NONLINEAR ENGINEERING AND ROBOTICS

MSC 2010: 37A30, 37C27

A Two-Contour System with Two Clusters of Different Lengths

M. V. Yashina, A. G. Tatashev

A system belonging to the class of dynamical systems such as Buslaev contour networks is investigated. On each of the two closed contours of the system there is a segment, called a *cluster*, which moves with constant velocity if there are no delays. The contours have two common points called *nodes*. Delays in the motion of the clusters are due to the fact that two clusters cannot pass through a node simultaneously. The main characteristic we focus on is the average velocity of the clusters with delays taken into account. The contours have the same length, taken to be unity. The nodes divide each contour into parts one of which has length d , and the other, length $1 - d$. Previously, this system was investigated under the assumption that the clusters have the same length. It turned out that the behavior of the system depends qualitatively on how the directions of motion of the clusters correlate with each other. In this paper we explore the behavior of the system in the case where the clusters differ in length.

Keywords: dynamical systems, Buslaev's contour network, spectral cycles, self-organization

Received November 30, 2020
Accepted May 24, 2021

This work has been supported by the RFBR, Grant No. 20-01-00222.

Marina V. Yashina
yash-marina@yandex.ru
Moscow Automobile and Road Construction State Technical University (MADI)
Leningradskiy prosp. 64, Moscow, 125319 Russia
Moscow Aviation Institute (National Research University)
Volokolamskoe sh. 4, Moscow, 125993 Russia

Alexander G. Tatashev
a-tatashev@yandex.ru
Moscow Automobile and Road Construction State Technical University (MADI)
Leningradskiy prosp. 64, Moscow, 125319 Russia

1. Introduction

In this paper we explore a dynamical system belonging to the class of contour networks developed by A. P. Buslaev. The introduction of this class of dynamical systems was motivated by the fact that almost no analytical results had been known for the network models of traffic flows and the network models had been mainly investigated only by simulation modeling. At the same time, for systems of such type the problems of exploring the properties of ergodicity and reversibility are also of much current interest. The structure and the rules of operation of contour networks were introduced in such a way that they could, on the one hand, capture the special features of real traffic networks and, on the other hand, that they could be investigated using analytical methods, in particular, the approaches of functional analysis, algebra, the theory of Markovian processes and other classical branches of mathematics. Along with the contour networks, A. P. Buslaev introduced the concept of cluster motion in discrete and continuous mathematical models of traffic and developed an approach to traffic modeling, which was called the deterministic-stochastic approach according to which the velocity of traffic flow on a highway is represented as the sum of the deterministic component, to which the constant velocity of motion of the cellular field corresponds, and the stochastic component, which corresponds to the average velocity of particles moving in this field.

The contour networks can also have other applications, in particular, they can be used in modeling the operation of information and communication systems.

Analytical results for mathematical traffic models in which particles move according to certain rules on a one-dimensional or two-dimensional grid and which can be interpreted in terms of cellular automata [1] or random processes with prohibitions [2] were obtained, for example, in [3–9].

The concept of cluster motion in the mathematical traffic model was introduced in [10]. In the discrete version, a cluster is a group of particles located in neighboring cells and moving simultaneously, which explains the use of the term “cluster”. In the continuous version, a cluster is a moving segment which, in the absence of delays, moves with constant velocity.

The concept of contour networks (*Buslaev networks*) was introduced in [11]. A contour network contains closed contours along which particles or clusters move. The neighboring contours have common points called *nodes*. Delays in the motion of the clusters arise due to a restriction under which no more than one particle (cluster) can intersect the node simultaneously. If a cluster approaches the node at the instant when another cluster passes through this node, then the approaching cluster comes to a stop and resumes its motion after the node gets free. If clusters approach the same node simultaneously, then competition arises and one of these clusters, which is chosen in accordance with the specified rule of competition settlement, is the first to pass through the node. The main problems include finding the average velocity of motion of the clusters (particles), finding conditions for the system to enter a state of free-flow traffic (from some time instant on, all clusters move without delays at the current instant and in the future) or a state of traffic breakdown (the motion of the particles ceases completely), and choosing a rule of competition settlement that is best-suited to maximize the average velocity of motion. Analytical results were obtained for two-contour systems with one [12–15] or two [16–19] nodes, for systems with several contours and one common node [20] and for contour networks with regular periodic one-dimensional [21–24] or two-dimensional [25–27] contour systems.

In [16] we investigated a contour network containing two contours on each of which there is one cluster. The length of the contour and the length of the cluster located on it do not depend on the contour. There are two nodes which divide each contour into two parts: the length of one

of them is d and that of the other is $1 - d$ (if the length of the contour is taken to be unity). We considered two versions of the system which differ from each other in the way the directions of motion of the clusters correlate to each other. In one of these versions, both clusters, after passage through one of the nodes, are on an arc of length d , and after passage through another node they reach an arc of length $1 - d$. In this version, the system is called a *system with codirectional motion* (Fig. 1). In the other version, after passage through the same node, one cluster reaches an arc of length d , and the other reaches an arc of length $1 - d$. In this version, the system is called a *system with one-directional motion* (Fig. 2), since in its geometric interpretation either both clusters move counterclockwise or both clusters move clockwise. In [16], we considered a system with the left-priority rule of competition settlement, according to which, during competition, priority is always given to the same cluster, which is located on the left. The main characteristic under study is the average velocity of motion, which is equal to the average distance traveled by the cluster in unit time, with delays taken into account. The results of [16] imply that, at any given values of d , any length of cluster l and any initial state of the system, the average velocity is the same for both clusters and, in the case of codirectional motion, does not depend on the initial state, and in the case of one-directional motion, for each given value of d and l , one or, depending on the initial state, two values of the average velocity are possible. In Ref. [16] we also considered discrete versions of systems with codirectional and one-directional motion. In [17] we examined a continuous version of the system with one-directional motion, and Ref. [19] dealt with a continuous version of the system with codirectional motion. It turned out that the behavior of the continuous version of the system and that of the discrete version are similar. By virtue of symmetry, the average velocity of the clusters in the system with codirectional motion does not depend on the competition settlement rule. In [18], for the system with one-directional motion, special attention was given to the problem of finding a rule of competition settlement that is best-suited to maximize the average velocity of clusters. The average velocity of clusters depends on the choice of the rule of competition settlement only in initial states from which the system enters a state such that competition takes place. In [18], it was proved that at any values of d there exists a rule of competition settlement according to which the average velocity of motion of the clusters is not lower than that prescribed by any other rule, and, depending on the values of the parameters d and l , such an optimal rule is either a rule according to which the competition is always won by a cluster that passes from a smaller arc to a larger one, or a rule according to which the competition is always won by a cluster that passes from a larger arc to a smaller one.

In this paper, we investigate a continuous two-contour system with two nodes. The novelty is that, in contrast to the papers [16–19], where the continuous and discrete two-contour systems were considered under the assumption that the clusters have the same length, in this paper we investigate a continuous two-contour system with two nodes, in which the clusters have

Fig. 1. A two-contour system with codirectional motion.

Fig. 2. A two-contour system with one-directional motion.

different lengths, l_1 and l_2 . It turned out that in this case the number of possible behavior scenarios increases significantly. In particular, it was found that, depending on the parameters of the system and its initial state, the clusters can move either with the same or with different velocities, whereas in the case of equal lengths the clusters always move with the same velocities. The continuous version of the system is considered. It is proved that in the case of codirectional motion, as in the case of contours of the same length, the average velocity does not depend on the number of the cluster or the initial state of the system. The dependence of the average velocity of the clusters on the values of d , l_1 and l_2 is found. It is shown that for a system with one-directional motion the clusters can, for given values of d , l_1 , l_2 and for a given initial state, have different average velocities and that for given values of d , l_1 and l_2 the average velocity of the clusters can depend on the initial state. Various behavior scenarios of the system with one-directional motion are described and sufficient conditions are found under which specific scenarios take place.

2. Description of a two-contour system with two nodes.

Definitions of the main concepts

2.1. Description of the system

The system contains two closed *contours*: contour C_1 and contour C_2 , Fig. 1. We assume that the length of each contour is 1. On the i th contour there is a *cluster* which is a moving segment of length $l_i < 1$, $i = 1, 2$. If there are no delays, the velocity of motion of the cluster is equal to 1, i.e., the cluster makes a complete circuit around the contour in an interval of 1 unit of time. The contours have two common points, called *nodes*. On each contour, a *coordinate system* is defined which is segment $[0, 1]$. The direction of the coordinate system coincides with that of the motion of the cluster. The state of the system at time $t \geq 0$ is a vector $A(t) = (x_1(t), x_2(t))$, where $x_i(t)$ is the coordinate of the front point of the cluster, $i = 1, 2$. When the system is in the state $(x_1(t), x_2(t))$, the cluster of the i th contour is located on the arc $(x_i(t) - l_i, x_i(t))$ (subtraction modulo 1), $i = 1, 2$.

We consider two versions of the system: a system *with codirectional motion* (Fig. 1) and a system *with one-directional motion* (Fig. 2).

In the system with codirectional motion, one of the nodes, called node 1, is located at the point with coordinate 0 on both contours, and node 2 is located on both contours at the point with coordinate $d \leq 1/2$. We will say that, when the system is in the state $(x_1(t), x_2(t))$, cluster i occupies node 1 if $0 < x_i(t) < l_i$, and occupies node 2 if $d < x_i < d + l_i$ or $0 \leq x_i < d + l_i - 1$, $i = 1, 2$. We will say that, when the system is in the state $(x_1(t), x_2(t))$, cluster i is located at node 1 if $x_i(t) = 0$, and cluster i is located at node 2 if $x_i(t) = d$, $i = 1, 2$.

In the system with one-directional motion, node i is located at point 0 on contour i and at point d on contour $i = 1, 2$, $j = 1, 2$, $j \neq i$. We will say that, when the system is in the state $(x_1(t), x_2(t))$, cluster i occupies node i if $0 < x_i(t) < l_i$, and occupies node $j \neq i$ if $d < x_i < d + l_i$ or $0 \leq x_i < d + l_i - 1$. We will say that, when the system is in the state $(x_1(t), x_2(t))$, cluster i is located at node i if $x_i(t) = 0$, and cluster i is located at node $j \neq i$ if $x_i(t) = d$, $i = 1, 2$, $j = 1, 2$.

The state of the system is admissible if in this state none of the nodes is simultaneously occupied by two clusters.

Both in the system with codirectional motion and in the system with one-directional motion, a delay in the motion of a cluster arises at the instant when this cluster is located at a node that at this instant is occupied by another cluster. The delay continues until the instant when this

node ceases to be occupied. If two clusters approach a node simultaneously, then *competition* arises. The cluster that *wins* the competition in accordance with the specified *rule of competition settlement* is the first to pass through the node.

For definiteness, we assume that in the system with codirectional motion competition is allowed by the rule according to which the competition is always won by cluster 1 (*left-priority rule*). All results for the system with codirectional motion which are proved in this paper hold true for any rule of competition settlement, since the proofs use no specific rule of competition settlement.

We will assume that, in the system with one-directional motion with $d < 1/2$, the competition is always won by a cluster that passes from an arc of length d to an arc of length $1 - d$, and when $d = 1/2$, the competition is settled in accordance with the left-priority rule.

We assume that $l_1 \leq l_2$. The case where $l_1 > l_2$ is considered in a completely similar way.

2.2. Limit cycles. The average velocity of a cluster.

States of free-flow traffic and traffic breakdown

As will be proved in this paper, from some time instant on, the system's states belonging to some set recur periodically. Thus, the invariant sets of the states of the dynamical system under consideration belong to the type called *limit cycles*.

Let $H_i(t)$ be the distance traveled by cluster i in the time interval $(0, t)$. Let us call the limit

$$\lim_{t \rightarrow \infty} \frac{H_i(t)}{t}, \quad i = 1, 2 \quad (1)$$

the average velocity v_i of cluster i . Since one of the limit cycles takes place, limit (1) exists and is equal to $v_i = A_i/T$, where A_i is the distance traveled by the cluster on the limit cycle and T is a period.

The average velocities of the clusters depend on the values of l_1 , l_2 , d and, in the general case, on the initial state of the system.

If at any time $t \geq t_0$ all particles move, then we say that from time t_0 on the system is in the state of free-flow traffic. We will call the system's property of entering the state of free-flow traffic from any initial state *self-organization*. If the system is in the state of free-flow traffic, then $v_1 = v_2 = 1$.

If no particle moves from time t_0 on, then we say that from time t_0 on the system is in the state of traffic breakdown. If the system is in the state of traffic breakdown, then $v_1 = v_2 = 0$.

3. The behavior of the system with codirectional motion

3.1. A necessary condition for the existence of free-flow traffic

We prove a lemma that holds both for the system with codirectional motion and for the system with one-directional motion.

Lemma 1. *If the inequality*

$$l_1 + l_2 > 1 \quad (2)$$

is satisfied, then the system cannot be in the state of free-flow traffic.

Proof. If on a limit cycle the system is in the state of free-flow traffic, then the period of the cycle is equal to 1 and during this period each node is occupied by cluster 1 over an interval

with duration l_1 and by cluster 2 over an interval with duration l_2 . But this is impossible if inequality (2) is not satisfied since a node cannot be occupied by two clusters simultaneously. This proves Lemma 1. $\square$

3.2. Lemmas for the system with codirectional motion

We prove lemmas that will be used in the proofs of the theorems of the system with codirectional motion.

Lemma 2. *If on a limit cycle the system with codirectional motion is neither in the state of free-flow traffic nor in the state of traffic breakdown, then the system enters on this limit cycle one of the states $(l_1, 0)$, $(0, l_2)$, $(d + l_1, d)$, $(d, d + l_2)$ (addition modulo 1).*

Proof. It follows from the condition of the lemma that delays of at least one cluster occur on the limit cycle. If a delay of cluster 2 occurs at node 1, then at the instant of termination of this delay the system is in the state $(l_1, 0)$ (Fig. 3). If a delay of cluster 1 occurs at node 1, then this delay ends when the system is in the state $(0, l_2)$ (Fig. 4). If a delay of cluster 2 occurs at node 2, then at the instant of termination of this delay the system is in the state $(d + l_1, d)$ (addition modulo 1), Fig. 5. If a delay of cluster 1 occurs at node 2, then this delay ends when the system is in the state $(d, d + l_2)$ (Fig. 6). This implies that on the limit cycle the system enters at least one of the states listed in the condition of the lemma. This proves Lemma 2. $\square$

Fig. 3. State $(l_1, 0)$. Termination of the delay of cluster 2 at node 1

Fig. 4. State $(0, l_2)$. Termination of the delay of cluster 1 at node 1

Fig. 5. State $(d + l_1, d)$. Termination of the delay of cluster 2 at node 2

Fig. 6. State $(d, d + l_2)$. Termination of the delay of cluster 1 at node 2

Lemma 3. *The system with codirectional motion can be in the state of traffic breakdown if and only if the following condition is satisfied:*

$$l_1 > 1 - d. \quad (3)$$

Proof. In accordance with the assumption that $l_1 \leq l_2$, inequality (3) also leads to the inequality $l_2 > 1 - d$. Consequently, the state $(0, d)$ (Fig. 7) and the state $(d, 0)$ (Fig. 8) are states of traffic breakdown.

Fig. 7. State $(0, d)$. Traffic breakdown

Fig. 8. State $(d, 0)$. Traffic breakdown

If the system is in the state of traffic breakdown, then the clusters are at different nodes, and each cluster, when it is at one node, occupies another, which is impossible if inequality (3) is not satisfied. This proves Lemma 3. $\square$

3.3. A necessary and sufficient condition for self-organization of the system with codirectional motion

The following theorem gives a necessary and sufficient condition for the system with codirectional motion to enter the state of free-flow traffic.

Theorem 1. *If the inequality*

$$l_1 + l_2 \leq 1 \quad (4)$$

is satisfied, then the system enters the state of free-flow traffic from any initial state of the system.

If inequality (2) is satisfied, then the system does not enter the state of free-flow traffic from any initial state.

Proof. If inequality (4) is satisfied, inequality (3) cannot hold, and in accordance with Lemma 3 the system cannot be in the state of traffic breakdown. It suffices to prove that from each of the states $(l_1, 0)$, $(0, l_2)$, $(l_1 + d, 0)$, $(0, l_2 + d)$ (addition modulo 1) the system enters the state of free-flow traffic and then, taking Lemma 2 into account, we obtain the statement of the theorem.

Consider the following possible cases.

1) Assume that $l_2 \leq d$.

Let

$$A(t_0) = (l_1, 0).$$

Then we have

$$A(t_0 + d - l_1) = (d, d - l_1), \quad A(t_0 + d) = (d + l_1, d),$$

$$A(t_0 + 1 - l_1) = (0, 1 - l_1), \quad A(t_0 + 1) = (l_1, 0).$$

Thus, the states $(l_1, 0)$, $(d + l_1, d)$ are states of free-flow traffic.

Similarly, we have the sequence of states

$$A(t_0) = (0, l_2), \quad A(t_0 + d - l_2) = (d - l_2, d), \quad A(t_0 + d) = (d, d + l_2),$$

$$A(t_0 + 1 - l_2) = (1 - l_2, 0), \quad A(t_0 + 1) = (0, l_2)$$

and thus the states $(0, l_2)$, $(d, d + l_2)$ are also states of free-flow traffic.

2) Assume that $l_1 \leq d < l_2 < 1 - d$, $l_1 + l_2 \leq 1$.

In this case we have the sequences of states

$$\begin{aligned} A(t_0) &= (l_1, 0), \quad A(t_0 + d - l_1) = (d, d - l_1), \quad A(t_0 + d) = (d + l_1, d), \\ A(t_0 + 1 - l_1) &= (0, 1 - l_1), \quad A(t_0 + 1) = (l_1, 0); \\ A(t_0) &= (0, l_2), \quad A(t_0 + d) = (d, d + l_2), \quad A(t_0 + 1 - l_2) = (1 - l_2, 0), \\ A(t_0 + 1 - l_2 + d) &= (1 - l_2 + d, d), \quad A(t_0 + 1) = (0, l_2). \end{aligned}$$

Thus, the states $(l_1, 0)$, $(d + l_1, d)$, $(0, l_2)$, $(d, d + l_2)$ are states of free-flow traffic.

3) Let $l_1 \leq d$, $l_2 \geq 1 - d$, $l_1 + l_2 \leq 1$.

We have the sequences of states

$$\begin{aligned} A(t_0) &= (l_1, 0), \quad A(t_0 + d - l_1) = (d, d - l_1), \quad A(t_0 + d) = (d + l_1, d), \\ A(t_0 + 1 - l_1) &= (0, 1 - l_1), \quad A(t_0 + 1) = (l_1, 0); \\ A(t_0) &= (0, l_2), \quad A(t_0 + 1 - l_2) = (1 - l_2, 0), \quad A(t_0 + d) = (d, d + l_2 - 1), \\ A(t_0 + d + 1 - l_2) &= (1 - l_2 + d, d), \quad A(t_0 + 1) = (0, l_2). \end{aligned}$$

Consequently, the states $(l_1, 0)$, $(d + l_1, d)$, $(0, l_2)$, $(d, d + l_2 - 1)$ are states of free-flow traffic.

4) Let $l_1 > d$, $l_1 + l_2 \leq 1$.

Then we have the sequences of states

$$\begin{aligned} A(t_0) &= (l_1, 0), \quad A(t_0 + d) = (l_1 + d, d), \quad A(t_0 + 1 - l_1) = (0, 1 - l_1), \\ A(t_0 + 1 - l_1 + d) &= (d, 1 - l_1 + d), \quad A(t_0 + 1) = (l_1, 0); \\ A(t_0) &= (0, l_2), \quad A(t_0 + d) = (d, d + l_2), \\ A(t_0 + 1 - l_2) &= (1 - l_2, 0), \quad A(t_0 + 1) = (0, l_2). \end{aligned}$$

Thus, the states $(l_1, 0)$, $(d + l_1, d)$, $(0, l_2)$, $(d, d + l_2)$ are states of free-flow traffic.

We have considered all possible cases for which $l_1 + l_2 \leq 1$. This proves Theorem 1. $\square$

3.4. Possible behavior scenarios of the system with codirectional motion

We prove a theorem that describes the behavior of the system with codirectional motion under conditions such the system enters neither the state of free-flow traffic nor the state of traffic breakdown.

Theorem 2. *Suppose that for the system with codirectional motion the condition*

$$l_1 \leq 1 - d, \quad l_1 + l_2 > 1$$

is satisfied. Then from any initial state the system enters the states of the same limit cycle with period

$$T = l_1 + l_2. \quad (5)$$

In this case, the average velocity of each cluster is

$$v_1 = v_2 = \frac{1}{l_1 + l_2}. \quad (6)$$

Proof. By Lemma 3, the system cannot be in the state of traffic breakdown, and by Lemma 1, it cannot be in the state of free-flow traffic. It suffices to prove that from each of the states $(l_1, 0)$, $(0, l_2)$, $(l_1 + d, 0)$, $(0, l_2 + d)$ (addition modulo 1) the system enters the state of free-flow traffic and then, taking Lemma 2 into account, we obtain the statement of the theorem. $\square$

Consider the following possible cases.

1) Assume that $l_1 \leq d$, $l_1 + l_2 > 1$.

We have the sequences of states

$$A(t_0) = (l_1, 0), A(t_0 + d - l_1) = (d, d - l_1), A(t_0 + d) = (d + l_1, d), A(t_0 + 1 - l_1) = (0, 1 - l_1),$$

$$A(t_0 + l_2) = (0, l_2), A(t_0 + 1) = (1 - l_2, 0), A(t_0 + l_1 + l_2) = (l_1, 0);$$

$$A(t_0) = (d, d + l_2 - 1), A(t_0 + 1 - l_2) = (d + 1 - l_2, d), A(t_0 + l_1) = (d + l_1, d).$$

Thus, the states $(l_1, 0)$, $(0, l_2)$, $(d + l_1, d)$ belong to a limit cycle with the period and the average clusters' velocity which are calculated from the formulae (5) and (6), and from the state $(d, d + l_2 - 1)$ the system enters the state of this cycle.

2) Let $l_1 > d$, $l_2 \leq 1 - d$, $l_1 + l_2 > 1$.

We have the sequences of states

$$A(t_0) = (l_1, 0), A(t_0 + d) = (d + l_1, d), A(t_0 + 1 - l_1) = (0, 1 - l_1),$$

$$A(t_0 + l_2) = (0, l_2), A(t_0 + 1) = (1 - l_2, 0), A(t_0 + l_1 + l_2) = (l_1, 0);$$

$$A(t_0) = (d, d + l_2), A(t_0 + 1 - d - l_2) = (1 - l_2, 0), A(t_0 + l_1 - d) = (l_1, 0).$$

Hence, the states $(l_1, 0)$, $(0, l_2)$, $(d + l_1, d)$ belong to a limit cycle with the period and the average clusters' velocity which are calculated from the formulae (5) and (6), and from the state $(d, d + l_2)$ the system enters the state of this cycle.

3) Let $d < l_1 \leq 1 - d$, $l_2 > 1 - d$.

In this case we have the sequences of states

$$A(t_0) = (l_1, 0), A(t_0 + d) = (d + l_1, d), A(t_0 + 1 - l_1) = (0, 1 - l_1), A(t_0 + l_2) = (0, l_2),$$

$$A(t_0 + 1) = (1 - l_2, 0), A(t_0 + l_1 + l_2) = (l_1, 0);$$

$$A(t_0) = (d, d + l_2 - 1), A(t_0 + 1 - l_2) = (d + 1 - l_2, d), A(t_0 + d) = (d + l_1, d).$$

Hence, the states $(l_1, 0)$, $(0, l_2)$, $(d + l_1, d)$ belong to a limit cycle with the period and the average clusters' velocity which are calculated from the formulae (5) and (6), and from the state $(d, d + l_2 - 1)$ the system enters the state of this cycle.

We have considered all possible cases in which the conditions of Theorem 2 are satisfied. This proves Theorem 2.

The following theorem gives a necessary and sufficient condition for the system with codirectional motion to enter the state of traffic breakdown.

Theorem 3. *If inequality (3) is satisfied, then the system enters the state of traffic breakdown from any initial state of the system.*

If inequality (3) is not satisfied, then the system does not enter the state of traffic breakdown from any initial state.

Proof. If inequality (3) is satisfied, inequality (2) is satisfied also and hence, by Lemma 3, the system cannot be in the state of free-flow traffic. Thus, by Lemma 2, in order to prove the statement of the theorem, it remains to prove that the system enters the state of traffic breakdown from each of the states $(l_1, 0)$, $(0, l_2)$, $(d + l_1 - 1, 0)$, $(0, d + l_2 - 1)$.

We have the sequences of states

$$\begin{aligned} A(t_0) &= (l_1, 0), \quad A(1 - l_1) = (0, 1 - l_1), \quad A(d) = (0, d); \\ A(t_0) &= (0, l_2), \quad A(1 - l_2) = (1 - l_2, 0), \quad A(d) = (d, 0); \\ A(t_0) &= (d + l_1 - 1, 0), \quad A(1 - l_1) = (d, 0); \\ A(t_0) &= (0, d + l_2 - 1), \quad A(1 - l_2) = (0, d). \end{aligned}$$

Given that the states $(0, d)$, $(d, 0)$ are states of traffic breakdown, we obtain the statement of the theorem. $\square$

In accordance with the following theorem, the behavior of the system with codirectional motion is described by one of the theorems (1, 2 or 3) for any parameters of the system.

Theorem 4. *For any set of values of l_1, l_2, d the average velocity of the clusters is the same for both clusters, does not depend on the initial state and takes one of the values, 1, $1/(l_1 + l_2)$ or 0.*

Proof. Given that the condition of one of the theorems (Theorem 1, 2 or 3) is satisfied for any set of the parameter values, we obtain the statement of Theorem 4. $\square$

4. The behavior of the system with one-directional motion

We prove the theorems of possible behavior scenarios of the system with one-directional motion. For each scenario considered, we formulate sufficient conditions for this scenario to take place.

Lemma 4. *If on a limit cycle the system with one-directional motion is neither in the state of free-flow traffic nor in the state of traffic breakdown, then the system enters on this limit cycle one of the states (l_1, d) , (d, l_2) , $(d + l_1, 0)$, $(0, d + l_2)$ (addition modulo 1), Figs. 9–12.*

Fig. 9. State (l_1, d) . Termination of the delay of cluster 2 at node 1

Fig. 10. State (d, l_2) . Termination of the delay of cluster 1 at node 2

Fig. 11. State $(d + l_1, 0)$. Termination of the delay of cluster 2 at node 2

Fig. 12. State $(0, d + l_2)$. Termination of the delay of cluster 1 at node 1

Lemma 5. *The system with one-directional motion can be in the state of traffic breakdown if and only if the following condition is satisfied:*

$$l_1 > d.$$

Lemma 4 is proved by Lemma 2, and Lemma 5, by Lemma 3. The states of traffic breakdown of the system with one-directional motion are shown in Figs. 13, 14.

Fig. 13. State $(0, 0)$. Traffic breakdown

Fig. 14. State (d, d) . Traffic breakdown

Theorem 5. *If the conditions*

$$l_2 \leq d, \quad l_1 + l_2 \leq 1 - 2d$$

are satisfied, then from some time instant on the system is in the state of free-flow traffic for any initial state of the system.

Proof. Under the assumptions of Theorem 1 and by Lemma 4, the system cannot be in the state of traffic breakdown. If on some limit cycle the system is not also in the state of free-flow traffic, then, by Lemma 3, on any limit cycle the system is at least in one of the states $(l_1 + d_1, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) . We prove that from any of these states the system enters over some time interval the state of free-flow traffic.

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then under the assumptions of the theorem we have the following sequence of states:

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + 1 - l_1) &= (d, 1 - l_1), & A(t_0 + 1) &= (l_1 + d, 0). \end{aligned}$$

Thus, in a unit of time the system returns into the state $(l_1 + d, 0)$, i.e., it is in the state of free-flow traffic.

Let

$$A(t_0) = (0, l_2 + d).$$

Then

$$\begin{aligned} A(t_0 + d) &= (d, l_2 + 2d), & A(t_0 + 1 - l_2 - d) &= (1 - l_2 - d, 0), \\ A(t_0 + 1 - l_1) &= (1 - l_2, d), & A(t_0 + 1) &= (0, l_2 + d). \end{aligned}$$

Thus, the state $(0, l_2 + d)$ is the state of free-flow traffic.

Assume that

$$A(t_0) = (l_1, d).$$

We have

$$\begin{aligned} A(t_0 + d - l_1) &= (d, 2d - l_1), & A(t_0 + 1 - d) &= (1 - d + l_1, 0), \\ A(t_0 + 1 - l_1) &= (0, 1 - l_1 + d), & A(t_0 + 1) &= (l_1, d). \end{aligned}$$

Thus, the state (l_1, d_1) is the state of free-flow traffic.

Similarly, the state of free-flow traffic is the state $(0, l_2)$ since, if

$$A(t_0) = (d, l_2),$$

then

$$\begin{aligned} A(t_0 + d - l_2) &= (2d - l_2, d), & A(t_0 + 1 - d) &= (0, 1 - d + l_2), \\ A(t_0 + 1 - l_2) &= (1 - l_2 + d, 0), & A(t_0 + 1) &= (d, l_2). \end{aligned}$$

This proves Theorem 5. $\square$

Theorem 6. *If the conditions*

$$l_2 \leq d, \quad l_2 \leq 1 - 2d, \quad l_1 + l_2 > 1 - 2d$$

are satisfied, then, depending on the initial state of the system, either the system enters the state of free-flow traffic or the average velocity of the clusters is

$$v = \frac{1}{l_1 + l_2 + 2d}. \quad (7)$$

Proof. Under the assumptions of Theorem 2, as follows from the statement of Lemma 4, the system cannot be in the state of traffic breakdown. If on some limit cycle the system is not also in the state of free-flow traffic, then, by Lemma 3, on any limit cycle the system is at least in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) . We prove that from any of these states the system enters over some time interval either the state of free-flow traffic or a limit cycle takes place with the average velocity of each cluster that is calculated from the formula (7).

Assume that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then we have

$$\begin{aligned} A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), & A(t_0 + l_2 + d) &= (0, l_2 + d), \\ A(t_0 + 1) &= (1 - l_2 - d, 0), & A(t_0 + l_1 + l_2 + 2d) &= (l_1 + d, 0). \end{aligned}$$

Thus, in an interval with duration $l_1 + l_2 + 2d$ all clusters make a complete circuit. Hence, on the limit cycle containing the state $(l_1 + d, 0)$, $(0, l_2 + d)$, the average velocity of the clusters is calculated from the formula (7).

If at time t_0

$$A(t_0) = (l_1, d),$$

then we have

$$A(t_0 + d - l_1) = (d, 2d - l_1), \quad A(t_0 + 1 - l_1) = (0, d - l_1), \quad A(t_0 + 1) = (l_1, d).$$

Consequently, the state (l_1, d) is the state of free-flow traffic.

Let

$$A(t_0) = (d, l_2).$$

Then

$$A(t_0 + d - l_2) = (2d - l_2, d), \quad A(t_0 + 1 - l_2) = (d - l_2, 0), \quad A(t_0 + 1) = (d, l_2).$$

Hence, the state (d, l_2) is the state of free-flow traffic.

Thus, each of the four initial states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) belongs to a limit cycle on which either the system is in the state of free-flow traffic or the average velocity of the clusters is defined by the formula (7). This proves Theorem 6. $\square$

Theorem 7. *If the conditions*

$$l_1 \leq d, \quad d < l_2 < 2d, \quad l_2 \leq 1 - 2d, \\ l_1 + l_2 > \max(2d, 1 - 2d)$$

are satisfied, then there exist exactly two limit cycles, one of which corresponds to the average clusters' velocity calculated from the formula (7), and the other corresponds to the average velocity equal to

$$v = \frac{1}{l_1 + l_2 + 1 - 2d}.$$

Proof. Under the assumptions of Theorem 7 and by Lemma 4, the system cannot be in the state of traffic breakdown. This implies that there exist time instants at which the front particle of contour 2 is in a cell with number 0. Suppose that at time t_0 the system is in the state

$$A(t_0) = (\alpha, 0).$$

Assume that

$$0 \leq \alpha < d.$$

Then we have

$$A(t_0 + d - \alpha) = (d, d - \alpha)$$

and hence at time $t = d - \alpha$ a delay of cluster 1 begins.

If

$$d \leq \alpha < l_1 + d,$$

then at time $t = 0$ cluster 2 does not move.

If

$$l_1 + d < \alpha \leq 1 - d,$$

then

$$A(t_0 + 1 - \alpha) = (0, 1 - \alpha), \quad d \leq 1 - \alpha < 1 - l_1 - d,$$

and so we find that at time $t = 1 - \alpha$ cluster 1 does not move.

If

$$\alpha > 1 - d,$$

then

$$A(t_0 + 1 - \alpha + d) = (0, \alpha + d - 1, d), \quad 0 < \alpha + d - 1 < d,$$

and hence at time $t = d$ cluster 1 does not move.

Thus, taking Lemma 3 into account, we find that there exists a time instant at which the system is in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) .

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then under the assumptions of the theorem we have the following sequence of states:

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + l_2 + d) &= (0, l_2 + d), & A(t_0 + l_2 + 2d) &= (d, l_2 + 2d), \\ A(t_0 + 1) &= (1 - l_2 - d, 0), & A(t_0 + l_1 + l_2 + 2d) &= (l_1 + d, 0). \end{aligned}$$

Thus, the states $(l_1 + d, 0)$, $(0, l_2 + d)$ belong to a limit cycle with the average clusters' velocity calculated from the formula (7).

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$\begin{aligned} A(t_0 + d - l_1) &= (d, 2d - l_1), & A(t_0 + l_2 - d) &= (d, l_2), \\ A(t_0 + 1 - d) &= (1 + d - l_2, 0), & A(t_0 + 1 + l_2 - 2d) &= (0, l_2 - d), \\ A(t_0 + 1) &= (2d - l_2, d), & A(t_0 + 1 + l_1 + l_2 - 2d) &= (l_1, d). \end{aligned}$$

Thus, the states (l_1, d) , (d, l_2) belong to a limit cycle with the average clusters' velocity calculated from the formula (8).

Theorem 8. *If the conditions*

$$\begin{aligned} l_1 &\leq d, \\ 2d &\leq l_2 \leq 1 - 2d, \\ l_1 + l_2 &> 1 - 2d \end{aligned}$$

are satisfied, then there exists a unique limit cycle with the average clusters' velocity calculated from the formula (7).

Proof. The system cannot be in the state of free-flow traffic. The proof of this is similar to that of the corresponding part of Theorem 7. Thus, taking Lemma 3 into account, we find that there exists a time instant at which the system is in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) .

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then under the assumptions of the theorem we have the following sequence of states:

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + l_2 + d) &= (0, l_2 + d), & A(t_0 + l_2 + 2d) &= (d, l_2 + 2d), \\ A(t_0 + 1) &= (1 - l_2 - d, 0), & A(t_0 + l_1 + l_2 + 2d) &= (l_1 + d, 0). \end{aligned}$$

Thus, the states $(l_1 + d, 0)$, $(0, l_2 + d)$ belong to a limit cycle with the average clusters' velocity calculated from the formula (7).

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$\begin{aligned} A(t_0 + d - l_1) &= (d, 2d - l_1), & A(t_0 + l_2 - d) &= (d, l_2), \\ A(t_0 + 1 - d) &= (1 + d - l_2, 0), & A(t_0 + 1 + l_2 - 2d) &= (0, l_2 - d), \\ A(t_0 + 1) &= (1 + 2d - l_2, d), & A(t_0 + 1 + l_2 - 2d) &= (0, l_2 - d), \\ & & A(t_0 + 1 + l_2) &= (0, l_2 + d). \end{aligned}$$

Thus, from the states (l_1, d) , (d, l_2) the system enters the state $(0, l_2 + d)$ belonging to a limit cycle with the average velocity of both clusters which is calculated from the formula (7).

Theorem 9. *If the conditions*

$$\begin{aligned} l_1 &\leq d, \quad l_1 \leq 1 - 2d, \quad 1 - 2d < l_2 < 1 - d, \\ d &< l_2 < 2d, \quad l_1 + l_2 > 2d \end{aligned}$$

are satisfied, then there exists a unique limit cycle corresponding to the average clusters' velocity calculated from the formula (8).

Proof. As in the proof of Theorem 7, it is proved that the system cannot be in the state of free-flow traffic. Thus, taking Lemma 4 into account, we find that there exists a time instant at which the system is in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) .

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$\begin{aligned} A(t_0 + d - l_1) &= (d, 2d - l_1), & A(t_0 + l_2 - d) &= (d, l_2), \\ A(t_0 + 1 - d) &= (1 + d - l_2, 0), & A(t_0 + 1 + l_2 - 2d) &= (0, l_2 - d), \\ A(t_0 + 1) &= (2d - l_2, d), & A(t_0 + 1 + l_1 + l_2 - 2d) &= (l_1, d). \end{aligned}$$

Thus, the states (l_1, d) , (d, l_2) belong to a limit cycle with the average clusters' velocity calculated from the formula (8).

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then under the assumptions of the theorem we have the following sequence of states:

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + l_2 + d) &= (0, l_2 + d), & A(t_0 + 1) &= (1 - l_2 - d, 0), \\ A(t_0 + 1) &= (1 - l_1 - d, 0), & A(t_0 + l_2 + 2d) &= (d, l_2 + 2d - 1), \\ & & A(t_0 + 1 + l_2) &= (d, l_2). \end{aligned}$$

Thus, from the state $(l_1 + d, 0)$ the system enters the state (d, l_2) belonging to a limit cycle with the average clusters' velocity calculated from the formula (8).

Suppose that at time t_0 the system is in the state

$$A(t_0) = (0, l_2 + d).$$

Then under the assumptions of the theorem we have the following sequence of states:

$$A(t_0 + 1 - l_2 - d) = (1 - l_2 - d, 0), \quad A(t_0 + d) = (d, l_2 + 2d - 1), \quad A(t_0 + 1 - d) = (d, l_2).$$

Thus, from the state $(0, l_2 + d)$ the system enters the state (d, l_2) belonging to a limit cycle with the average clusters' velocity calculated from the formula (8). This proves Theorem 9. $\square$

Theorem 10. *If the conditions*

$$\begin{aligned} l_1 &\leq d, \quad l_1 \leq 1 - 2d, \\ l_2 &> 1 - d, \quad l_1 + l_2 > 1 \end{aligned}$$

are satisfied, then for any initial state the velocities of motion of the clusters are calculated from the formulae

$$v_1 = \frac{1}{2(l_1 + l_2)}, \quad (9)$$

$$v_2 = \frac{1}{l_1 + l_2}. \quad (10)$$

Proof. Under the assumptions of Theorem 10 and by Lemmas 1 and 5, the system cannot be in the state of traffic breakdown or free-flow traffic. If on some limit cycle the system is not also in the state of free-flow traffic, then, in accordance with Lemma 4, on any limit cycle the system is at least in one of the states $(l_1 + d_1, 0)$, $(0, l_2 + d - 1)$, (l_1, d) , (d, l_2) . We prove that all these states belong to a limit cycle corresponding to the average clusters' velocities calculated from the formulae (9) and (10).

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then we have

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + 1) &= (0, 0), & A(t_0 + l_2 + d) &= (0, l_2 + d - 1), \\ A(t_0 + 1 + d) &= (1 - l_2, d), & A(t_0 + l_1 + l_2 + d) &= (l_1, d), \\ A(t_0 + l_2 + 2d) &= (d, 2d - l_1), & A(t_0 + l_1 + 2l_2) &= (d, l_2), \\ A(t_0 + 1 + l_1 + l_2) &= (1 + d - l_2, 0), & A(t_0 + 2(l_1 + l_2)) &= (l_1 + d, 0). \end{aligned}$$

Thus, there is a unique limit cycle with period $2(l_1 + l_2)$ on which cluster 1 makes two circuits around the contour and cluster 2 makes one circuit around it, whence the statement of Theorem 10 follows. $\square$

Theorem 11. *If*

$$l_1 \leq d, \quad l_1 \leq 1 - 2d, \quad l_2 > 1 - d, \quad l_2 \geq 2d, \quad l_1 + l_2 \leq 1,$$

then from any initial state the system enters the state belonging to a limit cycle on which the clusters move with the average velocities

$$v_1 = \frac{1}{2}, \quad v_2 = 1. \quad (11)$$

Proof. Under the assumptions of the theorem and by Lemma 1, the system cannot be in the state of free-flow traffic and, by Lemma 5, the system cannot be in the state of traffic breakdown. Assume that

$$A(t_0) = (l_1 + d, 0).$$

We have

$$\begin{aligned} A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), & A(t_0 + 1) &= (0, 0), \\ A(t_0 + l_2 + d) &= (0, l_2 + d - 1), & A(t_0 + 1 + d) &= (1 - l_2, d), \\ A(t_0 + l_2 + 2d) &= (d, l_2 + 2d - 1), & A(t_0 + 1 + l_2) &= (d, l_2), \\ A(t_0 + 2) &= (1 + d - l_2, 0), & A(2 + d) &= (1 + 2d - l_2, d), \\ A(2 + l_2 - d) &= (0, l_2 - d), & A(2 + l_2 + d) &= (0, l_2 + d - 1). \end{aligned}$$

Thus, the states $(0, l_2 + 2d - 1)$, (d, l_2) belong to a limit cycle on which the average velocities of the clusters are calculated from the formulae (11), and from the state $(l_1 + d, 0)$ the system enters such a state.

Let

$$A(t_0) = (l_1, d).$$

Then

$$A(t_0 + d - l_1) = (d, 2d - l_1), \quad A(t_0 + l_2 - d) = (d, l_2).$$

Thus, from the state (l_1, d) the system enters the state (d, l_2) belonging to a limit cycle with the average clusters' velocities calculated from the formula (11). This proves Theorem 11. $\square$

Theorem 12. *If the conditions*

$$d < l_1 < 2d, \quad d < l_2 < 2d, \quad l_1 + l_2 \leq 1 - 2d$$

are satisfied, then, depending on the initial state, the system enters either the state of free-flow traffic or the state of traffic breakdown.

Proof. If on some limit cycle the system is not in the state of free-flow traffic or traffic breakdown, then, by Lemma 3, on any limit cycle the system is at least in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) . We prove that from any of these states the system enters over some time interval the state of free-flow traffic or traffic breakdown.

Suppose that at time t_0 the system is in the state

$$\begin{aligned} A(t_0) &= (l_1 + d, 0), \quad A(t_0 + d) = (l_1 + 2d, d), \\ A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \quad A(t_0 + 1 - l_1) = (d, 1 - l_1), \\ A(t_0 + 1) &= (l_1 + d, 0). \end{aligned}$$

Thus, the state $(l_1 + d, 0)$ is the state of free-flow traffic.

Assume that

$$A(t_0) = (0, l_2 + d).$$

Then we have

$$\begin{aligned} A(t_0 + d) &= (d, l_2 + 2d), & A(t_0 + 1 - l_2 - d) &= (1 - l_2 - d, 0), \\ A(t_0 + 1 - l_2) &= (1 - l_2, d), & A(t_0 + 1) &= (0, l_2 + d). \end{aligned}$$

Thus, the state $(0, l_2 + d)$ is the state of free-flow traffic.

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$\begin{aligned} A(t_0 + 1 - l_1) &= (0, 1 - l_1 + d), \quad A(t_0 + 1 - l_1 + d) = (d, 2d - l_1), \\ A(t_0 + 1) &= (d, d). \end{aligned}$$

Thus, from the state (l_1, d) the system enters the state of traffic breakdown (d, d) .

Assume that

$$A(t_0) = (d, l_2).$$

Then

$$\begin{aligned} A(t_0 + 1 - l_2) &= (1 - l_2 + d, 0), \quad A(t_0 + 1 - l_2 + d) = (2d - l_2, d), \\ A(t_0 + 1) &= (d, d). \end{aligned}$$

Thus, from the state (d, l_2) the system enters the state of traffic breakdown (d, d) . This proves Theorem 12. $\square$

Theorem 13. *If the conditions*

$$l_1 > 2d, \quad l_2 \leq 1 - 2d, \quad 1 - 2d < l_1 + l_2 \leq 1$$

are satisfied, then, depending on the initial state, either the system enters the state of traffic breakdown or a limit cycle takes place with the average clusters' velocity calculated from the formula (7).

Proof. In accordance with Lemma 1, if the condition of the theorem is satisfied, the system enters from no initial state the state of free-flow traffic.

If on some limit cycle the system is not in the state of free-flow traffic or traffic breakdown, then, by Lemma 3, on any limit cycle the system is at least in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) . We prove that from any of these states the system enters over some time interval the state of traffic breakdown or a limit cycle takes place with the average velocity of each cluster which is calculated from the formula (7).

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

Then

$$\begin{aligned} A(t_0 + d) &= (l_1 + 2d, d), & A(t_0 + 1 - l_1 - d) &= (0, 1 - l_1 - d), \\ A(t_0 + l_2 + d) &= (0, l_2 + d), & A(t_0 + 1) &= (1 - l_2 - d, 0), \\ A(t_0 + l_2 + 2d) &= (d, l_2 + 2d - 1), & A(t_0 + l_1 + l_2 + 2d) &= (l_1 + d, 0). \end{aligned}$$

Thus, the states $(l_1 + d, 0)$ and $(0, l_2 + d)$ belong to a limit cycle with the average clusters' velocity calculated from the formula (7).

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$\begin{aligned} A(t_0 + 1 - l_1) &= (0, 1 - l_1 + d), & A(t_0 + 1 - l_1 + d) &= (d, 1 - l_1 + 2d), \\ A(1 - d) &= (l_1 - d, 0), & A(1 + d) &= (l_1 + d, 0). \end{aligned}$$

Thus, from the state (l_1, d) the system enters the state belonging to a limit cycle with the average clusters' velocity calculated from the formula (7).

Let

$$A(t_0) = (d, l_2).$$

Then

$$\begin{aligned} A(t_0 + 1 - l_2) &= (1 - l_2 + d, 0), & A(t_0 + 1 - l_2 + d) &= (1 - l_2 + 2d, d), \\ A(1 - d) &= (0, l_2 - d), & A(1 + d) &= (0, l_2 + d). \end{aligned}$$

Consequently, from the state (d, l_2) the system enters the state $(0, l_1 + d)$, from which the system enters the state of a limit cycle with the average clusters' velocity calculated from the formula (7).

The state of traffic breakdown is the state (d, d) . This proves Theorem 13. $\square$

Theorem 14. *If the conditions*

$$l_1 \geq 2d, \quad l_1 \leq 1 - 2d, \quad 1 - 2d < l_2 \leq 1 - d$$

are satisfied, then the system enters the state of traffic breakdown from any initial state.

Proof. In accordance with Lemma 1, if the condition of the theorem is satisfied, the system enters from no initial state the state of free-flow traffic.

If on some limit cycle the system is not in the state of free-flow traffic or traffic breakdown, then, by Lemma 4, on any limit cycle the system is at least in one of the states $(l_1 + d, 0)$, $(0, l_2 + d)$, (l_1, d) , (d, l_2) . We prove that from any of these states the system enters over some time interval the state of traffic breakdown.

Suppose that at time t_0 the system is in the state

$$A(t_0) = (l_1 + d, 0).$$

We have

$$A(t_0 + d) = (l_1 + 2d, 0), \quad A(t_0 + 1 - l_1 - d) = (0, 1 - l_1 - d),$$

$$A(l_2 + d) = (0, l_2 + d),$$

$$A(t_0 + l_2 + d) = (0, l_2 + d), \quad A(t_0 + 1) = (1 - l_2 - d, 0),$$

$$A(t_0 + l_2 + 2d) = (d, l_2 + 2d - 1), \quad A(t_0 + 1 + d) = (d, d).$$

Thus, from the states $(l_1 + d, 0)$ and $(0, l_2 + d)$ the system enters the state of traffic breakdown.

Assume that

$$A(t_0) = (l_1, d).$$

Then

$$A(t_0 + 1 - l_1) = (0, 1 - l_1 + d), \quad A(t_0 + l_2) = (0, l_2 + d).$$

Thus, from the state (l_1, d) the system enters the state $(0, l_2 + d)$, from which the system enters the state of traffic breakdown.

Assume that

$$A(t_0) = (d, l_2).$$

Then

$$A(t_0 + 1 - l_2) = (1 - l_2 + d, 0), \quad A(t_0 + l_1) = (l_1 + d, 0).$$

Thus, from the state (d, l_2) the system enters the state $(0, l_2 + d)$, from which the system enters the state of traffic breakdown. This proves Theorem 14. $\square$

Thus, we have proved that for the system with codirectional motion at any values of l_1 , l_2 , d the average velocity is the same for both clusters, and that three values of the average velocity 0, $1/(l_1 + l_2)$, 1 are possible.

Thus, for the system with one-directional motion we have revealed the following possible scenarios for given values of l_1 , l_2 , d .

- 1) The system enters the state of free-flow traffic from any initial state.
- 2) Depending on the initial state, either the system enters the state of free-flow traffic or the average velocity of the clusters is

$$v = \frac{1}{l_1 + l_2 + 2d}.$$

- 3) Depending on the initial state, the average velocity of the clusters is either

$$v = \frac{1}{l_1 + l_2 + 2d},$$

or

$$v = \frac{1}{l_1 + l_2 + 1 - 2d}.$$

4) For any initial state, the average velocity of the clusters is

$$v = \frac{1}{l_1 + l_2 + 2d}.$$

5) For any initial state, the average velocity of the clusters is

$$v = \frac{1}{l_1 + l_2 + 1 - 2d}.$$

6) For any initial state, the average velocity of cluster 1 is

$$v_1 = \frac{1}{2(l_1 + l_2)},$$

and the average velocity of cluster 2 is

$$v_2 = \frac{2}{l_1 + l_2}.$$

7) For any initial state, the average velocity of cluster 1 is

$$v_1 = \frac{1}{2},$$

and the average velocity of cluster 2 is

$$v_2 = 1.$$

8) Depending on the initial state, the system enters either the state of free-flow traffic or the state of traffic breakdown.

9) Depending on the initial state, either the system enters the state of traffic breakdown or the average velocity of the clusters is

$$v = \frac{1}{l_1 + l_2 + 2d}.$$

10) The system enters the state of traffic breakdown from any initial state.

5. Conclusion

A dynamical system belonging to the class of contour networks of Buslaev type, depending on the orientation, has been investigated. Two versions of the system are considered which differ by the ratio of the directions of motion of the clusters along the contours. The behavior of these versions of the system differs qualitatively. The behavior of the two-contour system with one-directional motion with different values of the system parameters is more diverse than that of the system with codirectional motion. In particular, in the case of one-directional motion the average velocity of the clusters depends, in the general case, on the initial state with fixed

values of the system parameters. In contrast to the case of clusters equal in length, which was considered previously and in which the average velocity is always the same for both clusters, for different lengths of the clusters, depending on the initial state, either the average velocity of the clusters is the same or the average velocity of one cluster is twice as large as that of the other. We note that the dependence of the average velocity of the clusters on the initial state distinguishes the two-contour system with one-directional motion from simple traffic models, which are systems with particles moving on a one-dimensional infinite or closed grid [1–5, 9], in which the average velocity of the particles does not depend on the initial state. But the mathematical traffic models [28] which are investigated by simulation modeling and agree with the three-phase traffic flow theory developed by B. S. Kerner, exhibit, as do the contour networks, a dependence of the average velocity of motion on the initial state. For stochastic versions of the contour networks and for models that agree with the three-phase theory, the average velocity of motion also depends on how the random process proceeds with the initial state being fixed.

References

- [1] Wolfram, S., Statistical Mechanics of Cellular Automata, *Rev. Mod. Phys.*, 1983, vol. 55, no. 3, pp. 601–644.
- [2] Spitzer, F., Interaction of Markov Processes, *Adv. Math.*, 1970, vol. 5, no. 2, pp. 246–290.
- [3] Belitsky, V. and Ferrari, P. A., Invariant Measures and Convergence Properties for Cellular Automaton 184 and Related Processes, *J. Stat. Phys.*, 2005, vol. 118, nos. 3–4, pp. 589–523.
- [4] Gray, L. and Griffeath, D., The Ergodic Theory of Traffic Jams, *J. Statist. Phys.*, 2001, vol. 105, nos. 3–4, pp. 413–452.
- [5] Kanai, M., Nishinari, K., and Tokihiro, T., Exact Solution and Asymptotic Behavior of the Asymmetric Behavior of the Asymmetric Simple Exclusion Process on a Ring, *J. Phys. A*, 2009, vol. 39, no. 29, 10 pp.
- [6] Blank, M., Metric Properties of Discrete Time Exclusion Type Processes in Continuum, *J. Stat. Phys.*, 2010, vol. 140, no. 1, pp. 170–197.
- [7] Middleton, A., Biham, O., and Levine, D., Self-Organization and a Dynamic Transition in Traffic-Flow Models, *Phys. Rev. A*, 1992, vol. 46, no. 10, R6124–R6127.
- [8] D'Souza, R. M., Coexisting Phases and Lattice Dependence of a Cellular Automaton Model for Traffic Flow, *Phys. Rev. E*, 2005, vol. 71, no. 6, 066112, 10 pp.
- [9] Yashina, M. and Tatashev, A., Traffic Model Based on Synchronous and Asynchronous Exclusion Processes, *Math. Methods Appl. Sci.*, 2020, vol. 43, no. 14, pp. 8136–8146.
- [10] Bugaev, A. S., Buslaev, A. P., Kozlov, V. V., and Yashina, M. V., Distributed Problems of Monitoring and Modern Approaches to Traffic Modeling, in *Proc. of the 14th Internat. IEEE Conf. on Intelligent Transportation Systems (ITSC, Washington, D.C., Oct 2011)*, pp. 477–481.
- [11] Kozlov, V. V., Buslaev, A. P., and Tatashev, A. G., On Synergy of Totally Connected Flows on Chain-mails, in *Proc. of the 13th Internat. Conf. on Computational and Mathematical Methods in Science and Engineering (Almeria, Spain, June 2013): Vol. 3*, pp. 861–874.
- [12] Buslaev, A. P. and Tatashev, A. G., Exact Results for Discrete Dynamical Systems on a Pair of Contours, *Math. Methods Appl. Sci.*, 2018, vol. 41, no. 17, pp. 7283–7294.
- [13] Tatashev, A. G. and Yashina, M. V., Behavior of Continuous Two-Contours System, *WSEAS Trans. Math.*, 2019, vol. 18, Art. 4, pp. 28–36.
- [14] Tatashev, A. G. and Yashina, M. V., Self-Organization of Two-Contours Dynamical System with Common Node and Cross Movement, *WSEAS Trans. Math.*, 2019, vol. 18, Art. 45, pp. 373–378.
- [15] Myshkis, P. A., Tatashev, A. G., and Yashina, M. V., Cluster Motion in a Two-Contour System with Priority Rule for Conflict Resolution, *J. Comput. Syst. Sci. Int.*, 2020, vol. 59, no. 3, pp. 311–321.

- [16] Yashina, M. V. and Tatashev, A. G., Spectral Cycles and Average Velocity of Clusters in Discrete Two-Contours System with Two Nodes, *Math. Methods Appl. Sci.*, 2020, vol. 43, no. 7, pp. 4303–4316.
- [17] Yashina, M. V. and Tatashev, A. G., Uniform Cluster Traffic Model on Closed Two-Contours System with Two Non-symmetrical Common Nodes, in *Traffic and Granular Flow 2019*, I. Zuriguel, A. Garcimartin, R. Cruz (Eds.), Springer Proc. in Phys., vol. 252, Cham: Springer, 2020, pp. 583–588.
- [18] Yashina, M. V., Tatashev, A. G., and Fomina, M. Yu., Optimization of Velocity Mode in Buslaev Two-Contour Networks via Competition Resolution Rules, *Int. J. Interact. Mob. Technol.*, 2020, vol. 14, no. 10, pp. 61–73.
- [19] Buslaev, A. P., Tatashev, A. G., and Yashina, M. V., Flow Spectrum of Closed/Open Contours Chains, *Comput. Math. Methods*, 2021, vol. 3, no. 3, e1087, 13 pp.
- [20] Buslaev, A. P., Tatashev, A. G., and Yashina, M. V., About Synergy of Flows on Flower, in *Dependability Engineering and Complex Systems: Proc. of the 11th Internat. Conf. on Dependability and Complex Systems (Brunow, Poland, 2016)*, W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, J. Kacprzyk (Eds.), Adv. Intell. Syst. Comput., vol. 470, Cham: Springer, 2016, pp. 75–84.
- [21] Kozlov, V. V., Buslaev, A. P., and Tatashev, A. G., Monotonic Walks on a Necklace and a Coloured Dynamical Vector, *Int. J. Comput. Math.*, 2015, vol. 92, no. 9, pp. 1910–1920.
- [22] Tatashev, A. G. and Yashina, M. V., Spectrum of Elementary Cellular Automata and Closed Chains of Contours, *Machines*, 2019, vol. 7, no. 2, Art. No. 28, 12 pp.
- [23] Buslaev, A. P., Fomina, M. Yu., Tatashev, A. G., and Yashina, M. V., On Discrete Flow Networks Model Spectra: Statements, Simulation, Hypotheses, *J. Phys. Conf. Ser.*, 2018, vol. 1053, no. 1, 012034, 7 pp.
- [24] Tatashev, A. G. and Yashina, M. V., Discrete Open Buslaev Chain with Heterogeneous Loading, in *Proc. of the 7th Internat. Conf. on Control, Mechatronics and Automation (ICCMA, Nov 6–8, 2019, Delft, Netherlands)*, pp. 283–288.
- [25] Buslaev, A. P., Tatashev, A. G., and Yashina, M. V., Qualitative Properties of Dynamical System on Toroidal Chainmail, in *AIP Conf. Proc.*, 2013, vol. 1558, no. 1, pp. 1144–1147.
- [26] Kuteynikov, I. A., Tatashev, A. G., and Yashina, M. V., On Properties of Closed/Open Two-Dimensional Network-Chainmail with Different Rules of Particle Movement, *Period. Eng. Nat. Sci.*, 2019, vol. 7, no. 1, pp. 447–457.
- [27] Kozlov, V. V., Buslaev, A. P., Tatashev, A. G., and Yashina, M. V., Dynamical Systems on Honeycombs, in *Traffic and Granular Flow'13*, M. Chraïbi, M. Boltes, A. Schadschneider, A. Seyfried (Eds.), Cham: Springer, 2015, pp. 441–452.
- [28] Kerner, B. S., Klenov, S. L., and Wolf, D. E., Cellular Automata Approach to Three-Phase Traffic Theory, *Phys. A.*, 2002, vol. 35, no. 47, pp. 9971–10014.