MATHEMATICAL PROBLEMS OF NONLINEARITY

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Research on the Motion of a Body in a Potential Force Field in the Case of Three Invariant Relations

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The problem of the motion of a rigid body with a fixed point in a potential force field is considered. A new case of three nonlinear invariant relations of the equations of motion is presented. The properties of Euler angles, Rodrigues–Hamilton parameters, and angular velocity hodographs in the Poincot method are investigated using an integrated approach in the interpretation of body motion.

Keywords: potential force field, Euler angles, Rodrigues–Hamilton parameters, Poincot method

The dynamics of a rigid body with a fixed point is based on the solution of two fundamental problems. One of them is certainly concerned with describing the motion of a body in the form of differential equations and with obtaining solutions to these equations in a certain class of functions. The other problem involves the study of motion properties (the kinematic interpretation of motion in the solutions of the former problem). When both problems are solved, we can talk about a complete comprehensive study of a specific problem of rigid body dynamics [1, 2]. In the monographs [3–6] and the articles [7, 8], a description is given of diverse classes of differential equations in rigid body dynamics, which were studied using different approaches, and contributions are noted, made by various scientists on problems of integration of equations of motion (L. Euler, J. Lagrange, S. V. Kovalevskaya, H. Poincaré, A. M. Lyapunov, S. A. Chaplygin,

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W. Hess, G. Kirchhoff, and others). In this analysis, we can highlight the problem of nonintegrability by quadratures of equations of rigid body dynamics. This property is proved for many problems of rigid body dynamics. Note that nonintegrability of the Euler–Poisson equations was shown by S. L. Ziglin [9, 10], this fact was established by V. V. Kozlov, D. A. Onishchenko [11] and A. V. Borisov [12] for Kirchhoff equations, the proper result was obtained by A. A. Burov [13] for equations of motion of a body in Cardan suspension, and for equations of the dynamics of linked rigid bodies such a result was proved by O. I. Bogoyavlensky [14].

The problem of nonintegrability of the equations of rigid body dynamics stimulated many scientists to search for partial solutions of these equations (W. Hess, V. A. Steklov, S. A. Chaplygin, N. Kovalevsky, and many others). The importance of this approach is noted in the monographs [3, 5, 6, 15], which fully present partial solutions of the equations of rigid body dynamics. To date, the method of invariant relations has been formed for constructing partial solutions. In [15], definitions of the invariant relations, proposed by A. Poincaré [16], T. Levi-Civita [17], S. A. Chaplygin [18], and P. V. Kharlamov [19], are presented.

To solve the second main problem (the kinematic interpretation of the motion of a body with a fixed point), the Poincot method, the apex method and other methods can be successfully used. The importance of studying the properties of motion of a body was noted by L. Poincot [20], N. E. Zhukovsky [1], and E. Routh [21]. Recently, a modified Poincot method for representing the motion of a body by rolling some characteristic axoids without sliding has been developed [22]. This method allowed a kinematic approach to be applied to the interpretation of motion of a body.

The purpose of this article is to construct new classes of solutions to the equations of motion of a rigid body under the action of potential forces, and to interpret motion using an integrated approach. One of the main results of the article is a new structure of invariant relations and conditions for their existence, which are expressed by the equations $A_2 = A_1 = A_3(n + 2)$ for the principal moments of inertia A_i of the body. These equations give Kovalevskaya conditions at $n = 0$, in the case $n = 2$ they give Goryachev–Chaplygin conditions, and in the case $n \in N$ ($n > 2$) they are **new** in the dynamics of a rigid body. The integrated approach to the interpretation of motion of the body provides a clear insight into the properties of kinematic interpretation for the solutions found.

The results obtained in this paper are fundamentally different from those of H. M. Yehya [23, 24], who studied the first integrals of the equations of motion of a body in the potential force field. Invariant relations considered by V. Y. Olshansky [25, 26] pertain to the problem of motion of a body with liquid filling. In the papers by R. G. Mukharlyamov [27, 28], program motions of Lagrangian dynamical systems are investigated, which can be linked with the terminology of invariant relations. Differential equations for these motions are constructed, and the problem of their stabilization is considered.

In the present paper, a new class of solutions to the equations of motion of a rigid body in a potential force field is constructed, which can be defined by three nonlinear invariant relations. Using the method of solving inverse problems of dynamics, the force function and the conditions for the principal moments of inertia are found, which depend on the natural exponent of the given invariant relations.

For the invariant relations under study, the integration problem of the equations of motion is reduced to quadratures. On the basis of the found solution, Euler angles, Rodrigues–Hamilton parameters, moving and fixed hodographs of the angular velocity vector are determined. The modified Poincot method is used for the kinematic interpretation of the motion of the body. The results obtained are complex, and therefore they can be used in applications of solid mechanics.

1. Problem statement. Consider the equations of motion of a heavy rigid body in a potential force field [29, 30]:

$$A_1 \dot{\omega}_1 = (A_2 - A_3) \omega_2 \omega_3 + \nu_3 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_2} - \nu_2 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_3}, \quad (1)$$

$$A_2 \dot{\omega}_2 = (A_3 - A_1) \omega_3 \omega_1 + \nu_1 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_3} - \nu_3 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_1}, \quad (2)$$

$$A_3 \dot{\omega}_3 = (A_1 - A_2) \omega_1 \omega_2 + \nu_2 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_1} - \nu_1 \frac{\partial U(\nu_1, \nu_2, \nu_3)}{\partial \nu_2}, \quad (3)$$

$$\dot{\nu}_1 = \omega_3 \nu_2 - \omega_2 \nu_3, \quad \dot{\nu}_2 = \omega_1 \nu_3 - \omega_3 \nu_1, \quad \dot{\nu}_3 = \omega_2 \nu_1 - \omega_1 \nu_2. \quad (4)$$

The first integrals of Eqs. (1)–(4) are

$$\begin{aligned} \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1, & A_1 \omega_1 \nu_1 + A_2 \omega_2 \nu_2 + A_3 \omega_3 \nu_3 &= k, \\ A_1 \omega_1^2 + A_2 \omega_2^2 + A_3 \omega_3^2 - 2U(\nu_1, \nu_2, \nu_3) &= 2E, \end{aligned} \quad (5)$$

where k and E are arbitrary constants. In formulas (1)–(5) the following notation is introduced: ω_i ($i = \overline{1, 3}$) are the components of the angular velocity vector $\boldsymbol{\omega}$ of a rigid body with a fixed point; ν_i ($i = \overline{1, 3}$) are the components of the unit vector $\boldsymbol{\nu}$ of the symmetry axis of the force field; A_i ($i = \overline{1, 3}$) are the principal moments of inertia of the body; $U(\nu_1, \nu_2, \nu_3)$ is the forcing function.

Consider conditions for the existence of invariant relations for Eqs. (1)–(4)

$$\omega_1 = \nu_3^{n-1} \left(-\frac{\mu_1 n}{n+2} \nu_1 + \beta_1 \mu_2 \right), \quad \omega_2 = \nu_3^{n-1} \left(-\frac{\mu_1 n}{n+2} \nu_2 + \beta_2 \mu_2 \right), \quad \omega_3 = \mu_1 \nu_3^n, \quad (6)$$

where $\mu_1, \mu_2, \beta_1, \beta_2$ are constant parameters, and n is a natural number or zero.

This article uses the method of solving the inverse problems of mechanics, that is, on the basis of invariant relations the forcing function $U(\nu_1, \nu_2, \nu_3)$ is constructed with the help of the third equality of system (5):

$$U(\nu_1, \nu_2, \nu_3) = \frac{1}{2} \left[A_1 \omega_1^2(\nu_1, \nu_2, \nu_3) + A_2 \omega_2^2(\nu_1, \nu_2, \nu_3) + A_3 \omega_3^2(\nu_1, \nu_2, \nu_3) \right] - E, \quad (7)$$

where the functions ω_i ($i = \overline{1, 3}$) are determined by Eqs. (6), which are used in Eqs. (1)–(3).

If $n = 0$ in (7), then the forcing function contains singular terms. Similar cases took place in [23, 29]. Physical interpretation of a potential function with singular terms was studied in quantum mechanics by I. V. Komarov and V. B. Kuznetsov [31, 32].

The aim of the article is to study the conditions for existence of invariant relations (6) in Eqs. (1)–(4), where the forcing function has the form (7) [33].

2. Integration of Eqs. (4) on invariant relations (6). Substitute ω_i ($i = \overline{1, 3}$) from (6) into Poisson equations (4):

$$\begin{aligned} \dot{\nu}_1 &= \nu_3^n \left[\frac{2(n+1)}{n+2} \mu_1 \nu_2 - \beta_2 \mu_2 \right], \\ \dot{\nu}_2 &= \nu_3^n \left[-\frac{2(n+1)}{n+2} \mu_1 \nu_1 + \beta_1 \mu_2 \right], \\ \dot{\nu}_3 &= \mu_2 \nu_3^{n-1} (\beta_2 \nu_1 - \beta_1 \nu_2). \end{aligned} \quad (8)$$

Statement 1. For any valid values of $\beta_1, \beta_2, \mu_1, \mu_2$ and $n \in N$, Eqs. (8) admit two invariant relations:

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad \beta_1 \nu_1 + \beta_2 \nu_2 = \frac{\mu_1}{\mu_2(n+2)}[n - (n+1)\nu_3^2]. \quad (9)$$

Proof. It is enough to differentiate both parts of (9) by virtue of (8). As a result, we obtain an identity in the variable ν_3 .

Statement 2. For any valid values of parameters $\beta_1, \beta_2, \mu_1, \mu_2$ and $n \in N$, Eqs. (8) are integrable by quadratures in the sense of Jacobi.

Proof. Let us take the third component of the vector ν as an independent variable. From Eqs. (9) we find $\nu_1(\nu_3), \nu_2(\nu_3)$:

$$\begin{aligned} \nu_1(\nu_3) &= \frac{1}{\mu_2 \chi_0^2(n+2)} [\mu_1 \beta_1 (n - (n+1)\nu_3^2) - \beta_2 \sqrt{F(\nu_3)}], \\ \nu_2(\nu_3) &= \frac{1}{\mu_2 \chi_0^2(n+2)} [\mu_1 \beta_2 (n - (n+1)\nu_3^2) + \beta_1 \sqrt{F(\nu_3)}]. \end{aligned} \quad (10)$$

Here $\chi_0^2 = \beta_1^2 + \beta_2^2$, and

$$\begin{aligned} F(\nu_3) &= -\varepsilon_2^2 \nu_3^4 + \varepsilon_1 \nu_3^2 + \varepsilon_0, \quad \varepsilon_2 = \mu_1(n+1), \\ \varepsilon_1 &= 2\mu_1^2 n(n+1) - \chi_0^2 \mu_2^2 (n+2)^2, \quad \varepsilon_0 = \chi_0^2 \mu_2^2 (n+2)^2 - n^2 \mu_1^2. \end{aligned} \quad (11)$$

To obtain the dependencies $\nu_i(t)$ ($i = \overline{1,3}$) on t , we turn to the third equation of (8). By virtue of the obvious identity

$$(\beta_2 \nu_1 - \beta_1 \nu_2)^2 + (\beta_1 \nu_1 + \beta_2 \nu_2)^2 = \chi_0^2 (1 - \nu_3^2)$$

and the second relation of (9), we find

$$\beta_2 \nu_1 - \beta_1 \nu_2 = -\frac{1}{\mu_2(n+2)} \sqrt{F\nu_3}. \quad (12)$$

Substitute the value (12) into the third equation of (8). Then we find that the function $\nu_3(t)$ is determined by inverting the integral

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\nu_3^{n-1} \sqrt{F(\nu_3)}} = -\frac{1}{n+2} (t - t_0), \quad (13)$$

where t_0 is the initial value of t , and $\nu_3^{(0)}$ is a constant. Let us prove that on the segment $[-1; 1]$ there exists a simply connected domain in which $F(\nu_3) > 0$. Consider the equation $F(\nu_3) = 0$:

$$\varepsilon_2^2 \nu_3^4 - \varepsilon_1 \nu_3^2 - \varepsilon_0 = 0. \quad (14)$$

Compute the discriminant of Eq. (14), which is a quadratic equation in ν_3^2 :

$$D = \mu_2^2 \chi_0^2 (n+2)^2 [\mu_2^2 (n+2)^2 + 4\mu_1^2 (n+1)] > 0. \quad (15)$$

By virtue of (15) the roots of Eq. (14) with respect to ν_3^2 are real. We will further consider three cases: $\varepsilon_0 = 0$, $\varepsilon_0 > 0$, $\varepsilon_0 < 0$.

The first case is characterized by the following properties:

$$n^2\mu_1^2 = \chi_0^2\mu_2^2(n+2)^2, \quad F(\nu_3) = \mu_1^2(n+1)^2(\lambda_0^2 - \nu_3^2), \quad (16)$$

where

$$\lambda_0 = \frac{\sqrt{n(n+2)}}{n+1} < 1. \quad (17)$$

The variable ν_3 varies on the segment

$$-\lambda_0 \leq \nu_3 \leq \lambda_0. \quad (18)$$

In the second case we have

$$n^2\mu_1^2 < \chi_0^2\mu_2^2(n+2)^2, \quad F(\nu_3) = \varepsilon_2^2(\alpha_2^2 - \nu_3^2)(\nu_3^2 + \alpha_1^2), \quad (19)$$

here α_1^2 and α_2^2 are the roots of Eq. (14) with respect to ν_3^2 . Variation of ν_3 is defined by the formula

$$-\alpha_2 \leq \nu_3 \leq \alpha_2. \quad (20)$$

It is presumed in (20) that $\alpha_2 > 0$.

The third case is

$$n^2\mu_1^2 > \chi_0^2\mu_2^2(n+2)^2, \quad F(\nu_3) = \varepsilon_2^2(\gamma_2^2 - \nu_3^2)(\nu_3^2 - \gamma_1^2), \quad (21)$$

where γ_1^2, γ_2^2 are the roots of Eq. (14) with respect to ν_3^2 . The variable ν_3 changes in the doubly connected scope

$$-\gamma_2 \leq \nu_3 \leq -\gamma_1, \quad \gamma_1 \leq \nu_3 \leq \gamma_2, \quad (22)$$

where it is presumed that $\gamma_1 > 0, \gamma_2 > 0$. Thus, the function $\nu_3(t)$, obtained by inverting the integral (13), is real for any $\mu_1, \mu_2, \beta_1, \beta_2, n \in N$. This proves Statement 2.

3. Integration of Eqs. (1)–(3). In the study of this problem, we first consider the angular momentum integral on the invariant relations (6) and (9). The following statement is true.

Statement 3. Sufficient conditions for the compatibility of the angular momentum integral from (5) and invariant relations (6), (9) are the conditions $k = 0$ and

$$A_2 = A_1 = A_3(n+2). \quad (23)$$

Proof is based on the following transformations. At the first stage, we substitute invariant relations (6) into the second relation of system (5). At the second stage, assuming in the resulting equality that conditions (23) and the equality $k = 0$ are satisfied, we obtain an identity in the variable ν_3 .

Statement 4. Assume that sufficient conditions (23) are satisfied, then relations (6), (10), (11), (13) are solutions of Eqs. (1)–(3), where the forcing function $U(\nu_1, \nu_2, \nu_3)$ of (7) is determined on the invariant relation (6) using (5).

This statement can be proved as follows. At first, using the invariant relation (9), we transform the function $U(\nu_1, \nu_2, \nu_3)$ of (7) to the form, where conditions (23) and invariant relation (9) are taken then into account. Next, we substitute the transformed function $U(\nu_1, \nu_2, \nu_3)$

and functions (6), (10) into Eqs. (1)–(3) and, taking (13) into account, we obtain an identity in ν_3 . The function for foundation equations is

$$U(\nu_1, \nu_2, \nu_3) = \frac{A_3 \nu_3^{2(n-1)}}{2(n+2)} [\mu_1^2 n^2 + \mu_2 \chi_0^2 (n+2)^2 - \mu_1^2 (n+1)(n-2) \nu_3^2 - 2\mu_1 \mu_2 n(n+2)(\beta_1 \nu_1 + \beta_2 \nu_2)] - E.$$

4. Analysis of integral (13).

4.1. Case $\varepsilon_0 = 0$. Substitute the expression for $F(\nu_3)$ from (16) into (13):

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\nu_3^n \sqrt{\lambda_0^2 - \nu_3^2}} = -\frac{\mu_1(n+1)}{n+2}(t-t_0). \quad (24)$$

Here and further we assume $\mu_1 > 0$ without loss of generality. Obviously, the initial value $\nu_3^{(0)} = 0$, which belongs to interval (18), is excluded from consideration.

Statement 5. The function $\nu_3(t)$, obtained by inversion of the integral from (24), possesses the following asymptotic property: $\nu_3 \rightarrow 0$ for $t \rightarrow \infty$. Integral (24) can be determined in finite form by using integrals of rational functions of an auxiliary variable.

Proof. The second part of Statement 5 is based on the known Euler replacement, the application of which reduces the integrand expression to a rational function of an auxiliary variable. We give an example of the Euler transform. Let $n = 2k + 1$ ($k \in \mathbb{N}$) in (24). Using the replacement $\lambda_0^2 - \nu_3^2 = z^2$, where z is a new variable, we reduce equation (24) to the form

$$\int_{z^{(0)}}^z \frac{dz}{(\lambda_0^2 - z^2)^{k+1}} = \frac{2\mu_1(k+1)}{2k+3}(t-t_0). \quad (25)$$

We write the integration result, which follows from (25) at $k = 0$ ($N = 1$):

$$\nu_3^2 = \frac{3e^w}{(e^w + 1)^2}, \quad w = \frac{2\mu_1(t-t_0)}{\sqrt{3}}. \quad (26)$$

According to relations (26), the initial value of ν_3 is assumed to be equal to $\frac{\sqrt{3}}{2}$. As $t \rightarrow \infty$, the variable $\nu_3 \rightarrow 0$, that is, the solution (6) for $n = 1$, is characterized by three linear invariant relations:

$$\omega_1 = -\frac{\mu_1}{3}\nu_1 + \beta_1\mu_2, \quad \omega_2 = -\frac{\mu_1}{3}\nu_2 + \beta_2\mu_2, \quad \omega_3 = \mu_1\nu_3 \quad (27)$$

and describes, in view of (10) and (26), the body motion that is asymptotic to the quiescent state. The mass distribution of this body satisfies the following conditions:

$$A_2 = A_1 = 3A_3, \quad (28)$$

which is new in rigid body dynamics.

For the proof of the first part of the statement, let us consider the integral in (24). Suppose, for example, $\nu_3^{(0)} \in (0, \lambda_0)$. In view of (24), at given $\nu_3^{(0)}$ we have $\frac{d\nu_3}{dt} < 0$, that is, ν_3 is decreasing. As $\nu_3 \rightarrow 0$ in (24), the integral diverges at $n > 0$. Therefore, the value $\nu_3 = 0$ is

reached in an infinite time interval. The proof of the first part of Statement 5 follows from this property.

Remark. Despite the fact that the nature of the functions $\nu_3(t)$ is known, it is not always possible to find the dependence $\nu_3(t)$ explicitly. Indeed, for $k = 1$ ($n = 3$), it follows from (25) that

$$\ln \frac{\lambda_0 + z}{\lambda_0 - z} + \frac{2z}{\lambda_0^2 - z^2} = -\frac{16\lambda_0^2(t - t_0)}{5}.$$

4.2. Case $\varepsilon_0 > 0$. We consider this option under the assumption that $\mu_1 > 0$. Substitute the expression for $A(\nu_3)$ from (19) into Eq. (13):

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\nu_3^{n-1} \sqrt{(\alpha_2^2 - \nu_3^2)(\nu_3^2 + \alpha_1^2)}} = -\frac{\mu_1(n+1)}{n+2}(t - t_0), \quad (29)$$

where $\nu_3^{(0)} \neq 0$, $\nu_3^{(0)} \neq |\alpha_2|$, because the function under integral in (29) is not defined at these values.

Statement 6. For any valid values of parameters characterizing the invariant relations (6), and $n > 0$ the function obtained by inversion of integral (29) is asymptotic.

Proof. Set the initial value of ν_3 from the interval (20) so that $0 < \nu_3^{(0)} < \alpha_2$ (this assumption does not limit the generality of the solution). Then it follows from (29) that $\frac{d\nu_3}{dt} < 0$, that is, ν_3 decreases to the value $\nu_3 = 0$, which is achieved in an infinite time period due to the divergence of the integral.

We give an example assuming $n = 2$ in (29). In this case, it follows from (23) that $A_2 = A_1 = 4A_3$. These equations characterize the Goryachev–Chaplygin solution of the classical problem of the motion of a heavy rigid body. From formula (29) we obtain

$$\nu_3^2 = \frac{\alpha_2^2 - \alpha_1^2 z^2}{1 + z^2}, \quad z(t) = \frac{\alpha_2(1 - e^{\varepsilon(t)})}{\alpha_1(1 + e^{\varepsilon(t)})}, \quad \varepsilon(t) = -\frac{3\mu_1\alpha_1}{2}(t - t_0). \quad (30)$$

The properties of the function $z(t)$ from (30) are obvious: $z(t_0) = 0$, $z \rightarrow \frac{\alpha_2}{\alpha_1}$ as $t \rightarrow \infty$. The function ν_3 is also asymptotic: $\nu_3 \rightarrow 0$ as $t \rightarrow \infty$. Since at $n = 2$ relations (6) take the form

$$\omega_1 = \nu_3(-\frac{\mu_1}{2}\nu_1 + \beta_1\mu_2), \quad \omega_2 = \nu_3(-\frac{\mu_1}{2}\nu_2 + \beta_2\mu_2), \quad \omega_3 = \mu_1\nu_3^2, \quad (31)$$

it follows from (6), (10), (30) and (31) that the body tends to a quiescent state as $t \rightarrow \infty$.

Statement 7. For any real values of the parameters characterizing the invariant relations (6), and for $n = 1$, the function $\nu_3(t)$ obtained by inversion of integral (29) is an elliptic Jacobi function.

Proof. Write (29) when $n = 1$:

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\sqrt{(\alpha_2^2 - \nu_3^2)(\nu_3^2 + \alpha_1^2)}} = -\frac{2\mu_1}{3}(t - t_0). \quad (32)$$

Using the well-known method for investigation of Legendre integrals, we obtain from formula (32)

$$\nu_3(t) = \alpha_2 \operatorname{cn}(k_*, \tau_1), \quad k_* = \frac{\alpha_2}{\sqrt{\alpha_1^2 + \alpha_2^2}}, \quad \tau_1 = \frac{2\mu_1}{3} \sqrt{\alpha_1^2 + \alpha_2^2} (t - t_0), \quad (33)$$

where $\operatorname{cn}(k_*, \tau_1)$ is the elliptic Jacobi function, and k_* is the modulus of this function. The period of this function is

$$T_1 = \frac{6K_1}{\mu_1 \sqrt{\alpha_1^2 + \alpha_2^2}}, \quad K_1 = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k_*^2 \sin^2 x}}. \quad (34)$$

So, the case $n = 1$ is characterized by conditions (33) and (34). From Eqs. (6), (10) and (33) it follows that the solution is periodic.

Case 4.3. Let $\varepsilon_0 < 0$. We introduce the value of the function $F(\nu_3)$ from (21) into the integral relation (13):

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\nu_3^{n-1} \sqrt{(\gamma_2^2 - \nu_3^2)(\nu_3^2 - \gamma_1^2)}} = -\frac{\mu_1(n+1)}{n+2} (t - t_0). \quad (35)$$

If $n = 1$ in (35), then we have

$$\nu_3(t) = \gamma_2 \operatorname{dn}(k^*, \tau_2), \quad k^* = \frac{\sqrt{\gamma_2^2 - \gamma_1^2}}{\gamma_2}, \quad \tau_2 = \frac{2\mu_1\gamma_2}{3} (t - t_0). \quad (36)$$

When getting formulas (36) it was assumed that the variable ν_3 ranges within the segment $\gamma_1 \leq \nu_3 \leq \gamma_2$ ($\gamma_1 > 0$, $\gamma_2 > 0$). The second case from (22) can be studied similarly. The function $\nu_3(t)$ in (36) is expressed in terms of the elliptic function $\operatorname{dn}(k^*, \tau_2)$, where k^* is its modulus. The period of this function is equal to

$$T_2 = \frac{3K_2}{\mu_1\beta_2}, \quad K_2 = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - (k^*)^2 \sin^2 x}}. \quad (37)$$

Due to (10) it follows from (36) that

$$\nu_1(\nu_3) = \frac{1}{3\mu_2\chi_0^2} [\mu_1\beta_1(1 - 2\nu_3^2) - \beta_2\sqrt{F(\nu_3)}], \quad (38)$$

$$\nu_2(\nu_3) = \frac{1}{3\mu_2\chi_0^2} [\mu_1\beta_2(1 - 2\nu_3^2) + \beta_1\sqrt{F(\nu_3)}], \quad (39)$$

where $\nu_3(t)$ takes values from (36), and the function $\sqrt{F(\nu_3)}$ is

$$\sqrt{F(\nu_3)} = 2\mu_1(\gamma_2^2 - \gamma_1^2) \operatorname{sn}(k^*, \tau_2) \operatorname{cn}(k^*, \tau_2). \quad (40)$$

Based on (6), (36) and (38)–(40), we conclude that the following statement is true.

Statement 8. If $\varepsilon_0 < 0$ and $n = 1$, then, for any valid values of parameters $\beta_1, \beta_2, \mu_1, \mu_2$, the solution of Eqs. (1)–(4) on the invariant relations (6) is periodic with the period specified in (37).

Consider the case where $n = 2k$ ($k \in N$) in (35). We write the integral relation (35) for $n = 2k$:

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3^2}{(\nu_3^2)^k \sqrt{(\gamma_1^2 - \nu_3^2)(\nu_3^2 - \gamma_1^2)}} = -\frac{\mu_1(2k+1)}{2(k+1)}(t-t_0). \quad (41)$$

Let us make a change of variables using the formula

$$\nu_3^2 = \frac{1}{2}[(\gamma_1^2 + \gamma_2^2) + (\gamma_1^2 - \gamma_2^2) \sin \varphi]. \quad (42)$$

By virtue of (42) and (41) we have

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\varphi}{[(\gamma_1^2 + \gamma_2^2) + (\gamma_1^2 - \gamma_2^2) \sin \varphi]^k} = \frac{\mu_1(2k+1)}{(k+1)2^{k+1}}(t-t_0). \quad (43)$$

It is obvious that the integral of (43) can be reduced to the integral of a rational function of some auxiliary variable. This transformation is quite complicated for arbitrary k . We present the result of analysis of the integral of (43) for $k = 1$ ($n = 2$):

$$\varphi = 2 \operatorname{arctg} \frac{\gamma_2 + \gamma_1 \operatorname{tg} \tau}{\gamma_2 - \gamma_1 \operatorname{tg} \tau}, \quad \tau = \frac{3\mu_1\gamma_1\gamma_2}{4}(t-t_0). \quad (44)$$

Based on formulas (6), (10), (42) and (44), we come to the following statement.

Statement 9. In the case where $\varepsilon_0 < 0$ and $n = 2$, the solution of the equations of motion of the body is described by elementary periodic functions of time.

Remark. The periodicity property of the solution (6), (10), where $\nu_3(t)$ is a function obtained by inversion of the integral (35), is preserved in the general case (that is, at $n \neq 1$ and at $n \neq 2$). To prove this property, consider, for example, the case where $\nu_3 \in [\gamma_1, \gamma_2]$ (see formulas (22)). Let the initial value $\nu_3^{(0)}$ be enclosed between the values γ_1, γ_2 . It follows from formula (35) that $\frac{d\nu_3}{dt} < 0$ for $\nu_3 = \nu_3^{(0)}$ (as before, we consider $\mu_1 > 0$). Therefore, the function $\nu_3(t)$ decreases to the value γ_1 , reaching it in a finite time interval (the integral in (35) converges as $\nu_3 \rightarrow \gamma_1$). Further, the function ν_3 will increase due to the fact that the radical $\sqrt{\nu_3 - \gamma_1}$ reverses sign. That is, $\nu_3(t)$ increases from γ_1 to γ_2 . At the point $\nu_3 = \gamma_2$, the radical $\sqrt{\gamma_2 - \nu_3}$ reverses sign, and the function $\nu_3(t)$ decreases from γ_2 to γ_1 . Then the process is repeated.

Thus, we have the general **assertion** that the function $\nu_3(t)$ obtained by inversion of the integral from (35) is periodic. Since the solution constructed in this paper is expressed in terms of this auxiliary variable without singularity, it is periodic also.

5. The Euler angles. The Rodrigues–Hamilton parameters. To find the Euler angles, we use the known formulas (see, for example, [6]) written in vector form

$$\begin{aligned} \theta &= \arccos(\boldsymbol{\nu} \cdot \boldsymbol{\mathfrak{z}}_3), & \varphi &= \operatorname{arctg} \frac{\boldsymbol{\nu} \cdot \boldsymbol{\mathfrak{z}}_1}{\boldsymbol{\nu} \cdot \boldsymbol{\mathfrak{z}}_2}, \\ \frac{d\psi}{d\tau} &= \frac{(\boldsymbol{\omega} \times \boldsymbol{\mathfrak{z}}_3) \cdot (\boldsymbol{\nu} \times \boldsymbol{\mathfrak{z}}_3)}{(\boldsymbol{\nu} \times \boldsymbol{\mathfrak{z}}_3)^2}, \end{aligned} \quad (45)$$

where $\mathfrak{a}_i(t)$ ($i = \overline{1, 3}$) are unit vectors of the moving coordinate system. Using relations (6), (10), (12) and turning in the third formula (45) to differentiation in the independent variable $\nu_3(t)$ using (13), we obtain from (45)

$$\begin{aligned}\theta(\nu_3) &= \arccos \nu_3, & \varphi(\nu_3) &= \operatorname{arctg} \frac{\mu_1 \beta_1 [n - (n+1)\nu_3^2] - \beta_2 \sqrt{F(\nu_3)}}{\mu_1 \beta_2 [n - (n+1)\nu_3^2] + \beta_1 \sqrt{F(\nu_3)}}, \\ \psi(\nu_3) &= \mu_1 \int_{\nu_3^{(0)}}^{\nu_3} \frac{\nu_3^2 d\nu_3}{(1 - \nu_3^2) \sqrt{F(\nu_3)}}.\end{aligned}\quad (46)$$

Let us write the Rodrigues–Hamilton parameters [34, 35] as functions

$$\begin{aligned}\tilde{\lambda}_0(\nu_3) &= \cos \frac{\theta(\nu_3)}{2} \cos \frac{\psi(\nu_3) + \varphi(\nu_3)}{2}, & \tilde{\lambda}_1(\nu_3) &= \sin \frac{\theta(\nu_3)}{2} \cos \frac{\psi(\nu_3) - \varphi(\nu_3)}{2}, \\ \tilde{\lambda}_2(\nu_3) &= \sin \frac{\theta(\nu_3)}{2} \sin \frac{\psi(\nu_3) - \varphi(\nu_3)}{2}, & \tilde{\lambda}_3(\nu_3) &= \cos \frac{\theta(\nu_3)}{2} \sin \frac{\psi(\nu_3) + \varphi(\nu_3)}{2}.\end{aligned}\quad (47)$$

The interest of the formula for the function $\psi(\nu_3)$ from (46) is that at $\varepsilon_0 = 0$ we can specify the explicit form of this function

$$\psi(\nu_3) = -\operatorname{arctg} \sqrt{n(n+2) - (n+1)^2 \nu_3^2} + \psi_0, \quad (48)$$

where ψ_0 is a constant, $n \in N$. It is important that this result takes place at $\varepsilon_0 = 0$ for arbitrary natural values of n . Consider, for example, the case where $n = 1$.

Substitute the value ν_3^2 from (26) into formula (48):

$$\psi(t) = -\operatorname{arctg} \frac{\sqrt{3}(e^w - 1)}{2(e^w + 1)} + \psi_0, \quad (49)$$

where w is specified in (26). If $t \rightarrow \infty$, then the angle of precession (49) tends to the value

$$\psi_0 - \operatorname{arctg} \frac{\sqrt{3}}{2}, \quad (50)$$

that is, (50) shows the limit value $\psi(\nu_3)$.

To obtain the time dependences for the angles θ and φ and for the Rodrigues–Hamilton parameters also, we substitute the function $\nu_3(t)$ obtained by the inversion of the integral from (13) into (46) and (47). We present the result for $\psi(t)$ using formulas (33) and (36):

$$\psi(\tau_1) = -\frac{\alpha_2 k_*}{2} \int_0^{\tau_1} \frac{\operatorname{cn}^2(k_*, \tau_1) \cdot d\tau_1}{1 - \alpha_2^2 \operatorname{cn}^2(k_*, \tau_1)}, \quad (51)$$

$$\psi(\tau_1) = -\frac{1}{2} \int_0^{\tau_2} \frac{\operatorname{dn}^2(k^*, \tau_2) \cdot d\tau_2}{1 - \gamma_2^2 \operatorname{dn}^2(k^*, \tau_2)}. \quad (52)$$

Statement 10. In cases (51) and (52) the time dependence of the precession angle is given by the formula $\psi(t) = g_0 t + l(t)$, where g_0 is the constant (the average value of the function $\psi(t)$ on the corresponding period, see formulas (34), (37)), and $l(t)$ is a periodic function of this period.

Proof. The proof is based on the known method of integration of periodic functions.

The formula presented in Statement 10 allows us to obtain a qualitative property of the precession angle. If, for example, $g_0 = 0$, then the function $\psi(t)$ will be a periodic function; at $g_0 \neq 0$ this formula shows the secular component in the motion of a body in a fixed space.

Remark. On the basis of the results established in Section 4, formulas (46) make it possible to obtain the properties of Euler angles and, consequently, of Rodrigues–Hamilton parameters (see formulas (47)).

6. Equations of the fixed hodograph of the angular velocity. For application of the Poincaré theorem for the kinematic interpretation of the motion of a body with a fixed point, the equations of the fixed hodograph, presented by P. V. Kharlamov [36], are in use. These equations are obtained in the cylindrical coordinate system (ρ, ζ, α) . If the dynamic equations (1)–(3) and the Poisson equations (4) are integrated, that is, the following functions are obtained

$$\boldsymbol{\omega} = \omega_1(t) \boldsymbol{\mathfrak{a}}_1 + \omega_2(t) \boldsymbol{\mathfrak{a}}_2 + \omega_3(t) \boldsymbol{\mathfrak{a}}_3, \quad (53)$$

$$\boldsymbol{\nu} = \nu_1(t) \boldsymbol{\mathfrak{a}}_1 + \nu_2(t) \boldsymbol{\mathfrak{a}}_2 + \nu_3(t) \boldsymbol{\mathfrak{a}}_3, \quad (54)$$

then the following equations [36] take place for the components of the vector $\boldsymbol{\omega}$ in a fixed coordinate system with unit vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3 = \boldsymbol{\nu}$:

$$\begin{aligned} \omega_\zeta(t) &= \omega_\rho(t) \cos \alpha(t), \quad \omega_\eta(t) = \omega_\rho(t) \sin \alpha(t), \quad \omega_\zeta(t) = \sum_{i=1}^3 \omega_i(t) \nu_i(t), \\ \omega_\rho^2(t) &= \sum_{i=1}^3 \omega_i^2(t) - \omega_\zeta^2(t), \quad \alpha(t) = \int_{\tau_0}^t \frac{1}{\omega_\rho^2(\tau)} [\dot{\boldsymbol{\omega}}(\tau) \cdot (\boldsymbol{\nu}(\tau) \times \boldsymbol{\omega}(\tau))] d\tau. \end{aligned} \quad (55)$$

A simpler formula is proved for the polar angle in [22]:

$$\operatorname{tg}(\alpha(t) - \psi(t)) = \frac{(\boldsymbol{\omega}(t) \times \boldsymbol{\nu}(t)) \cdot (\boldsymbol{\nu}(t) \times \boldsymbol{\mathfrak{a}}_3)}{\boldsymbol{\mathfrak{a}}_3 \cdot (\boldsymbol{\omega}(t) \times \boldsymbol{\nu}(t))}. \quad (56)$$

Consider formula (45) for $d\psi/dt$, then we obtain from (56)

$$\alpha(t) = \alpha_0^* + \int_{t_0}^t \frac{(\boldsymbol{\omega}(\tau) \times \boldsymbol{\mathfrak{a}}_3) \cdot (\boldsymbol{\nu}(\tau) \times \boldsymbol{\mathfrak{a}}_3)}{(\boldsymbol{\nu}(\tau) \times \boldsymbol{\mathfrak{a}}_3)^2} d\tau + \operatorname{arctg} \frac{(\boldsymbol{\omega}(t) \times \boldsymbol{\nu}(t)) \cdot (\boldsymbol{\nu}(t) \times \boldsymbol{\mathfrak{a}}_3)}{\boldsymbol{\mathfrak{a}}_3 \cdot (\boldsymbol{\omega}(t) \times \boldsymbol{\nu}(t))}. \quad (57)$$

Unlike formula (55) for $\alpha(t)$, expressions (57) for $\alpha(t)$ do not contain the derivative $\dot{\boldsymbol{\omega}}$.

Let us find equations of the fixed hodograph for the solution of (6), (10), (13). As an auxiliary function, we take ν_3 again. Substituting the values (6), (10) into the formula for $\omega_\theta, \omega_\rho^2$ of (55) and formula (57), we obtain

$$\omega_\zeta(\nu_3) = \frac{\mu_1(n+1)\nu_3^{n+1}}{n+2}, \quad (58)$$

$$\omega_\rho^2(\nu_3) = \frac{\nu_3^{2(n-1)}}{(n+2)^2} [-\mu_1^2(n+1)^2\nu_3^4 + 2\mu_1^2(n+1)(n+2)\nu_3^2 + \varepsilon_0], \quad (59)$$

$$\alpha(\nu_3) = \mu_1 \int_{\nu_3^{(0)}}^{\nu_3} \frac{\nu_3^2 d\nu_3}{(1-\nu_3^2)\sqrt{F(\nu_3)}} + \operatorname{arctg} \frac{\mu_1\nu_3[(n+1)\nu_3^2 - (n+2)]}{\sqrt{F(\nu_3)}} + \alpha_0. \quad (60)$$

Further, the following formula will be used:

$$\alpha(\nu_3) = -\mu_1 \int_{\nu_3^{(0)}}^{\nu_3} \frac{G(\nu_3) d\nu_3}{H(\nu_3) \sqrt{F(\nu_3)}}, \quad (61)$$

$$\begin{aligned} G(\nu_3) &= -\mu_1^2(n+2)(n+1)^2\nu_3^4 - 2(n+1)\varepsilon_0\nu_3^2 + (n+2)\varepsilon_0, \\ H(\nu_3) &= -\mu_1^2(n+1)^2\nu_3^4 + 2\mu_1^2(n+1)(n+2)\nu_3^2 + \varepsilon_0. \end{aligned} \quad (62)$$

Integral (61) is derived from Kharlamov's formula (the last formula of (55)). To obtain the dependences of the components of a fixed hodograph on time, it is necessary to consider the integral relation (13). Note that formula (60) is much simpler than formula (61).

Example. Consider the case of use of relations (58)–(62). Let $\varepsilon_0 = 0$. Then we find from (58)–(60)

$$\omega_\zeta(\nu_3) = \frac{\mu_1(n+1)\nu_3^{n+1}}{n+2}, \quad (63)$$

$$\omega_\rho = \mu_1 \frac{\sqrt{n+1}}{n+2} \nu_3^n \sqrt{2(n+2) - (n+1)\nu_3^2}, \quad (64)$$

$$\alpha(\nu_3) = \operatorname{arctg} \frac{(n+2)}{\sqrt{n(n+2) - (n+1)^2\nu_3^2}} + \alpha_0. \quad (65)$$

In relations (63)–(65), the function $\nu_3(t)$ satisfies the integral relation (24). It was shown earlier that, for any values $n > 1$, $n \in N$, this function has an asymptotic character ($\nu_3 \rightarrow 0$ as $t \rightarrow \infty$). This circumstance can be used in formulas (63)–(65). The following property is true.

Statement 11. For any values $n > 1$, $n \in N$, the solution (6), (10), (13) under the condition $n^2\mu_1^2 = \chi_0^2\mu_2^2(n+2)^2$ characterizes the asymptotic motion of the body to a state of rest.

Proof. Consider the functions (63)–(65) for $t \rightarrow \infty$ ($\nu_3 \rightarrow 0$). In this case, $\omega_\zeta \rightarrow 0$, $\omega_\rho \rightarrow 0$, $\alpha \rightarrow \operatorname{arctg} \sqrt{\frac{n+2}{n}}$. In (65) we can always achieve the fulfillment of the condition $\alpha_0 = 0$ by choosing the coordinate system $O\xi\eta\zeta$. A property of particular importance is that α_0 tends to a finite limit. This property is proved in [37] for many solutions of the equations of dynamics of a rigid body with a fixed point.

Remark. In the cases where the function $\nu_3(t)$ is periodic in time (these cases are described in Sec. 4), it follows from Eqs. (58)–(60) that the functions $\omega_\zeta(t)$, $\omega_\rho(t)$ are periodic in time, and, in view of Statement 10, the function $\alpha(t)$ can be characterized as follows: $\alpha(t) = G_0t + L(t)$, where G_0 is a constant depending on the average value of a certain function, and $L(t)$ is a periodic function of t . In this case, the motion of the body is often called α -conditionally periodic.

7. A modified method for interpretation of the body motion in the solution (6), (10), (13). Formulas (6), (9), (58)–(62) show that the application of the classical Poincaré method for the interpretation of motion leads to the study of higher-order surfaces. Therefore, we use the modified method [22]. According to this method, we introduce the vector $\mathbf{b}(t) = b(t)\boldsymbol{\omega}$, where $b(t) = \frac{1}{\nu_3^n}$. The function $b(t)$ has a singularity at $\nu_3 = 0$, so we consider it

for $\nu_3 \neq 0$. Then by virtue of formula (53) and the equation $\mathbf{b}(t) = \frac{1}{\nu_3^n} \boldsymbol{\omega}$ it follows from (53) that

$$b_i(t) = \frac{1}{\nu_3^n} \omega_i(t). \quad (66)$$

We consider relations (66) under the assumption that the variable ν_3 is taken as an independent variable. Write an expression for $\mathbf{b}(t)$ in vector form:

$$\mathbf{b}(t) = (\omega_1 \mathfrak{a}_1 + \omega_2 \mathfrak{a}_2 + \omega_3 \mathfrak{a}_3), \quad (67)$$

where ω_i are specified by the invariant relations (6). In the fixed coordinate system we obtain, based on (55),

$$\mathbf{b}(t) = \frac{1}{\nu_3^n} \left(\omega_\rho(\nu_3) \cos \alpha(\nu_3) \mathbf{i}_1 + \omega_\rho(\nu_3) \sin \alpha(\nu_3) \mathbf{i}_2 + \omega_\zeta(\nu_3) \mathbf{i}_3 \right). \quad (68)$$

According to the method of [22], we will represent the motion of the body by moving an axoid rolling without slipping along a fixed axoid. The moving axoid has the guide curve (67), and the fixed axoid has the guide curve (68).

Consider the case where $\varepsilon_0 = 0$. It was previously shown that the function $\nu_3(t)$, defined by integral (24), is asymptotic, that is, $\nu_3 \rightarrow 0$ as $t \rightarrow \infty$.

Statement 12. Under the condition $\varepsilon_0 = 0$, the motion of the body can be reproduced by rolling without slipping the conical surface with the circular guide along the conical surface with the circular guide.

Proof. We convert solution (6), taking account of the condition $\varepsilon_0 = 0$ in (10). Then the components $\nu_1(\nu_3)$, $\nu_2(\nu_3)$ are as follows:

$$\begin{aligned} \nu_1(\nu_3) &= \frac{1}{\chi_0} [\beta_1(n - (n+1)\nu_3^2) - \beta_2\nu_3(n+1)\sqrt{\lambda_0^2 - \nu_3^2}], \\ \nu_2(\nu_3) &= \frac{1}{\chi_0} [\beta_2(n - (n+1)\nu_3^2) + \beta_1\nu_3(n+1)\sqrt{\lambda_0^2 - \nu_3^2}]. \end{aligned} \quad (69)$$

Substitute the values (69) into the invariant relations (6):

$$\begin{aligned} \omega_1(\nu_3) &= \frac{\mu_2\nu_3^n(n+1)}{n} [\beta_1\nu_3 + \beta_2\sqrt{\lambda_0^2 - \nu_3^2}], \\ \omega_2(\nu_3) &= \frac{\mu_2\nu_3^n(n+1)}{n} [\beta_2\nu_3 - \beta_1\sqrt{\lambda_0^2 - \nu_3^2}], \\ \omega_3(\nu_3) &= \mu_1\nu_3^n. \end{aligned} \quad (70)$$

As long as $b(\nu_3) = \frac{1}{\nu_3^n}$, we obtain from (70), taking (67) into account

$$\begin{aligned} b_1(\nu_3) &= \frac{\mu_2(n+1)}{n} [\beta_1\nu_3 + \beta_2\sqrt{\lambda_0^2 - \nu_3^2}], \\ b_2(\nu_3) &= \frac{\mu_2(n+1)}{n} [\beta_2\nu_3 - \beta_1\sqrt{\lambda_0^2 - \nu_3^2}], \\ b_3(\nu_3) &= \mu_1. \end{aligned} \quad (71)$$

It follows from relations (71) that the moving hodograph of the vector $\mathbf{b}(\nu_3)$ is a plane curve located in the plane $b_3(\nu_3) = \mu_1$. Excluding the variable ν_3 from the first two equations of (71), we obtain

$$b_1^2 + b_2^2 = \frac{n\mu_1^2}{n+2}. \quad (72)$$

Thus, by virtue of (72), the first part of Statement 12 is proved. Obviously, in order to bring Eq. (72) to the canonical form, it is necessary to rotate the coordinate system $O_*b_1b_2$ by an angle $\arctg \frac{\beta_2}{\beta_1}$.

Now we study the fixed hodograph of the vector $\mathbf{b}(t)$ using formulas (63)–(65), (68)

$$\mathbf{b}(t) = \frac{\mu_1(n+1)}{n+2} \sqrt{\lambda_0^2 - \nu_3^2} \mathbf{i}_1 + \mu_1 \mathbf{i}_2 + \frac{\mu_1(n+1)}{n+2} \nu_3 \mathbf{i}_3. \quad (73)$$

The fixed hodograph $\mathbf{b}(t)$ from (73) is also a plane curve lying in the plane $b_\eta = \mu_1$. Consider the expressions for b_ξ , b_ζ that follow from (73):

$$b_\xi = \frac{\mu_1(n+1)}{n+2} \sqrt{\lambda_0^2 - \nu_3^2}, \quad b_\zeta = \frac{\mu_1(n+1)\nu_3}{n+2}. \quad (74)$$

Excluding the variable ν_3 in these equations, we obtain

$$b_\xi^2 + b_\zeta^2 = \frac{\mu_1^2 n}{n+2}. \quad (75)$$

Thus, the fixed hodograph $\mathbf{b}(t)$ is a circle (75) lying in the plane $b_\zeta = \mu_1$. The change of the variable ν_3 is characterized by the integral relation (24). Taking an initial value $\nu_3^{(0)}$ that satisfies the condition $0 < \nu_3^{(0)} < \lambda_0$ and using the previously obtained properties of conical surfaces, which are defined by guide curves (71) and (73), it is possible to get a picture of the motion of the body.

Remark. The result of the kinematic interpretation of the motion of the body is very simple (rolling of two plane curves without sliding), and it is valid for any values of n , which characterize the invariant relations (6). If we apply the classical Poinsoth method, then, since the guide curves of the moving and fixed axoids depend on n , it is impossible to develop a general study for any properties of these curves.

References

- [1] Zhukovskiy, N. E., On Importance of the Geometrical Interpretation in the Theoretical Mechanics, in *Collected Works: Vol. 7*, Moscow: Gostekhizdat, 1950, pp. 9–15 (Russian).
- [2] Kharlamov, P. V., New Methods in the Dynamics of Systems of Rigid Bodies, in *Dynamics of Multi-body Systems: International Union of Theoretical and Applied Mechanics Proc.*, K. Magnus (Ed.), Berlin: Springer, 1978, pp. 133–143.
- [3] Leimanis, E., *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*, Springer Tracts in Natural Philosophy, vol. 7, Berlin: Springer, 1965.
- [4] Gorr, G. V., Kudryashova, L. V., and Stepanova, L. A., *Classical Problems in the Theory of Solid Bodies, Their Development and Current State*, Kiev: Naukova Dumka, 1978 (Russian).
- [5] Borisov, A. V. and Mamaev, I. S., *Rigid Body Dynamics: Hamiltonian Methods, Integrability, Chaos*, Izhevsk: R&C Dynamics, Institute of Computer Science, 2005 (Russian).

- [6] Gashenenko, I. N., Gorr, G. V., and Kovalev, A. M., *Classical Problems of Rigid Body Dynamics*, Kiev: Naukova Dumka, 2012. (Russian).
- [7] Kharlamov, P. V., Current State and Development Prospects of Classical Problems of Rigid Body Dynamics, *Mekh. Tverd. Tela*, 2000, vol. 30, pp. 1–12 (Russian).
- [8] Kovalev, A. M. and Gorr, G. V., Donetsk School of Mechanics and Stability, *PNAES*, 2014, vol. 20, no. 2(42), pp. 96–126.
- [9] Ziglin, S. L., Branching of Solutions and Nonexistence of First Integrals in Hamiltonian Mechanics: 1, *Funct. Anal. Appl.*, 1982, vol. 16, no. 3, pp. 181–189; see also: *Funktsional. Anal. i Prilozhen.*, 1982, vol. 16, no. 3, pp. 30–41.
Ziglin, S. L., Branching of Solutions and Nonexistence of First Integrals in Hamiltonian Mechanics: 2, *Funct. Anal. Appl.*, 1983, vol. 17, no. 1, pp. 6–17; see also: *Funktsional. Anal. i Prilozhen.*, 1983, vol. 17, no. 1, pp. 8–23.
- [10] Ziglin, S. L. Splitting of the Separatrices and the Nonexistence of First Integrals in Systems of Differential Equations of Hamiltonian Type with Two Degrees of Freedom, *Math. USSR-Izv.*, 1988, vol. 31, no. 2, pp. 407–421; see also: *Izv. Akad. Nauk SSSR Ser. Mat.*, 1987, vol. 51, no. 5, pp. 1088–1103, 1118–1119.
- [11] Kozlov V. V., Onishchenko D. A. Nonintegrability of Kirchhoff's Equations, *Sov. Math. Dokl.*, 1982, vol. 26, pp. 495–498; see also: *Dokl. Akad. Nauk SSSR*, 1982, vol. 266, no. 6, pp. 1298–1300.
- [12] Borisov, A. V., Necessary and Sufficient Conditions for Integrability of Kirchhoff Equations, *Regul. Chaotic Dyn.*, 1996, vol. 1, no. 2, pp. 61–76 (Russian).
- [13] Burov, A. A., Nonintegrability of the Equations of Gyrostat Motions in Cardan Suspension, in *Research Problems of Stability and Stabilisation of Motion*, Moscow: Computing Centre of the USSR Acad. Sci., 1986, pp. 3–10 (Russian).
- [14] Bogoyavlenskii, O. I., Integrable Problems of the Dynamics of Coupled Rigid Bodies, *Russian Acad. Sci. Izv. Math.*, 1993, vol. 41, no. 3, pp. 395–416; see also: *Izv. Ross. Akad. Nauk Ser. Mat.*, 1992, vol. 56, no. 6, pp. 1139–1164.
- [15] Gorr, G. V., *Invariant Relations of Equations of Rigid Body Dynamics: Theory, Results, Comments*, Izhevsk: R&C Dynamics, Institute of Computer Science, 2017 (Russian).
- [16] Poincaré, H., *Les méthodes nouvelles de la mécanique céleste: In 3 Vols.*, Paris: Gauthier-Villars, 1892, 1893, 1899. (New York: Dover, 1957; reprint.)
- [17] Levi-Civita, T. and Amaldi, U., *Lezioni di meccanica razionale: Vol. 1. Cinematica — principi e statica*, Bologna: Zanichelli, 1950.
Levi-Civita, T. and Amaldi, U., *Lezioni di meccanica razionale: Vol. 2. Dinamica dei sistemi con un numero finito di gradi di liberta*, Bologna: Zanichelli, 1951, 1952.
- [18] Chaplygin, S. A., On the Last Multiplier Principle, *Mat. Sb.*, 1900, vol. 21, no. 3, pp. 479–489 (Russian).
- [19] Kharlamov, P. V., On Invariant Relations of a System of Ordinary Differential Equations, *Mekh. Tverd. Tela*, 1974, vol. 6, pp. 15–24 (Russian).
- [20] Poincot, L., Théorie nouvelle de la rotation des corps, *J. Math. Pures Appl.*, 1851, vol. 16, pp. 9–129, 289–336.
- [21] Routh, E. J., *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies: Being Part II of a Treatise on the Whole Subject*, 6th ed., New York: Dover, 1955.
- [22] Gorr, G. V., On an Approach in the Use of the Poincot Theorem of kinematic Interpretation of the Body Motion with Fixed Point, *Mekh. Tverd. Tela*, 2012, vol. 42, pp. 26–36 (Russian).
- [23] Yehia, H. M., Transformations of Mechanical Systems with Cyclic Coordinates and New Integrable Problems, *J. Phys. A*, 2001, vol. 34, no. 49, pp. 11167–11183.
- [24] Yehia, H. M., New Solvable Problems in the Dynamics of a Rigid Body about a Fixed Point in a Potential Field, *Mech. Res. Commun.*, 2014, vol. 57, pp. 44–48.
- [25] Ol'shanskiy, V. Yu., Linear Invariant Relations of Kirchhoff's Equations, *J. Appl. Math. Mech.*, 2015, vol. 79, no. 4, pp. 334–349; see also: *Prikl. Mat. Mekh.*, 2015, vol. 79, no. 4, pp. 476–497.

- [26] Ol'shanskiy, V. Yu., A New Linear Invariant Relation of the Poincaré–Zhukovskii Equations, *J. Appl. Math. Mech.*, 2012, vol. 76, no. 6, pp. 636–645; see also: *Prikl. Mat. Mekh.*, 2012, vol. 76, no. 6, pp. 883–894.
- [27] Mukharlyamov, R. G., Stabilization of the Motions of Mechanical Systems in Prescribed Phase-Space Manifolds, *J. Appl. Math. Mech.*, 2006, vol. 70, no. 2, pp. 210–222; see also: *Prikl. Mat. Mekh.*, 2006, vol. 70, no. 2, pp. 236–249.
- [28] Mukharlyamov, R. G., Differential-Algebraic Equations of Programmed Motions of Lagrangian Dynamical Systems, *Mech. Solids*, 2011, vol. 46, no. 4, pp. 534–543; see also: *Izv. Akad. Nauk. Mekh. Tverd. Tela*, 2011, no. 4, pp. 50–61.
- [29] Goryachev, D. N., *Certain General Integrals in the Problem on the Motion of a Rigid Body*, Warsaw, 1910 (Russian).
- [30] Goryachev, D. N., New Cases of Motion of a Rigid Body around a Fixed Point, *Warshav. Univ. Izv.*, 1915, vol. 3, pp. 3–14 (Russian).
- [31] Komarov, I. V. and Kuznetsov, V. B., Generalization of the Goryachev–Chaplygin Gyrostat in Quantum Mechanics, in *Zap. Nauchn. Sem. LOMI*, 1987, vol. 164, pp. 134–141 (Russian).
- [32] Komarov, I. V. and Kuznetsov, V. B., Semiclassical Quantization of Kowalewski Top, *Theoret. and Math. Phys.*, 1987, vol. 73, no. 3, pp. 1255–1263; see also: *Teoret. Mat. Fiz.*, 1987, vol. 73, no. 3, pp. 335–347.
- [33] Gorr, G. V. and Maznev, A. V., Integration of the Rigid Body Dynamics on the Invariant Manifold, *Mekh. Tverd. Tela*, 2016, vol. 46, pp. 25–36 (Russian).
- [34] Koshlyakov, V. N., *Rodrigues – Hamilton Parameters and Their Applications in Rigid Body Mechanics*, Kiev: Institute of Mathematics NAS of Ukraine, 1994 (Russian).
- [35] Lurie, A. I., *Analytical Mechanics*, Berlin: Springer, 2002.
- [36] Kharlamov, P. V., Kinematic Interpretation of the Motion of a Body with a Fixed Point, *J. Appl. Math. Mech.*, 1964, vol. 28, no. 3, pp. 615–621; see also: *Prikl. Mat. Mekh.*, 1964, vol. 28, no. 3, pp. 502–507.
- [37] Gorr, G. V. and Nosyreva, E. P., Behavior of the Immovable Hodograph for Asymptotically Permanent Motions of the Gyrostat in the Generalized Problem of Dynamics, *Mekh. Tverd. Tela*, 1994, vol. 26, pp. 13–20 (Russian).