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On Synchronization of Quasiperiodic Oscillations

A. D. Morozov, K. E. Morozov

We study the role of quasi-periodic perturbations in systems close to two-dimensional Hamiltonian ones. Similarly to the problem of the influence of periodic perturbations on a limit cycle, we consider the problem of the passage of an invariant torus through a resonance zone. The conditions for synchronization of quasi-periodic oscillations are established. We illustrate our results using the Duffing–van der Pol equation as an example.

Keywords: resonances, quasi-periodic, periodic, synchronization, averaged system, phase curves, equilibrium states

1. Problem statement

In 1930 A. Andronov and A. Witt investigated the problem of oscillation synchronization using the quasi-linear van der Pol equation as an example [1]. In other words, they found what is happening to a limit cycle when the system undergoes a periodic (harmonic) perturbation. Then a more general problem of synchronization in systems close to two-dimensional nonlinear Hamiltonian ones was considered [2, 3]. It was established that, in contrast to the quasi-linear case, when the synchronization interval with respect to a parameter is small along with the perturbation, in the nonlinear case it is of the order $O(1)$. Recently, physicists found the effect of synchronization of *quasi-periodic* oscillations in the case of more sophisticated self-oscillating systems (see, for example, [4]).

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Albert D. Morozov

morozov@mm.unn.ru

Kirill E. Morozov

kirwamath@gmail.com

Lobachevsky State University of Nizhny Novgorod

prosp. Gagarina 23, Nizhny Novgorod, 603950, Russia

In the present paper we shall focus on the problem of synchronization of quasi-periodic oscillations in nonlinear systems close to Hamiltonian ones. We shall follow the paper [5]. So, let us consider the system

$$\begin{aligned}\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \varepsilon g(x, y, t), \\ \dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \varepsilon f(x, y, t),\end{aligned}\tag{1.1}$$

where ε is a small positive parameter, H is the Hamiltonian function and g, f are sufficiently smooth (analytic) and uniformly bounded in some domain $G \subset R^2$ or $G \subset R^1 \times S^1$ along with partial derivatives up to the second order. Besides, functions g, f are continuous and quasi-periodic (as functions of time) uniformly with respect to $(x, y) \in G$ with incommensurable frequencies $\omega_1, \omega_2, \dots, \omega_m$.

Let us assume that the unperturbed system is nonlinear and has a cell $D \subset G$ filled with closed phase curves $H(x, y) = h$, $h \in [h_-, h_+]$. We also suppose that D does not contain any equilibria and separatrices. So D is a connected domain homeomorphic to a ring.

In D we change the variables from x, y to the action I -angle θ variables. Then the system (1.1) takes the form

$$\begin{aligned}\dot{I} &= \varepsilon F_1(I, \theta, \theta_1, \dots, \theta_m), \\ \dot{\theta} &= \omega(I) + \varepsilon F_2(I, \theta, \theta_1, \dots, \theta_m), \\ \dot{\theta}_k &= \omega_k, \quad k = 1, 2, \dots, m,\end{aligned}\tag{1.2}$$

where

$$\begin{aligned}F_1 &= f(x(I, \theta), y(I, \theta), \theta_1, \dots, \theta_n) x'_\theta - g(x(I, \theta), y(I, \theta), \theta_1, \dots, \theta_n) y'_\theta, \\ F_2 &= -f(x(I, \theta), y(I, \theta), \theta_1, \dots, \theta_n) x'_I + g(x(I, \theta), y(I, \theta), \theta_1, \dots, \theta_n) y'_I,\end{aligned}\tag{1.3}$$

the functions $x(I, \theta)$, $y(I, \theta)$ define the change of variables.

System (1.2) is defined on the direct product of the interval $K \times T^{m+1}$, where $K = [I_-(h_-), I_+(h_+)]$ and T^{m+1} is an $(m+1)$ -dimensional torus, where h_- , h_+ determine the boundaries of D . Here $\omega(I)$ is a natural frequency of the unperturbed system. We shall suppose $\omega(I)$ to be monotonic over the interval $[I_-, I_+]$.

Along with the system (1.1), let us consider the autonomous system

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial y} + \varepsilon \bar{g}(x, y) \\ \dot{y} &= -\frac{\partial H}{\partial x} + \varepsilon \bar{f}(x, y),\end{aligned}\tag{1.4}$$

where

$$\begin{aligned}\bar{g}(x, y) &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} g(x, y, \theta_1, \dots, \theta_m) d\theta_1 \dots d\theta_m, \\ \bar{f}(x, y) &= \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} f(x, y, \theta_1, \dots, \theta_m) d\theta_1 \dots d\theta_m.\end{aligned}$$

Suppose that this system has a rough limit cycle. It implies that the generating Poincaré–Pontryagin function $B(h)$ has a simple zero $h = h_0$ [3].

In [5] the solutions behavior in neighborhoods of individual levels of energy (both resonance and nonresonance ones) was established. Here we shall study the solutions' behavior in a neighborhood of the level $h = h_0$ (such that $B(h_0) = 0$) when it is close to the resonance one. Actually, we consider the synchronization problem in the case of quasi-periodic perturbations.

As an example we take the following system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x - x^3 + \varepsilon[(1 - p_1 x^2)y + p_2 F(t)],\end{aligned}\tag{1.5}$$

which is equivalent to the Duffing–van der Pol equation. Here $p_1 > 0$, p_2 are parameters, $F(t) = \sin \omega_1 t \sin \omega_2 t$, $\omega_1 = 1$, $\omega_2 = \sqrt{5}$.

2. Behavior of solutions in neighborhoods of resonance levels

For $\varepsilon = 0$, the $(m + 2)$ -dimensional phase space of system (1.2) is foliated by $(m + 1)$ -dimensional tori T_{m+1} , the motion on which is conditionally periodic with frequencies $\omega_1, \dots, \omega_m$. For $\varepsilon \neq 0$, the invariant tori are destroyed owing to the nonconservativity of the perturbation and/or the presence of an integer combination of the frequencies $\omega, \omega_1, \dots, \omega_m$:

$$n\omega(I) - (\mathbf{k}, \Omega) = 0, \quad \mathbf{k} = (k_1, \dots, k_m), \quad \Omega = (\omega_1, \dots, \omega_m).\tag{2.1}$$

For given Ω and fixed $\mathbf{k}, n$, relation (2.1) can be viewed as an equation for I . If this equation has a real solution $I = I_{n\mathbf{k}}$ on the interval $[I_-, I_+]$, then the level $I = I_{n\mathbf{k}}$ (the closed phase curve $H(x, y) = h_{n\mathbf{k}}$ of the unperturbed system) will be called a resonance level.

Let us proceed to the study of the behavior of solutions in the neighborhoods $U_\mu = \{(I, \theta) : I_{n\mathbf{k}} - C\mu < I < I_{n\mathbf{k}} + C\mu, 0 \leq \theta < 2\pi, C = \text{const} > 0, \mu = \sqrt{\varepsilon}\}$ of the individual resonance levels $I = I_{n\mathbf{k}}$. In system (1.2), we make the change of variables

$$\theta = \psi + \sum_{j=1}^m k_j \theta_j / n, \quad I = I_{n\mathbf{k}} + \mu W, \quad \mu = \sqrt{\varepsilon}.\tag{2.2}$$

As a result, system (1.2) becomes

$$\begin{aligned}\dot{W} &= \mu F_1 \left(I_{n\mathbf{k}}, \psi + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) + \\ &\quad + \mu^2 \left[\partial F_1 \left(I_{n\mathbf{k}}, \psi + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) / \partial I \right] W + O(\mu^3), \\ \dot{\psi} &= \mu b_1 W + \mu^2 F_2 \left(I_{n\mathbf{k}}, \psi + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) + O(\mu^3), \\ \dot{\theta}_j &= \omega_j, \quad j = 1, \dots, m,\end{aligned}\tag{2.3}$$

where the functions $F_1, F_2, \partial F_1 / \partial I$ are $2\pi n$ -periodic in $\theta_1, \theta_2, \dots, \theta_m$. Note that since the function $\omega(I)$ is monotone, one has $b_1 = \omega'(I_{n\mathbf{k}}) \neq 0$ (the nondegeneracy condition for the resonance). The variables in system (2.3) are divided into the slow variables W, ψ and the fast variables $\theta_1, \dots, \theta_m$. One can represent this system in the standard form of the averaging method

and apply the Bogolyubov theorem ([6, p. 379]; see also [7, p. 210]). Indeed, by substituting $\theta_j = \omega_j t$ into the first two equations in system (2.3), we obtain

$$\dot{\eta} = \mu F(\eta, \omega_1 t, \dots, \omega_m t; \mu),$$

where $\eta = (W, \psi)$, $F = (F_1 + \mu(\partial F_1 \partial I)W + O(\mu^2), b_1 W + \mu F_2 + O(\mu^2))$. Then the averaged system acquires the form $\dot{\xi} = \mu F_0(\xi; \mu)$, where $\xi = (u, v)$, $u = W + O(\mu)$, $v = \psi + O(\mu)$,

$$F_0 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F(\xi, \omega_1 t, \dots, \omega_m t; \mu) dt = \frac{1}{(2\pi n)^m} \int_0^{2\pi n} \dots \int_0^{2\pi n} F(\xi, \theta_1, \dots, \theta_m; \mu) d\theta_1 \dots d\theta_m$$

owing to the incommensurability of the frequencies.

The two-dimensional averaged system becomes

$$\begin{aligned} u' &= A(v, I_{n\mathbf{k}}) + \mu P_0(v, I_{n\mathbf{k}})u, \\ v' &= b_1 u + \mu(b_2 u^2 + Q_0(v, I_{n\mathbf{k}})), \end{aligned} \quad (2.4)$$

where the prime stands for the derivative with respect to the slow time $\tau = \mu t$ and

$$\begin{aligned} A &= \frac{1}{(2\pi n)^m} \int_0^{2\pi n} \dots \int_0^{2\pi n} F_1 \left(I_{n\mathbf{k}}, v + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) d\theta_1 \dots d\theta_m, \\ P_0 &= \frac{1}{(2\pi n)^m} \int_0^{2\pi n} \dots \int_0^{2\pi n} \left[\partial F_1 \left(I_{n\mathbf{k}}, v + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) \partial I \right] d\theta_1 \dots d\theta_m, \\ Q_0 &= \frac{1}{(2\pi n)^m} \int_0^{2\pi n} \dots \int_0^{2\pi n} F_2 \left(I_{n\mathbf{k}}, v + \left(\sum_{j=1}^m k_j \theta_j \right) / n, \theta_1, \dots, \theta_m \right) d\theta_1 \dots d\theta_m, \\ b_1 &= d\omega(I_{n\mathbf{k}})/dI, \quad b_2 = (d^2\omega(I_{n\mathbf{k}})/dI^2)/2. \end{aligned} \quad (2.5)$$

By analogy with the case of periodic perturbations, system (2.4) can be reduced modulo $O(\mu^2)$ terms to the form (see [3, 9])

$$\begin{aligned} u' &= A(v, I_{n\mathbf{k}}) + \mu\sigma u, \\ v' &= b_1 u + \mu b_2 u^2, \end{aligned} \quad (2.6)$$

where

$$\sigma = P_0 + \frac{dQ_0}{dv} = \frac{1}{(2\pi n)^m} \int_0^{2\pi n} \dots \int_0^{2\pi n} (g'_x + f'_y) d\theta_1 \dots d\theta_m. \quad (2.7)$$

Here $x = x(I_{n\mathbf{k}}, v + (\sum_{j=1}^m k_j \theta_j)/n)$, $y = y(I_{n\mathbf{k}}, v + (\sum_{j=1}^m k_j \theta_j)/n)$. The functions $A(v, I_{n\mathbf{k}})$ and $\sigma(v, I_{n\mathbf{k}})$ are sufficiently smooth and periodic with least period $2\pi/n$ [8, p. 158]. The phase space of system (2.6) is the cylinder $\{u, v \bmod (2\pi/n)\}$.

Associated with a simple stable equilibrium of the averaged system (2.6) is a stable quasi-periodic resonance solution with frequencies $k_1\omega_1/n, \dots, k_m\omega_m/n$ [6].

Let us represent $A(v, I_{nk})$ in the form¹

$$A(v, I_{nk}) = \tilde{A}(v, I_{nk}) + B(I_{nk}),$$

where $B(I_{nk}) = \frac{n}{2\pi} \int_0^{2\pi/n} A(v, I_{nk}) dv$ is the mean of $A(v, I_{nk})$. Suppose that the level $I = I_{nk}$ is an impassable one and it is close to the level $I = I_0$, in the neighborhood of which the system (1.4) has the rough limit cycle.

3. Passage of the $(m + 1)$ -dimensional torus through the resonance zone

Let us introduce in (2.6) the “detuning” γ which measures the deviation of the resonance level $I = I_{nk}$ from the level I_0 . Following [8] and [9], we observe the transition from the exact resonance to the nonresonance case, as we change γ . In order to do that, we consider the system on a cylinder

$$\begin{aligned} \frac{du}{d\tau} &= \tilde{A}(v, I_{nk}) + \mu(\sigma(v, I_{nk})u + \gamma), \\ \frac{dv}{d\tau} &= bu + \mu b_1 u^2, \end{aligned} \quad (3.1)$$

obtained from (2.6) by letting

$$B(I_{nk}) = (dB_0(I_0)/dI)(I_{nk} - I_0) = \mu\gamma.$$

Along with (2.6) consider the conservative system

$$\begin{aligned} \frac{du}{d\tau} &= \tilde{A}(v, I_{nk}) \\ \frac{dv}{d\tau} &= bu. \end{aligned} \quad (3.2)$$

We call the separatrices of the saddles of (3.1) embracing the phase cylinder $(v \bmod 2\pi, u)$ *external separatrices*.

Let us describe the relative positions of the external separatrices when the terms $O(\mu)$ are taken into account. We make use of the formula $\Delta_\mu = \mu\Delta_1 + O(\mu^2)$ for the distance Δ_μ between the corresponding separatrices of (2.6) [10]. Let us consider (3.2) as the unperturbed system and substitute in (3.1)

$$v = \xi + v_0 - \mu\gamma/\tilde{A}'(v_0), \quad u = \eta, \quad (3.3)$$

where v_0 is the coordinate of the saddle of (3.2).

As a result, (3.1), up to the terms $O(\mu^2)$, assumes the form

$$\left. \begin{aligned} \frac{d\eta}{d\tau} &= \tilde{A}(\xi + v_0) + \mu[\gamma + \sigma(\xi + v_0)\eta - \gamma\tilde{A}'(\xi + v_0)/\tilde{A}'(v_0)] \\ \frac{d\xi}{d\tau} &= b\eta + \mu b_1 \eta^2. \end{aligned} \right\} \quad (3.4)$$

¹An example of computation of the function $A(v, I_{nk})$ is given below in Section 4.

Evidently, the right-hand sides of (3.4) vanish when $\xi = \eta = 0$. Therefore, according to [10], we have

$$\Delta_1 = - \int_{-\infty}^{\infty} b_1 \eta^2 \frac{d\eta}{d\tau} d\tau + \int_{-\infty}^{\infty} \left[\sigma(\xi + v_0) \eta + \gamma - \gamma \frac{\tilde{A}'(\xi + v_0)}{\tilde{A}'(v_0)} \right] \frac{d\xi}{d\tau} d\tau. \quad (3.5)$$

Here $(\xi(\tau), \eta(\tau))$ is the solution of (3.2) on the separatrix. From the integral of (3.2) we obtain the following relation between η and ξ :

$$\eta = \pm \left[\frac{2}{b} (V(\xi, v_0) - V(0, v_0)) \right]^{1/2}, \quad V(\xi, v_0) = \int \tilde{A}(\xi + v_0) d\xi. \quad (3.6)$$

Substituting (3.6) in (3.5), we find

$$\Delta_1^{\pm} = \frac{2\pi\gamma}{n} \pm \int_0^{2\pi/p} \sigma(\xi + v_0) \left[\frac{2}{b} (V(\xi, v_0) - V(0, v_0)) \right]^{1/2} d\xi. \quad (3.7)$$

According to [8, 9], the following propositions hold.

Proposition 1. *For the exact resonance (i.e., for $\gamma = 0$) when the function $\sigma(v)$ has constant sign the external separatrices are split, i.e. $\Delta_{\mu} \neq 0$.*

Proposition 2. *When function $\sigma(v)$ has constant sign, system (3.1) with $\gamma = 0$ has no limit cycles.*

When $\sigma(v)$ has constant sign, from the conditions $\Delta_1^{\pm} = 0$ we find the (bifurcational) values of the “detuning” $\gamma = \gamma^{\pm}$ corresponding to the appearance in (3.1) of a separatrix going from a saddle to a saddle. The plus in $\gamma^{\pm}$ corresponds to the domain $\eta > 0$ and the minus — to $\eta < 0$. As γ goes past $\gamma^{\pm}$, a noncontractible limit cycle is born in (3.1).

Theorem 1. *Let $\sigma(v, I_{n\mathbf{k}})$ have constant sign, and assume, for concreteness, that $\varepsilon\sigma < 0$. Then there exists $\gamma_{n\mathbf{k}} = |\gamma^{\pm}| + O(\mu)$ such that:*

- 1) *for $\gamma > \gamma_{n\mathbf{k}}$ Eq. (3.1) has a noncontractible stable limit cycle;*
- 2) *for $\gamma = \gamma_{n\mathbf{k}}$ the limit cycle “sticks” to the separatrix contour Γ_n^+ formed by n saddles and external separatrices connecting one saddle to another, while the other unstable separatrices of the saddles approach stable focuses as $t \rightarrow \infty$;*
- 3) *for $-\gamma_{n\mathbf{k}} < \gamma < \gamma_{n\mathbf{k}}$ there are no noncontractible limit cycles;*
- 4) *for $\gamma = -\gamma_{n\mathbf{k}}$ there appears a contour Γ_n^- , formed by n saddles and their external separatrices, whose direction and position differ from those of Γ_n^+ ;*
- 5) *for $\gamma < -\gamma_{n\mathbf{k}}$ there is a stable noncontractible limit cycle.*

Figure 1 illustrates this theorem.

In fact, the theorem describes the “passage” of the $(m+1)$ -dimensional torus through a resonance zone. Indeed, the limit cycle of (3.1), embracing the phase cylinder, corresponds to the $(m+1)$ -dimensional torus in the initial (1.1) as well as the simple equilibrium corresponds to the m -dimensional torus (there is a quasi-periodic solution with the frequencies $k_1\omega_1/n, k_2\omega_2/n, \dots, k_m\omega_m/n$) [5, 6].

Fig. 1. Phase portraits of (3.1) under variation of γ .

4. Example

Let us study system (1.5). The unperturbed system ($\varepsilon = 0$) admits the following integral of energy $H(x, y) = y^2/2 + x^2/2 + x^4 = h$, $h > 0$. Denote the annular region of the phase plane $0 < h_- \leq h \leq h_+ < \infty$ by D .

The unperturbed system is equivalent to the second-order equation that has the following solution:

$$x(\theta, I) = x_1 \operatorname{cn}(2\mathbf{K}\theta/\pi), \quad \theta = \omega t, \quad (4.1)$$

where $\omega = \pi(1 + 4h)^{1/4}/(2\mathbf{K})$ is the frequency of motion on closed phase curves $H = h(I)$, x_1 is a positive solution of the equation $x^2/2 + x^4/4 = h$, $\operatorname{cn}(u)$ is the Jacobi elliptic function, $\mathbf{K}$ is the complete elliptic integral of first kind, and $k = k(h)$ is its modulus.

4.1. Autonomous system

First let us consider the autonomous system

$$\dot{x} = y, \dot{y} = -x - x^3 + \varepsilon[(1 - p_1 x^2)y]. \quad (4.2)$$

The Poincaré–Pontryagin generating function has the form (up to a positive factor) [9]

$$B(\rho) = [(1 - \rho)(1 - 2\rho)\mathbf{K} - (1 - 2\rho)^2\mathbf{E}] + \frac{2p_1}{5}[(\rho - 1)(2 - \rho)\mathbf{K} + 2(\rho^2 - \rho + 1)\mathbf{E}], \quad (4.3)$$

where $\mathbf{E}$ is the complete elliptic integral of the second kind, $\rho = k^2 = \frac{-1 + \sqrt{1+4h}}{2\sqrt{1+4h}}$. For $p_1 > 0$ it has a unique zero (denote it by $\rho_* \in (0, 1/2)$). As is well known [9], the simple zeros of the generating function determine the energy levels, in the neighborhood of which system (4.2) has a rough limit cycle for small enough $\varepsilon > 0$. It is not hard to see that $\rho_* \rightarrow 0$ as $p_1 \rightarrow \infty$ and $\rho_* \rightarrow 1/2$ as $p_1 \rightarrow 0+$. Since $\sigma = B'(\rho_*) < 0$, the cycle is stable, where

$$\sigma = 1 - p_1 \frac{2}{(1-2\rho)\mathbf{K}} [\mathbf{E} - (1-\rho)\mathbf{K}]. \quad (4.4)$$

From the condition $B(\rho, p_1) = 0$ we find

$$p_1(\rho) = \frac{5}{2} \frac{(1-\rho)(1-2\rho)\mathbf{K} - (1-2\rho^2)\mathbf{E}}{(\rho-1)(2-\rho)\mathbf{K} + 2(\rho^2 - \rho + 1)\mathbf{E}}. \quad (4.5)$$

Figure 2 shows the fragments of the dependence $p_1(\rho)$ in neighborhoods of resonances with $n = 1$ (a) and $n = 3$ (b).

Fig. 2. Fragments of the dependence $p_1(\rho)$.

4.2. Resonance zones and synchronization

Let us consider the resonance level $I = I_{n,k_1k_2}$ determined by the condition $n\omega(h(I)) = k_1\omega_1 + k_2\omega_2$.

The pendulum type equation describing the topology of the resonance zone has the form

$$\frac{d^2v}{d\tau^2} - A(v) = \mu\sigma \frac{dv}{d\tau}. \quad (4.6)$$

Just as before, we represent the function $A(v)$ in the form

$$A(v; I_{nk_1k_2}) = B(I_{nk_1k_2}) + p_2 S(v; I_{nk_1k_2}), \quad (4.7)$$

where

$$S(v; I_{nk_1k_2}) = \frac{1}{4\pi^2 n^2} \int_0^{2\pi n} \int_0^{2\pi n} x'_\theta(\theta, I_{nk_1k_2}) \sin \theta_1 \sin \theta_2 d\theta_1 d\theta_2.$$

Fig. 3. Phase portraits of (4.6) in the (x, y) -plane in the neighborhood of the level h_{311} : (a), (b), (c) and h_{111} : (d), (e).

Computing the function $S(v)$, we obtain the following expressions:

$$A(v; I_{nk_1k_2}) = \begin{cases} p_2\sqrt{2}(\omega_1 + \omega_2)\frac{a^{n/2}}{1+a^n}\sin nv + B(I_{nk_1k_2}); & \text{if } n \text{ is odd, } k_1 = k_2 = 1, \\ B(I_{nk_1k_2}); & \text{if } n \text{ is even or } k_1 \neq 1, k_2 \neq 1; \end{cases} \quad (4.8)$$

where $a = \exp\left(-\frac{\pi\mathbf{K}(\sqrt{1-k^2})}{\mathbf{K}(k)}\right)$.

From the conditions $\omega = (1 + \sqrt{5})/n$ and $\omega > 1$ it follows that there can be only two splittable resonance levels ($n = 1, n = 3$).

Figure 3 shows the phase portraits of Eq. (4.5) in the (x, y) -plane under variation of p_1 in the neighborhood of resonance levels of energy.

The intervals of synchronization of quasi-periodic oscillations with respect to p_1 (see on Fig. 2) could be obtained similarly to the case of periodic perturbations [8, pp. 184–187].

Rough equilibria of (4.6) correspond to the resonance quasi-periodic solutions with frequencies $\omega_1/n, \omega_2/n, n = 1, n = 3$ of the initial system (1.5). Such a solution for $p_1 = 18, p_2 = 1, \varepsilon = 0.05$ and $n = 3$ is presented in Fig. 4a. The projection of this solution onto the (x, y) -plane is shown in Fig. 4b.

Fig. 4

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