body axis.

If the solution $\theta = \tilde{\theta}(\tau)$ corresponds to rotations of the body axis, then, depending on the initial conditions, the angle φ tends, as $\tau \rightarrow \infty$ and $\tau \rightarrow -\infty$, to one of the equilibrium values satisfying the condition $\cos \varphi = 0$. For example, for the rotation from the upper half-plane of the phase plane of the model system ($\theta' > 0$, see Fig. 1) and the initial value $\varphi = \varphi_0$ from the interval $-\pi/2 < \varphi_0 < \pi/2$, we find that $\varphi \rightarrow \mp\pi/2$ as $\tau \rightarrow \pm\infty$.

4.2. The case $c \neq 0$

We now consider the general case $c \neq 0$.

4.2.1. Motions of the model system

The corresponding model system is described by the canonical equations with the Hamiltonian (4.2). In contrast to the case $c = 0$, the body axis Oz' cannot pass through the vertical, the angle θ varies in the range $(0, \pi)$.

Equilibrium positions. The equilibrium positions of the system are stationary points of the reduced potential energy

$$\Pi_1 = \cos \theta - \frac{\sigma}{2} \cos^2 \theta + \frac{c^2}{2 \sin^2 \theta}$$

satisfying the equation

$$\sin^4 \theta (\sigma \cos \theta - 1) - c^2 \cos \theta = 0.$$

Let us rewrite this equation in algebraic form with respect to the value $u = \cos \theta$ ($|u| < 1$), presenting it as

$$\sigma = G(u), \quad G(u) = \frac{c^2 u + (1 - u^2)^2}{(1 - u^2)^2 u}. \quad (4.21)$$

The equilibrium values of the model system are the abscissas of the intersection points of the graph of the function $y = G(u)$ and the straight line $y = \sigma$. Considering the parameter c fixed, we find the number of such points depending on the parameter σ .

Consider the properties of the function $G(u)$. Its derivative is

$$G'(u) = \frac{4c^2 u^3 - (1 - u^2)^3}{(1 - u^2)^3 u^2}.$$

Equating this derivative to zero, we find the equation for the extremum points of the function $G(u)$

$$u^2 + c_1 u - 1 = 0, \quad c_1 = (2|c|)^{2/3} \quad (c_1 > 0), \quad (4.22)$$

which has a unique solution

$$u = u_* = \frac{\sqrt{c_1^2 + 4} - c_1}{2} \quad (0 < u_* < 1)$$

on the interval $(-1, 1)$. This solution corresponds to the minimum point of the function $G(u)$. At this point we have

$$G(u_*) = \frac{1}{8} [c_1(c_1^2 + 6) + (c_1^2 + 4)^{3/2}] > 1. \quad (4.23)$$

On the interval $-1 < u < 0$, the derivative $G'(u)$ retains a constant sign (negative), and the function $G(u)$ monotonically decreases, vanishing at the point $u = u_{**}$ determined by the equation $c^2 u = -(1 - u^2)^2$.

For a fixed value of c , the graph of the function $y = G(u)$ on the intervals $(-1, 0)$ and $(0, 1)$ is shown in Fig. 3 by the solid lines.

For $\sigma < G(u_*)$, the straight line $y = \sigma$ intersects this graph at one point, and for $\sigma > G(u_*)$, at three points. In these cases, the model system with the Hamiltonian (4.2) has one and three equilibrium positions, respectively.

Fig. 3. Graph of the function $G(u)$.

To study the stability of the equilibrium positions found, we calculate the second derivative of the function $\Pi_1(\theta)$; subject to condition (4.21), we obtain

$$\Pi''_{1\theta\theta} = u(1 - u^2)G'_u(u).$$

For the equilibrium points located on the left (marked with g_1) branch of the graph of the function $y = G(u)$, as well as on the increasing part (g_3) of the right branch of the graph of the function (see Fig. 3), the inequality $\Pi''_{1\theta\theta} > 0$ is fulfilled, and these points correspond to the points of minimum of the potential energy and to the stable equilibria of the model system. The equilibrium points located on the descending part (g_2) of the right branch of the graph of the function $y = G(u)$ correspond to the points of maximum of the potential energy (for them $\Pi''_{1\theta\theta} < 0$) and to the unstable equilibrium positions of the model system.

The dotted line (marked with g_4) in Fig. 3 is the branch of the function $y = G(u)$ on the interval $u > 1$. It will be needed to determine the total number of roots of the polynomial $f(u)$ from relation (4.4) for integrating Eq. (4.5).

Oscillations and asymptotic motions. Let us describe the motions of the model system different from the equilibrium positions. Consider the inequality $f(u) \geq 0$ defining admissible regions of motion. Since $f(\pm 1) = -c^2 < 0$ and $f(\pm\infty) \sim -\sigma u^4$, the system motions exist if the graph of the function $y = f(u)$ has one of the types shown in Figs. 4a–4c and 4d for the cases $\sigma > 0$ and $\sigma < 0$, respectively. Note that the case similar to that shown in Fig. 4b, when one of the two positive intervals of the function $f(u)$ is located to the left of the interval $(-1, 1)$, is not implemented in this problem.

Figure 4 does not show the cases of double roots of the function $f(u)$, which correspond to the equilibrium positions of the model system or extraneous “equilibria” lying outside the interval $(-1, 1)$. These cases are boundary cases; when passing through them, the number of zeros of the function $f(u)$ changes, which leads to a qualitative change in the behavior of the model system and/or (in the case of extraneous “equilibria”) to a change in the integration technique.

Fig. 4. Different forms for the function $f(u)$.

The boundary cases are given by the conditions $f(u) = 0$, $f'_u(u) = 0$. Two polynomials $f(u)$ and $f'_u(u)$ have common roots if their resultant in u equal to

$$\begin{aligned} Rs(c^2) = & -16\sigma\{16\sigma^3c^6 - [8\sigma^4 + 64\sigma^3h + (32h^2 - 12)\sigma^2 - 72\sigma h + 27]c^4 + \\ & + [\sigma^5 + 24\sigma^4h + (88h^2 - 15)\sigma^3 + 2h(48h^2 - 31)\sigma^2 + \\ & + (16h^4 - 196h^2 + 12)\sigma - 8h(h^2 - 9)]c^2 + (1 - 2\sigma h)[(2h + \sigma)^2 - 4]^2\} \end{aligned} \quad (4.24)$$

vanishes.

Consider the equation $Rs(c^2) = 0$ (with $\sigma \neq 0$) as cubic with respect to c^2 . It can have one or three roots, depending on the sign of the discriminant of the equation equal to

$$\begin{aligned} D = & -(\sigma^2 + 2\sigma h - 1)^2 D_1, \\ D_1 = & 64\sigma h^3 - (96\sigma^2 + 36)h^2 + 12\sigma(21 + 4\sigma^2)h - 8\sigma^4 - 9\sigma^2 - 108. \end{aligned} \quad (4.25)$$

The expression D_1 , in turn, is a cubic polynomial with respect to h , with a negative discriminant equal to $-20155392(4\sigma^2 + 1)^3$. Therefore, for all values of σ , the polynomial D_1 has a unique real root $h = h_*(\sigma)$.

If $h > h_*(\sigma)$, then $D_1 > 0$, and $D < 0$; in this case, the polynomial $Rs(c^2)$ has a unique real root. If $h < h_*(\sigma)$, then $D > 0$, and this polynomial has three real roots. Only positive roots should be taken into account; they will correspond to real or extraneous (in the above sense) equilibria of the model system.

The cases $\sigma > 1$, $0 < \sigma \leq 1$, $-1 \leq \sigma < 0$, and $\sigma < -1$ are qualitatively different. The corresponding discriminant curves (at a fixed value of σ) are shown in Figs. 5a–5d in the parameter plane c^2, h . The points on the h -axis marked in this figures correspond to the energy levels of the equilibrium positions of the model system at $c = 0$. Some of them are generators for equilibrium curves for $c^2 > 0$; of the rest, the curves going to the region $c^2 < 0$ are initiated.

The symbols g_i that mark the various equilibrium curves in Fig. 5 correspond to the same symbols on different branches of the graph of the function $y = G(u)$ in Fig. 3. Comparing Figs. 3 and 5 for fixed values of c^2 and σ , it is possible to determine the equilibrium values of $u = u_{(i)}$ (we will mark them with the same index) and the corresponding energy levels $h = h_{(i)}$.

 Fig. 5. Discriminant curves in the parameter plane c^2, h at $c \neq 0$.

The point A_* in Fig. 5a corresponds to the minimum point in Fig. 3. Its abscissa $c^2 = c_A^2$ is the result of reversing the relation $\sigma = G(u_*)$ (the quantity $G(u_*)$ is defined in Eq. (4.23)) with respect to c^2 , taking into account the relation between the quantities c and c_1 from Eq. (4.22), and the ordinate $h_A = h_*(\sigma)$. The point B_* corresponds to the case of a double root $h = h_B = (1 - \sigma^2)/(2\sigma)$ of the discriminant D from Eq. (4.25) and a double root $c^2 = c_B^2 = 1/\sigma$ of the polynomial $Rs(c^2)$.

In regions 0 in Fig. 5 motions of the model system are not possible.

For points (c^2, h) located in Fig. 5a to the left of the line $c^2 = c_A^2$ and above region 0, the phase trajectories of the model system are depicted in Fig. 6a in the plane of the variables θ, p_2 (for $\theta \in (0, \pi)$). The energy level $h = h_{(1)}$ corresponds to the stable equilibrium $\theta = \theta_{(1)}$ (hereinafter $\theta_{(i)} = \arccos u_{(i)}, i = 1, 2, 3$). For $h_{(1)} < h < h_{(2)}$ (region 1 and part of region 2 below the curve g_2), we have oscillations in the vicinity of this equilibrium. The energy level $h = h_{(2)}$ corresponds to one of these oscillations and the stable equilibrium state $\theta = \theta_{(2)}$. For the values in the interval $h_{(2)} < h < h_{(3)}$ (region 3), two oscillations occur in the system in the vicinity of two stable equilibrium positions. For $h = h_{(3)}$, we have the unstable equilibrium $\theta = \theta_{(3)}$ and asymptotic motion (separatrix in Fig. 6a). For values $h > h_{(3)}$ (part of region 2 above

Fig. 6. Phase portraits of the model system at $c \neq 0$.

the curve g_3) oscillations occur in the system, which in Fig. 6a correspond to the closed curves covering all three singular points (equilibrium positions) of the model system.

For points (c^2, h) located in Fig. 5a to the right of the line $c^2 = c_A^2$, as well as for points in Figs. 5b–5d (above regions 0), for $h = h_{(1)}$ we have the stable equilibrium $\theta = \theta_{(1)}$, and for $h > h_{(1)}$ (regions 1, 2 and 4 in these figures) oscillations in its vicinity. The corresponding phase portrait is shown in Fig. 6b.

For the values of the parameters lying inside regions 1, 2, 3, and 4 in Fig. 5, the graphs of the function $y = f(u)$ have the form shown, respectively, in Figs. 4a, 4b, 4c, 4d.

Remark. In the case $\sigma = 0$ when vibrations of the suspension point are absent, we obtain the following equation of the discriminant curve from the formula (4.24):

$$27c^4 + 8h(h^2 - 9)c^2 - 16(h^2 - 1)^2 = 0. \quad (4.26)$$

The corresponding graphical representation of this case is Fig. 5c, with $h_0 = -h_\pi = 1$. Previously, an equation of a similar curve (written in parametric form and using parameters different from our parameters by constant factors) and a diagram similar to Fig. 5c were given in [19].

Using the notation given in Fig. 4 for the roots of the polynomial $f(u)$, we find [28, 30] that the frequency ω_1 of oscillations of the model system in region 1 and of both types of oscillations in region 3 are defined using the first formula in (4.13), where

$$\xi_1 = \frac{1}{2} \sqrt{|\sigma|(u_4 - u_3)(u_3 - u_1)}, \quad k_1 = \sqrt{\frac{(u_4 - u_3)(u_2 - u_1)}{(u_4 - u_2)(u_3 - u_1)}},$$

and $K(k_1)$, as before, is the complete elliptic integral of the first kind. The frequency ω_2 of oscillations in region 4 is calculated by the same formula in which we assume

$$\xi_2 = \xi_1, \quad k_2 = \sqrt{\frac{(u_3 - u_2)(u_4 - u_1)}{(u_4 - u_2)(u_3 - u_1)}}.$$

For points of region 2, the function $f(u)$ has a pair of complex conjugate roots $u_{3,4} = m \pm i n$. Let be $M^2 = (u_2 - m)^2 + n^2$, $N^2 = (u_1 - m)^2 + n^2$, then the frequency ω_3 of oscillations in this region is determined by the second formula in (4.13) (with replacement of the index 5 by 3), where

$$\xi_3 = \sqrt{\sigma MN}, \quad k_3 = \frac{(u_2 - u_1)^2 - (M - N)^2}{4MN}.$$

On the boundary curves $h = h_{(i)}$ ($i = 1, 2, 3$), Eq. (4.5) is integrated in elementary functions.

4.2.2. On body motions at $c \neq 0$

We now consider body motions corresponding to the solution $\theta = \tilde{\theta}(\tau)$ of the model system. If this solution describes one of the oscillations of the system, then, according to Eq. (4.6), the angular velocity of precession of the body axis changes periodically, with a period equal to the period of this oscillation, and retains a constant sign. This means that the angle ψ increases or decreases monotonically. The trajectory of the apex on the Poisson sphere is a curve enclosed between two parallels corresponding to the limits in variation of the angle θ for the oscillation under study (Fig. 7a).

Fig. 7. Trajectories of the apex on the Poisson sphere.

For the equilibrium solutions $\theta = \theta_0 = \text{const}$ of the model system, the angular velocity of precession of the body axis is constant, according to Eq. (4.6). This motion corresponds to the stationary rotation of the system with the Hamiltonian (3.27) (under conditions (4.1)). The apex trajectory is one of the parallels on the Poisson sphere (Fig. 7b). If the generating solution $\theta = \theta_0$ of the model system is unstable, then the stationary rotation is also unstable (with respect to perturbations of the quantities θ and p_2), since the characteristic equation of linearized equations of perturbed motion has a positive real root. If the generating solution $\theta = \theta_0$ is stable, then, based on the Routh theorem on stability of stationary rotations of a holonomic conservative system with cyclic coordinates [31, 32], this stationary rotation is stable with respect to perturbations θ and p_2 . And on the basis of Lyapunov's addition to the Routh theorem (see *ibid.*), it is also stable with respect to small perturbation of the quantity p_1 .

The body motions is precession in this case, for which the angular velocity of its proper rotation is generally variable, according to the equation

$$\varphi' = -\frac{c(a \sin \theta_0 \sin \varphi + \cos \theta_0)}{\sin^2 \theta_0}. \quad (4.27)$$

Under the condition $|\cot \theta_0| < a$, Eq. (4.27) has two constant solutions $\varphi = \varphi_0 = \text{const}$ (equilibrium points) defined by the equality $\sin \varphi_0 = -\cot \theta_0/a$, one of them is asymptotically stable and the other is unstable. If $\cot \theta_0 = \pm a$, then there is one equilibrium point $\varphi_0 = \mp \pi/2$, and it is unstable.

The solutions $\theta = \theta_0$, $\varphi = \varphi_0$, $\psi' = \text{const}$ correspond to stationary rotations of the body around the vertical. Necessary and sufficient conditions for their stability are the simultaneous fulfillment of the conditions of stability and asymptotic stability of the corresponding equilibrium points of the model system and Eq. (4.27). To verify the sufficiency, we choose the Lyapunov function in the form $V = V_1^2 + V_2^2 + \xi^2/2$, where V_1 and V_2 are the energy and area integrals on the disturbed motion, and $\xi = \varphi - \varphi_0$. By virtue of the equations of motion, the derivative $V' = d\xi^2 + O(\xi^3)$ ($d < 0$) of the function V is a negative sign-constant function for sufficiently small values of ξ . Hence, on the basis of the Lyapunov stability theorem, the stability of this motion follows. The necessity of the above stability conditions is examined when considering the characteristic equation of the linearized equations of perturbed motion.

Nonconstant solutions of Eq. (4.27) are determined from the relation

$$\int \frac{d\varphi}{\alpha + \beta \sin \varphi} = -\frac{c}{\sin^2 \theta} (\tau - \tau_0), \quad (4.28)$$

where the notations $\alpha = \cos \theta_0$, $\beta = a \sin \theta_0$ are introduced, and τ_0 is an arbitrary constant.

If $\alpha^2 > \beta^2$, then the derivative φ' retains a constant sign, and the angle φ monotonously increases or decreases. Integrating (4.28), we obtain the equation

$$\begin{aligned} \cos \varphi &= \frac{\cos^2[w(\tau - \tau_0)] - \sin^2[w(\tau - \tau_0) + \delta_0]}{\cos^2[w(\tau - \tau_0)] + \sin^2[w(\tau - \tau_0) + \delta_0]}, \\ w &= \frac{c\sqrt{\alpha^2 - \beta^2}}{2\sin^2 \theta_0}, \quad \delta_0 = \arctan \frac{\beta}{\sqrt{\alpha^2 - \beta^2}}. \end{aligned}$$

The rotation of the body around the Oz' axis is periodic, with a period equal to π/w . During this period, the body makes a turn of 2π , returning to its starting position.

In the case of $\alpha^2 < \beta^2$, the integration of Eq. (4.28) leads to the relation

$$\begin{aligned} \tan \frac{\varphi}{2} &= \frac{-\tan(\varphi_0/2) + \cot(\varphi_0/2) e^{-w_1(\tau - \tau_0)}}{e^{-w_1(\tau - \tau_0)} - 1}, \\ w_1 &= \frac{c\sqrt{\beta^2 - \alpha^2}}{\sin^2 \theta_0}, \quad \tan \frac{\varphi_0}{2} = \frac{\beta - \sqrt{\beta^2 - \alpha^2}}{\alpha}, \end{aligned}$$

from which it follows that, as $\tau \rightarrow \infty$ and $\tau \rightarrow -\infty$, the angle φ tends to the asymptotically stable and unstable equilibrium points of Eq. (4.27), respectively.

If $\alpha^2 = \beta^2$, then we obtain

$$\tan \frac{\varphi}{2} = \mp \left(\frac{1}{w_2(\tau - \tau_0)} + 1 \right), \quad w_2 = \frac{\alpha c}{2\sin^2 \theta_0},$$

where the upper and lower signs correspond to the cases $\alpha = \beta$ and $\alpha = -\beta$. As $\tau \rightarrow \pm\infty$, the angle φ tends to the equilibrium value $\varphi_0 = \mp \pi/2$.

5. On the dynamics of a body with Hess mass geometry in nonintegrable cases

We make some remarks on nonintegrable cases in the studied approximate problem of dynamics of a body with the Hess mass geometry in the presence of the suspension point vibrations. The reduced system with two degrees of freedom with the Hamiltonian (3.27) is nonintegrable if the vibration parameters do not satisfy conditions (4.1). If these conditions are satisfied, but relation (3.28) is violated, then the problem is also nonintegrable. However, due to the existing symmetry, which is characteristic in the Hess case (in the first of these cases), and the orthogonality of the vibration moment vector to the radius vector $\mathbf{OG}$ (in both cases), the body motion in these nonintegrable cases has a number of specific properties.

5.1. On the body motions in the general case of vibrations

As stated above, the Hamiltonian (3.27) of the reduced system for a body in the Hess case coincides with the Hamiltonian of a similar reduced system (with $p_\varphi = 0$) in the problem of dynamics of a Lagrange top with a vibrating suspension. This means that the motions of the axis of these two bodies (containing their centers of mass) are the same.

Hence, in particular, the equilibrium positions $\theta = \text{const}$, $\psi = \text{const}$ of both reduced systems are the same, being stationary points of (the same) reduced potential energy. Sufficient conditions for their stability (with respect to perturbations of the quantities θ , ψ , p_1 , p_2), which are defined as conditions for the minimum of the potential energy, also coincide. However, in the Lagrange case, each equilibrium position of the reduced system corresponds to the stationary rotation of the body, while in the Hess case it corresponds to the family of equilibrium positions of the body, for which the angle φ takes an arbitrary value. A similar property was noted in Section 4.1 when considering the integrable case.

The stationary rotations in the approximate problem of the dynamics of a Lagrange top with a vibrating suspension were investigated in detail in [26]. In connection with the foregoing, the results obtained in the article cited can be applied to describe the families of equilibrium positions of a body with the Hess mass geometry. We present the corresponding results, restricting ourselves to the case $a_{XY} = a_{XZ} = a_{YZ} = 0$.

In this case, there are two families of equilibrium positions of the body, corresponding to the vertical position of the axis Oz' , when the center of mass G is located above ($\theta = 0$) and below ($\theta = \pi$) of the suspension point. These equilibrium positions are stable (with respect to the above variables) when the condition

$$a_Z \mp \frac{Bg}{m\ell} > \max(a_X, a_Y)$$

holds, and are unstable when changing the inequality signs to opposite ones. Here the upper and lower signs correspond to $\theta = 0$ and $\theta = \pi$.

There can also be families of equilibria for which the body axis is inclined to the vertical. They are described by the relations

$$\psi = \frac{\pi}{2}, \cos \theta = -\frac{Bg}{m\ell a_1} \left(|a_1| > \frac{Bg}{m\ell} \right); \quad \psi = 0, \cos \theta = -\frac{Bg}{m\ell a_2} \left(|a_2| > \frac{Bg}{m\ell} \right),$$

with the quantities a_1 and a_2 introduced by the formulae (3.17). The conditions for the existence of these families are given in parentheses. If, for definiteness, $a_X > a_Y$ ($a_1 > a_2$), then the first one is stable for $a_1 > Bg/(m\ell)$ and unstable for $a_1 < -Bg/(m\ell)$; for this family, the axis of the

body is located in the OXZ plane. The second equilibrium family, for which the body axis lies in the plane OYZ , is unstable in the existence region.

For other vibration options considered in [26], the results obtained can be similarly applied to the problem studied in this paper.

5.2. The vibration symmetry case with nonequality in relation (3.28)

If the vibration parameters satisfy conditions (4.1), then for any body mass geometry, the vector $\mathbf{m}_v$ of the vibration moment is always horizontal as well as the vector $\mathbf{m}_g$ of the gravitational moment. For the Hess case at hand, these two vectors are collinear, since relation (3.3), after performing some transformations, is represented as

$$\mathbf{m}_v = \frac{m^2(a_Z - a_X)}{B} (\mathbf{OG} \cdot \boldsymbol{\gamma}) (\mathbf{OG} \times \boldsymbol{\gamma}).$$

We confine ourselves to considering the question of the existence of the particular body motions, being its stationary rotations around the axes fixed in the body and in the coordinate system $OXYZ$. They are analogs of Staude's permanent rotations in the dynamics of a heavy rigid body with a fixed point (see [33], as well as the monograph [34] and references therein).

For these motions, $\boldsymbol{\omega} = \text{const}$, and Eq. (2.1) takes the form

$$\boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{m}_g + \mathbf{m}_v. \quad (5.1)$$

From this relation it follows that the axis of permanent rotation of a body with any mass geometry can only be vertical if conditions (4.1) hold, and, therefore, $\boldsymbol{\omega} = \omega \boldsymbol{\gamma}$ ($\omega = \text{const}$). The permissible axes of permanent rotations lie on the surface of an elliptic cone in the space of the quantities $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ (projections of the vector $\boldsymbol{\gamma}$ onto the principal inertia axes Ox, Oy, Oz), which is an analogue of Staude's cone for a body with a fixed point. The equation of this cone is obtained by scalar multiplication of both parts of Eq. (5.1) by the vector $\mathbf{OG}$, which gives (taking into account the equality $\mathbf{m}_g \cdot \mathbf{OG} = 0$)

$$\omega^2 (\boldsymbol{\gamma} \times \mathbf{I}\boldsymbol{\gamma}) \cdot \mathbf{OG} = \mathbf{m}_v \cdot \mathbf{OG}. \quad (5.2)$$

For a body with an arbitrary mass geometry and a body with a center of mass in the principal plane of inertia (but not satisfying the Hess conditions), the vectors $\mathbf{m}_v$ and $\mathbf{OG}$ are not orthogonal, therefore Eq. (5.2) depends both on the angular velocity ω of permanent rotation, and on the vibration parameters. For example, for the second of these cases (when $y_G = 0$), this equation, written in terms of the projections of vectors in the system of principal axes of inertia, has the form

$$\begin{aligned} \omega^2 \tilde{\gamma}_2 [(B - C)x_G \tilde{\gamma}_3 + (A - B)z_G \tilde{\gamma}_1] = \\ = -\frac{m^2(a_Z - a_X)}{ABC} \tilde{\gamma}_2 (x_G \tilde{\gamma}_3 - z_G \tilde{\gamma}_1) [(A - B)Cz_G^2 - A(B - C)x_G^2]. \end{aligned} \quad (5.3)$$

This and the general cases are very difficult to study, since even the description of the set of permissible axes of permanent rotations is a very cumbersome problem.

If the body satisfies the Hess conditions, then the right-hand sides of Eqs. (5.2) and (5.3) vanish (due to the property $\mathbf{m}_v \cdot \mathbf{OG} = 0$, proven in Section 2), and these equations coincide exactly with the analogous equations for a body with a fixed point:

$$(\boldsymbol{\gamma} \times \mathbf{I}\boldsymbol{\gamma}) \cdot \mathbf{OG} = 0 \quad \text{or} \quad \tilde{\gamma}_2 [(B - C)x_G \tilde{\gamma}_3 + (A - B)z_G \tilde{\gamma}_1] = 0.$$

This equation describes a set of two planes

$$\tilde{\gamma}_2 = 0 \quad \text{and} \quad (B - C)x_G \tilde{\gamma}_3 + (A - B)z_G \tilde{\gamma}_1 = 0$$

fixed in the body, in which the axes of the body permanent rotation can lie.

In the case $\tilde{\gamma}_2 = 0$ these axes lie in the plane Oxz containing the center of mass of the body. The equation of the second plane, with regard to the second condition in (2.6), can be rewritten as $Ax_G\tilde{\gamma}_1 + Cz_G\tilde{\gamma}_3 = 0$. Hence, due to the relations $p = \omega\tilde{\gamma}_1$, $r = \omega\tilde{\gamma}_3$, and $y_G = 0$, we obtain equality (2.7). Thus, rotations around the axes from the second plane are realized in the integrable case, they were obtained in a different way in Section 4.2.

For the case $\tilde{\gamma}_2 = 0$, we project Eq. (5.1) onto the principal inertia axes Ox , Oy , Oz . The first and third equations are satisfied identically, and the second one has the form

$$(A - C)\tilde{\gamma}_1\tilde{\gamma}_3\omega^2 = -mg(z_G\tilde{\gamma}_1 - x_G\tilde{\gamma}_3)\left[1 + \frac{\sigma}{\ell}(x_G\tilde{\gamma}_1 + z_G\tilde{\gamma}_3)\right].$$

From here we obtain the relation describing the set of admissible axes of rotations, it is the condition for the positivity of the value ω^2 in this equation. Since $A > C$, we have

$$\tilde{\gamma}_1\tilde{\gamma}_3(z_G\tilde{\gamma}_1 - x_G\tilde{\gamma}_3)\left[1 + \frac{\sigma}{\ell}(x_G\tilde{\gamma}_1 + z_G\tilde{\gamma}_3)\right] < 0.$$

Further research is not carried out here, it is an independent problem.

Note that in the case considered in this section it is more convenient to choose the system of principal axes of inertia; the advantage in using the system of nonprincipal axes is lost.

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