MATHEMATICAL PROBLEMS OF NONLINEARITY

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On a Class of Isotopic Connectivity of Gradient-like Maps of the 2-sphere with Saddles of Negative Orientation Type

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We consider the class G of gradient-like orientation-preserving diffeomorphisms of the 2-sphere with saddles of negative orientation type. We show that for every diffeomorphism $f \in G$ every saddle point is fixed. We show that there are exactly three equivalence classes (up to topological conjugacy) $G = G_1 \cup G_2 \cup G_3$ where a diffeomorphism $f_1 \in G_1$ has exactly one saddle and three nodes (one fixed source and two periodic sinks); a diffeomorphism $f_2 \in G_2$ has exactly two saddles and four nodes (two periodic sources and two periodic sinks) and a diffeomorphism $f_3 \in G_3$ is topologically conjugate to a diffeomorphism f_1^{-1} . The main result is the proof that every diffeomorphism $f \in G$ can be connected to the “source-sink” diffeomorphism by a stable arc and this arc contains at most finitely many points of period-doubling bifurcations.

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1. Introduction

In 1976 S. Newhouse, J. Palis and F. Takens [1] introduced the notion of a stable arc connecting two structurally stable systems on a manifold, the arc being stable in the sense that any nearby arc exhibits the same type of behavior. Soon S. Newhouse and M. Peixoto [2] showed the existence of a stable arc between any two Morse–Smale flows. On the other hand, there are examples of Morse–Smale diffeomorphisms on manifolds of any dimension that cannot be joined by a stable arc. Therefore, it is important to find an invariant which defines the equivalence class of Morse–Smale diffeomorphisms with respect to relation of connectivity by a stable arc (*stable connectivity component*).

For orientation-preserving structurally stable maps of the circle the Poincaré rotation number is this invariant [3]. P. Blanchard found [4] some necessary conditions for existence of a stable arc connecting two Morse–Smale diffeomorphisms on surfaces. These conditions, in particular, imply that even on the 2-sphere there are infinitely many stable connectivity components. The simplest example of a 2-dimensional diffeomorphism is the “source-sink” system on the 2-sphere. The main result of this paper is the proof that a gradient-like orientation-preserving diffeomorphism of the 2-sphere with saddles of negative orientation type can be joined by a stable arc to the “source-sink” diffeomorphism.

2. Dynamics of G diffeomorphisms

Let M^n be a smooth closed orientable manifold with a metric d and let $f : M^n \rightarrow M^n$ be an orientation-preserving diffeomorphism. A point $x \in M^n$ is said to be *wandering* for f if there is an open neighborhood U_x of x such that $f^n(U_x) \cap U_x = \emptyset$ for all $n \in \mathbb{N}$. Otherwise the point x is called *nonwandering*. The set of all nonwandering points of f is called the *nonwandering set*, denoted by Ω_f .

If Ω_f is finite, then each $p \in \Omega_f$ is periodic and its period $m_p \in \mathbb{N}$. A point $p \in \Omega_f$ is called *hyperbolic* if the Jacobian matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ has no eigenvalues on the unit circle. If all the eigenvalues lie inside (outside) the unit circle, then p is a *sink* (a *source*). Sinks and sources are called *nodes*. If a hyperbolic periodic point is not a node, then it is called a *saddle*.

Fig. 1. (a) source, (b) sink, (c) saddle.

The hyperbolic structure of a periodic point p implies the existence of the *stable* W_p^s and the *unstable* W_p^u manifolds defined by

$$W_p^s = \{x \in S_g : \lim_{k \rightarrow +\infty} d(f^{k \cdot \text{per}(p)}(x), p) = 0\},$$

$$W_p^u = \{x \in S_g : \lim_{k \rightarrow +\infty} d(f^{-k \cdot \text{per}(p)}(x), p) = 0\};$$

W_p^s and W_p^u are diffeomorphic to $\mathbb{R}^{n-q_p}, \mathbb{R}^{q_p}$, respectively, where q_p is the number of the eigenvalues of the Jacobian matrix outside the unit circle.

Stable and unstable manifolds are called *invariant manifolds*. A path-component of the set $W_p^u \setminus p$ ($W_p^s \setminus p$) is called an *unstable (stable) separatrix*.

Periodic data of the periodic orbit $\mathcal{O}_p$ of a periodic point p is the collection (m_p, q_p, ν_p) where m_p is the period of p , $q_p = \dim W_p^u$ and ν_p is the type of orientation, meaning p : $p = +1$ ($p = -1$) if $f^{m_p}|_{W_p^u}$ preserves (reverses) the orientation. For orientation-preserving diffeomorphisms the orientation type of each node is $+1$, whereas the orientation type of a saddle can be $+1$ as well as -1 .

A diffeomorphism $f : M^n \rightarrow M^n$ is called a *Morse–Smale diffeomorphism* if

- 1) the nonwandering set Ω_f consists of a finite number of hyperbolic orbits;
- 2) the manifolds W_p^s, W_q^u of each two distinct nonwandering points p and q intersect transversally.

A Morse–Smale diffeomorphism is called *gradient-like* if from $W_{\sigma_1}^s \cap W_{\sigma_2}^u \neq \emptyset$ for two distinct points $\sigma_1, \sigma_2 \in \Omega_f$ it follows that $\dim W_{\sigma_1}^u < \dim W_{\sigma_2}^u$.

A *Morse–Smale flow* on a manifold M^n is defined in the same way. A Morse–Smale flow is said to be *gradient-like* if it has no periodic trajectories.

For $n = 2$ the dynamics of gradient-like diffeomorphisms is closely interconnected with the dynamics of periodic homeomorphisms. Recall that a homeomorphism $\phi : M^2 \rightarrow M^2$ is *periodic of order* $m \in \mathbb{N}$ if $\phi^m = \text{id}$ and $\phi^\mu \neq \text{id}$ for each natural $\mu < m$.

Proposition 1 ([5], Theorem 3.3). *Every orientation-preserving gradient-like diffeomorphism $f : M^2 \rightarrow M^2$ is topologically conjugate to the composition of a periodic homeomorphism and the time-1 map of a gradient-like flow.*

According to the classification by B. von Kerekjarto [6], periodic points of an orientation-preserving periodic homeomorphism of period m on the 2-sphere can only be of periods 1 and m , whereas the set of fixed points of this diffeomorphism is not empty. Hence, he have the following corollary of Proposition 1.

Proposition 2. *Periodic points of an orientation-preserving gradient-like diffeomorphism of the 2-sphere can only be of period 1 or of period m (the case $m = 1$ is possible). The set of fixed points of this diffeomorphism is not empty.*

For an orientation-preserving gradient-like diffeomorphism $f : M^2 \rightarrow M^2$ the following proposition is also true.

Proposition 3 ([5], Lemmas 3.1, 3.3). *Let $f : M^2 \rightarrow M^2$ be an orientation-preserving gradient-like diffeomorphism and let m_f be the least natural for which $\Omega_{f^{m_f}}$ consists of the fixed points of positive orientation type. Then the period of each saddle separatrix of f equals m_f .*

Thus, from Propositions 2 and 3 we have

Proposition 4. *Let f be an orientation-preserving gradient-like diffeomorphism of the 2-sphere. Then*

- 1) $m_f = m$;
- 2) *each saddle with negative orientation type is fixed.*

Recall that the 2-sphere is a manifold diffeomorphic to $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Denote by G the set of gradient-like orientation-preserving diffeomorphisms of the 2-sphere with saddles of negative orientation type.

In this paper we give a topological classification of diffeomorphisms of G , namely, we prove the following theorem.

Theorem 1. *There are three equivalence classes $G = G_1 \cup G_2 \cup G_3$ up to topological conjugacy, where a diffeomorphism $f_1 \in G_1$ has exactly one saddle and three nodes: one fixed source and two periodic sinks (Fig. 2); a diffeomorphism $f_2 \in G_2$ has exactly two saddles and four nodes: two periodic sources and two periodic sinks (Fig. 3); a diffeomorphism $f_3 \in G_3$ is topologically conjugate to a diffeomorphism f_1^{-1} (Fig. 4).*

Fig. 2. $f_1 \in G_1$.

Fig. 3. $f_2 \in G_2$.

Fig. 4. $f_3 \in G_3$.

3. Stable arcs in the space of diffeomorphisms

Consider a 1-parametric family of diffeomorphisms (an *arc*) $\varphi_t : M^n \rightarrow M^n, t \in [0, 1]$. Denote by $\mathcal{Q}$ the set of arcs $\{\varphi_t\}$ whose end points are Morse-Smale diffeomorphisms and such that:

- 1) every diffeomorphism φ_t has a finite limit set for every $t \in [0, 1]$;
- 2) the arc $\{\varphi_t\}$ contains a finite set of bifurcation points $b_1, \dots, b_m \in (0, 1)$.

Following [7], an arc $\{\varphi_t\}$ is called *stable* if it is an inner point of the equivalence class with respect to the following relation: two arcs $\{\varphi_t\}, \{\varphi'_t\} \in \mathcal{Q}$ are called *conjugate* if there are homeomorphisms $h : [0, 1] \rightarrow [0, 1], H_t : M^n \rightarrow M^n$ such that $h(b_i) = b'_i, i \in \{1, \dots, m\}$, $H_t \varphi_t = \varphi'_{h(t)} H_t, t \in [0, 1]$ and H_t continuously depend on t .

It is shown in [7] that an arc $\{\varphi_t\} \in \mathcal{Q}$ is stable if and only if

- each diffeomorphism $\varphi_{b_i}, i \in \{1, \dots, m\}$ has no cycles and it has exactly one nonhyperbolic periodic orbit, this orbit being a flip or noncritical saddle-node, and the arc passes through the bifurcation point in the typical way;
- stable and unstable manifolds of any two periodic points of the diffeomorphism $\varphi_t, t \in [0, 1]$ intersect transversally (Fig. 5).

Fig. 5. An arc from $\mathcal{Q}$.

We say that two diffeomorphisms f_0, f_1 are in the same class of *stable isotopic connectivity* if they can be joined in the space of diffeomorphisms by an arc of $\mathcal{Q}$.

An arc $\{\varphi_t\} \in \mathcal{Q}$ is said to *pass through a saddle-node bifurcation* φ_{b_i} in the typical way if in some neighborhood of the nonhyperbolic point (p, b_i) the arc φ_t is conjugate to

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \left(x_1 + \frac{1}{2}x_1^2 + \tilde{t}, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2} \right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

Fig. 6. Saddle-node.

An arc $\{\varphi_t\} \in \mathcal{Q}$ is said to *pass through period-doubling bifurcation (flip)* φ_{b_i} in the typical way if in some neighborhood of the nonhyperbolic point (p, b_i) the arc φ_t is conjugate to the arc

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \left(-x_1(1 \pm \tilde{t}) + x_1^3, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2} \right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_i| < 1/2$, $|\tilde{t}| < 1/10$.

The main result of this paper is the following theorem.

Theorem 2. Every diffeomorphism $f \in G$ is connected to the “source-sink” diffeomorphism by a stable arc.

Fig. 7. A period-doubling bifurcation (flip).

4. Proof of Theorem 1

In this section we prove that up to topological conjugacy there are exactly three equivalence classes $G = G_1 \cup G_2 \cup G_3$ where a diffeomorphism $f_1 \in G_1$ has exactly one saddle and three nodes — one fixed source and two periodic sinks; a diffeomorphism $f_2 \in G_2$ has exactly two saddles and four nodes — two periodic sources and two periodic sinks (Fig. 3); a diffeomorphism $f_3 \in G_3$ is topologically conjugate to a diffeomorphism f_1^{-1} (Fig. 4).

Proof.

Let $f \in G$. It follows from the properties of G that any saddle σ of f is of negative orientation type. Then, according to Proposition 4, the point σ is fixed. Denote by $\ell_\sigma^1, \ell_\sigma^2$ the stable separatrices of σ and denote by $\gamma_\sigma^1, \gamma_\sigma^2$ the unstable separatrices of σ . Consider two cases

- 1) there is a saddle σ such that $cl(\ell_\sigma^1) \setminus \ell_\sigma^1 = cl(\ell_\sigma^2) \setminus \ell_\sigma^2$ or $cl(\gamma_\sigma^1) \setminus \gamma_\sigma^1 = cl(\gamma_\sigma^2) \setminus \gamma_\sigma^2$;
- 2) there are no saddles satisfying 1).

In the case 1) to be definite consider $cl(\ell_\sigma^1) \setminus \ell_\sigma^1 = cl(\ell_\sigma^2) \setminus \ell_\sigma^2$ (Fig. 2). Since f is gradient-like, the set $cl(\ell_\sigma^1) \setminus \ell_\sigma^1 = cl(\ell_\sigma^2) \setminus \ell_\sigma^2$ is the source; denote it by α . Thus,

$$\alpha = cl(\ell_\sigma^1) \setminus \ell_\sigma^1 = cl(\ell_\sigma^2) \setminus \ell_\sigma^2$$

and then the set $K = \alpha \cup \ell_\sigma^1 \cup \ell_\sigma^2 \cup \sigma$ is the circle which is invariant under f . On the ambient 2-sphere this circle is the border of two 2-disks D_1, D_2 such that $\gamma_\sigma^1 \subset D_1, \gamma_\sigma^2 \subset D_2$. Since σ is of negative orientation type, we have $f(D_1) = D_2, f(D_2) = D_1$.

Hence, the disk D_1 is of period two and, consequently, cannot contain saddle points because they should be fixed. The Euler characteristic of the 2-sphere equals 2, therefore, there are exactly two sinks $\omega_1 \in D_1, \omega_2 \in D_2$ lying in the closures of the unstable separatrices $\gamma_\sigma^1, \gamma_\sigma^2$, respectively, and such that $f(\omega_1) = \omega_2, f(\omega_2) = \omega_1$.

So, in this case the nonwandering set of f is $\Omega_f = \{\alpha, \omega, f(\omega), \sigma\}$. Having applied the same reasoning for the case $cl(\gamma_\sigma^1) \setminus \gamma_\sigma^1 = cl(\gamma_\sigma^2) \setminus \gamma_\sigma^2$, we come to a diffeomorphism conjugate to f^{-1} .

Consider case 2). Let σ_1 be a saddle of f . Then all the four saddle separatrices have four distinct nodes in their closures. Let $cl(\ell_{\sigma_1}^1) \setminus \ell_{\sigma_1}^1 = \alpha, cl(\ell_{\sigma_1}^2) \setminus \ell_{\sigma_1}^2 = f(\alpha)$ and $cl(\gamma_{\sigma_1}^1) \setminus \gamma_{\sigma_1}^1 = \omega, cl(\gamma_{\sigma_1}^2) \setminus \gamma_{\sigma_1}^2 = f(\omega)$ (Fig. 8).

Since the Euler characteristic of the 2-sphere equals 2, the nonwandering set of f contains at least one saddle distinct from σ_1 . Since the set $\mathbb{S}^2 \setminus P$ is a connected manifold (here $\mathbb{S}^2$ is the 2-sphere and P is the set of sources of f), the union of the unstable manifolds of all saddles is connected. Therefore, there exists a saddle σ_2 such that

$$cl(\gamma_{\sigma_2}^1) \setminus \gamma_{\sigma_2}^1 = cl(\gamma_{\sigma_1}^1) \setminus \gamma_{\sigma_1}^1 = \omega.$$

Fig. 8. The saddle σ_1 .

Since the saddles are of negative orientation type, we have $f(\gamma^1_{\sigma_2}) = \gamma^2_{\sigma_2}$ and

$$cl(\gamma^2_{\sigma_2}) \setminus \gamma^2_{\sigma_2} = cl(\gamma^2_{\sigma_1}) \setminus \gamma^2_{\sigma_1} = f(\omega).$$

Thus, the set $K = \omega \cup f(\omega) \cup \gamma^1_{\sigma_1} \cup \gamma^2_{\sigma_1} \cup \gamma^1_{\sigma_2} \cup \gamma^2_{\sigma_2}$ is an f -invariant circle. By the same reasoning as in case 1) we find that there are no saddles of f distinct from σ_1, σ_2 and, hence, $\Omega_f = \{\sigma_1, \sigma_2, \omega, f(\omega), \alpha, f(\alpha)\}$. $\square$

5. Reduction of diffeomorphisms to the standard form

To decrease the number of periodic orbits of a diffeomorphism $f \in G$, we are going to construct an arc that passes through a period-doubling bifurcation in the typical way. In order to do this, the diffeomorphisms must be reduced to the standard form. To that end we reduce the dynamics in the neighborhood of a sink to the canonical contraction (Lemma 1) and we map the unstable saddle separatrix in the basin of the canonical sink to a smooth arc (Lemma 2). An important tool for all the constructions is

Proposition 5 (Thom's isotopy extension theorem, [8], Theorem 5.8). *Let Y be a smooth manifold without border, let X be a smooth submanifold of Y and let $\{f_t : X \rightarrow Y, t \in [0, 1]\}$ be a smooth isotopy such that f_0 is the inclusion map of X into Y . Then for every compact set $A \subset Y$ containing the support¹ $\text{supp}\{f_t\}$ there is a smooth isotopy $\{g_t \in \text{Diff}(Y), t \in [0, 1]\}$ such that $g_0 = \text{id}$, $g_t|_X = f_t|_X$ for every $t \in [0, 1]$ and $\text{supp}\{g_t\}$ lies in A .*

The details of the construction.

Denote by $O(0, 0)$ the origin of coordinates on the plane $\mathbb{R}^2$. For every $r > 0$ let $B_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. Denote by $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the diffeomorphism defined by $g(x, y) = (x/2, y/2)$.

Lemma 1. *Let a diffeomorphism $\varphi_0 : M^2 \rightarrow M^2$ have a hyperbolic sink ω_0 of period m and let (U_0, ψ_0) be a local chart of the manifold M^2 such that $\omega_0 \in U_0$, $\psi_0(\omega_0) = O$ and $\varphi_0^m(U_0) \subset U_0$.*

¹The support $\text{supp}\{f_t\}$ of an isotopy $\{f_t\}$ is the closure of the set $\{x \in X : f_t(x) \neq f_0(x) \text{ for some } t \in [0, 1]\}$.

Then there are neighborhoods U_1, U_2 of ω_0 such that $U_2 \subset U_1 \subset U_0$ and there is a bifurcation free arc $\varphi_t : M^2 \rightarrow M^2, t \in [0, 1]$ such that

- 1) the diffeomorphism $\varphi_t, t \in [0, 1]$ coincides with φ_0 outside the set $\bigcup_{k=0}^{m-1} \varphi_0^k(U_1)$ and the set $\bigcup_{k=0}^{m-1} \varphi_0^k(\omega_0)$ is the hyperbolic sink orbit of period m for every φ_t ;
- 2) the diffeomorphism $\psi_0 \varphi_1^m \psi_0^{-1}$ coincides with g on the set $\psi_0(U_2)$.

Proof.

Let $\phi_0 = \varphi_0^m$ and $\bar{\phi}_0 = \psi_0 \phi_0 \psi_0^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. To prove the lemma it suffices to construct neighborhoods $\bar{U}_1, \bar{U}_2$ of O such that $\bar{U}_2 \subset \bar{U}_1 \subset \mathbb{R}^2$ and a bifurcation free arc $\bar{\phi}_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2, t \in [0, 1]$ such that

- 1') the diffeomorphism $\bar{\phi}_t, t \in [0, 1]$ coincides with $\bar{\phi}_0$ outside the set $\bar{U}_1$ and O is the hyperbolic sink for all diffeomorphisms $\bar{\phi}_t$;
- 2') the diffeomorphism $\bar{\phi}_1$ coincides with g on the set $\bar{U}_2$.

Indeed, one can get the arc $\varphi_t : M^2 \rightarrow M^2$ from the arc $\bar{\phi}_t$ in the following way. Let $U_1 = \psi_0^{-1}(\bar{U}_1), U_2 = \psi_0^{-1}(\bar{U}_2)$ and $\phi_t = \psi_0^{-1} \bar{\phi}_t \psi_0$ on U_1 . Then for every $t \in [0, 1]$ φ_t coincides with φ_0 outside $\bigcup_{k=0}^{m-1} \varphi_0^k(U_1)$, $\varphi_t(z) = \varphi_0(z)$ for $z \in \varphi_0^k(U_1), k \in \{0, \dots, m-2\}$ and $\varphi_t(z) = \phi_t(\varphi_0^{1-m}(z))$ for $z \in \varphi_0^{m-1}(U_1)$.

Now we construct the arc $\bar{\phi}_t$.

Denote by $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the differential of the map $\bar{\phi}_0$ at the point O . Since O is a hyperbolic sink of $\bar{\phi}_0$, there exists a number $0 < \lambda < 1$ such that for every $v \in \mathbb{R}^2$

$$\|Q(v)\| \leq \lambda \|v\|,$$

where $\|\cdot\|$ is the Lyapunov norm in $\mathbb{R}^2$ (see, for example, [9]). For every $r > 0$ let $\tilde{B}_r = \{v \in \mathbb{R}^2 : \|v\| < r\}$. Since $\bar{\phi}_0$ is a diffeomorphism with the fixed point O and since all norms in $\mathbb{R}^2$ are equivalent, there is $\varepsilon_1 > 0$ such that for every $v \in \tilde{B}_{\varepsilon_1}$ the equation $\bar{\phi}_0(v) = Q(v) + \|v\|a(v)$ holds, where $\lim_{\|v\| \rightarrow 0} a(v) = 0$. Therefore, there exists $0 < \varepsilon_2 < \varepsilon_1$ such that for every

nonzero $v \in \tilde{B}_{\varepsilon_2}$ the inequality $\|\bar{\phi}_0(v)\| < \|v\|$ is true. Let $\bar{U}_1 = \tilde{B}_{\varepsilon_2}$. Since $g(cl \bar{U}_1) \subset \bar{U}_1$, the family of the maps $\chi_t : \bar{U}_1 \rightarrow \mathbb{R}^2$ is well-defined on the set $\bar{U}_1$ by

$$\chi_t = (1-t)\bar{\phi}_0 + tg.$$

By construction, $\|\chi_t(v)\| < \|v\|$ for every nonzero $v \in \bar{U}_1$. Notice that the origin of coordinates is a fixed point of χ_t , $\chi_t(cl \bar{U}_1) \subset \bar{U}_1$ for every $t \in [0, 1]$ and the isotopy $\xi_t = \bar{\phi}_0^{-1} \chi_t$ joins the identity map with $\bar{\phi}_0^{-1}g$. Let $\bar{U}_2 = \bar{\phi}_0(\bar{U}_1)$ (Fig. 9). Then $\xi_t(cl \bar{U}_2) \subset \bar{U}_1$ for all $t \in [0, 1]$. By Proposition 5, there is an isotopy $\Xi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which coincides with ξ_t on $\bar{U}_2$ and which is the identity outside $\bar{U}_1$. The desired arc is defined by

$$\bar{\phi}_t = \bar{\phi}_0 \Xi_t.$$

□

Fig. 9. Illustration for Lemma 1.

Lemma 2. Let a diffeomorphism $\varphi_0 : M^2 \rightarrow M^2$ have a hyperbolic sink ω_0 and a hyperbolic saddle σ_0 such that the unstable separatrix γ_{φ_0} of σ_0 lies in the basin of the sink $W_{\omega_0}^s$ and let σ_0 and ω_0 be of the same period m . Let (U_0, ψ_0) be a local chart of the manifold M^2 for which $\omega_0 \in U_0$, $\psi_0(\omega_0) = O$ and $\varphi_0^m(U_0) \subset U_0$. Then there are neighborhoods V_1, V_2 of ω_0 such that $V_2 \subset V_1 \subset U_0$ and there is a bifurcation free arc $\varphi_t : M^2 \rightarrow M^2, t \in [0, 1]$ with the following properties:

- 1) the diffeomorphism $\varphi_t, t \in [0, 1]$ coincides with φ_0 outside the set $\bigcup_{k=0}^{m-1} \varphi_0^k(V_1)$ and the set $\bigcup_{k=0}^{m-1} \varphi_0^k(\omega_0)$ is the hyperbolic sink orbit of period m for every φ_t ;
- 2) $\psi_0(\gamma_{\varphi_1} \cap V_2) \subset OX_1$ where γ_{φ_1} is the unstable separatrix of σ_0 with respect to φ_1 (Fig. 10).

Fig. 10. Smoothing of the separatrix.

Proof.

Let $\phi_0 = \varphi_0^m$ and $\bar{\phi}_0 = \psi_0 \phi_0 \psi_0^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. By Lemma 1, without loss of generality, suppose $\bar{\phi}_0 = g$ on the disk B_{2r_0} for some $r_0 > 0$. Let $K_0 = B_{2r_0} \setminus B_{r_0}$ and $\gamma_{\bar{\phi}_0} = \psi_0(\gamma_{\varphi_0})$.

Denote by E_g the set of contractions $\bar{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ coinciding with $\bar{\phi}_0$ outside B_{2r_0} and coinciding with g on $B_{r_{\bar{\phi}}}$ where $r_{\bar{\phi}} \leq 2r_0$. For every $\bar{\phi} \in E_g$ let

$$\gamma_{\bar{\phi}} = \bigcup_{k \in \mathbb{Z}} \bar{\phi}^k(\gamma_{\bar{\phi}_0} \cap K_0).$$

By construction, the $\bar{\phi}$ -invariant curve $\gamma_{\bar{\phi}}$ coincides with the $\bar{\phi}_0$ -invariant curve $\gamma_{\bar{\phi}_0}$ outside the disk B_{r_0} . Then to prove the lemma it suffices to construct a curve of contractions $\bar{\phi}_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2, t \in [0, 1]$ such that

- 1') the diffeomorphism $\bar{\phi}_t, t \in [0, 1]$ coincides with $\bar{\phi}_0$ outside the set B_{r_0} ;
- 2') $(\gamma_{\bar{\phi}_1} \cap B_{r_{\bar{\phi}_1}}) \subset OX_1$.

The arc $\varphi_t : M^2 \rightarrow M^2$ is constructed from the arc $\bar{\phi}_t$ in the same way as in Lemma 1 if one sets $V_1 = \psi_0^{-1}(B_{r_0})$ and $V_2 = \psi_0^{-1}(B_{r_{\bar{\phi}_1}})$.

In order to construct the arc $\bar{\phi}_t$, we now introduce the following notations for an arbitrary diffeomorphism $\bar{\phi} \in E_g$.

Represent the 2-torus $\mathbb{T}^2$ as the orbit space of the diffeomorphism g on the set $\mathbb{R}^2 \setminus O$ and denote by $p : \mathbb{R}^2 \setminus O \rightarrow \mathbb{T}^2$ the natural projection. Let $\hat{a} = p(OX_1)$ and $\hat{b} = p(\mathbb{S}^1)$ be generators of $\mathbb{T}^2$, let $K_{\bar{\phi}} = B_{r_{\bar{\phi}}} \setminus B_{r_{\bar{\phi}}/2}$, $\hat{\gamma}_{\bar{\phi}} = p(\gamma_{\bar{\phi}} \cap K_{\bar{\phi}})$ and $\hat{\gamma}_{\bar{\phi}} = p(\gamma_{\bar{\phi}} \cap K_{\bar{\phi}})$. Then the curve $\hat{\gamma}_{\bar{\phi}}$ is the knot on the torus $\mathbb{T}^2$ with coordinates $\langle 1, -n_{\bar{\phi}} \rangle$, $n_{\bar{\phi}} \in \mathbb{Z}$ in the basis $\hat{a}, \hat{b}$ (see, for example, [5]).

The arc $\bar{\phi}_t$ is the smooth product of the arcs η_t and ζ_t where

- I) the arc $\eta_t, t \in [0, 1]$ consists of contractions, it coincides with $\bar{\phi}_0$ outside the set B_{r_0} and it joins the diffeomorphism $\eta_0 = \bar{\phi}_0$ with some diffeomorphism $\eta_1 \in E_g$ such that the knot $\hat{\gamma}_{\eta_1}$ has coordinates $\langle 1, 0 \rangle$ in the basis $\hat{a}, \hat{b}$;
- II) the arc $\zeta_t \in E_g, t \in [0, 1]$ joins the diffeomorphism $\zeta_0 = \eta_1$ to the diffeomorphism ζ_1 such that $\hat{\gamma}_{\zeta_1} = \hat{a}$.

Fig. 11. Illustration to Lemma 2, Part I.

I) If $n_{\bar{\phi}} = 0$, then let $\eta_t = \bar{\phi}_0$ for every $t \in [0, 1]$. Otherwise define the diffeomorphism $\theta_t, t \in [0, 1] : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ in such a way that $\theta_t(O) = O$ and

$$\theta_t(\rho e^{i\varphi}) = \begin{cases} \rho e^{i\varphi}, \rho > r_0, \\ \rho e^{i(\varphi + 4n_{\bar{\phi}}\pi t(1 - \frac{\rho}{r_0}))}, \frac{r_0}{2} \leq \rho \leq r_0; \\ \rho e^{i(\varphi + 2n_{\bar{\phi}}\pi t)}, \rho < \frac{r_0}{2}. \end{cases}$$

Then $\eta_t = \theta_t \bar{\phi}_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the desired arc (Fig. 11).

II) By construction, the diffeomorphism $\eta_1 \in E_g$ and the knot $\hat{\gamma}_{\eta_1}$ has the coordinates $\langle 1, 0 \rangle$ in the basis $\hat{a}, \hat{b}$. According to [10], there is a diffeomorphism $\hat{h} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ which is smoothly isotopic to the identity and such that $\hat{h}(\hat{\gamma}_{\eta_1}) = \hat{a}$. For $r > 0$ let $K_r = B_r \setminus B_{r/2}$. Pick an open cover $D = \{D_1, \dots, D_q\}$ of the torus $\mathbb{T}^2$ such that the connected component $\bar{D}_i$ of the set $p^{-1}(D_i)$ is a subset of K_{r_i} for some $r_i < \frac{r_0-1}{2}$ and $r_1 \leq r_0/2$. According to [11, Lemma de fragmentation] there are diffeomorphisms $\hat{w}_1, \dots, \hat{w}_q : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ smoothly isotopic to the identity, with the following properties:

- i) for each $i \in \{1, \dots, q\}$ there exists a smooth isotopy $\{\hat{w}_{i,t}\}$ which is the identity outside D_i and which joins the identity and $\hat{w}_i$;
- ii) $\hat{h} = \hat{w}_1 \dots \hat{w}_q$.

Let $w_{i,t} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism which coincides with $(p|_{K_{r_i}})^{-1} \hat{w}_{i,t} p$ on K_{r_i} and which coincides with the identity outside K_{r_i} (Fig. 12). Then the desired arc is defined by

$$\zeta_t = \nu_1 w_{1,t} \dots w_{q,t}.$$

□

Fig. 12. Illustration to Lemma 2, Part II.

6. Proof of Theorem 2

In this section we construct a stable arc that joins the diffeomorphism $f \in G$ and “source-sink” diffeomorphism on the 2-sphere. The key technical tool to do this is the following lemma.

Lemma 3. *Let the nonwandering set of a gradient-like diffeomorphism $\varphi : M^2 \rightarrow M^2$ contain a fixed saddle σ of negative orientation type and such that $cl(W_\sigma^u) = W_\sigma^u \cup \omega \cup \varphi(\omega)$,*

where ω is a sink of period two. Then there exists a stable arc $H_{\varphi, \varphi_\sigma^u, t}$ joining φ to a gradient-like diffeomorphism φ_σ^u whose nonwandering set is the same as Ω_f but with the points $\sigma, \omega, f(\omega)$ removed.

Then the desired arc is:

- 1) $H_{f_1, (f_1)_\sigma^u, t}$ for $f_1 \in G_1$;
- 2) $H_{f_2, (f_2)_{\sigma_1}^u, t} * H_{\psi_{\sigma_2}^u, t}^{-1}$ where $\psi = ((f_2)_{\sigma_1}^u)^{-1}$ for $f_2 \in G_2$;
- 3) $H_{f_3^{-1}, (f_3^{-1})_\sigma^u, t}^{-1}$ for $f_3 \in G_3$.

Proof of Lemma 3.

From Lemmas 1 and 2 it follows that there exists a local chart (U, ψ) of the manifold M^2 such that $\omega \in U$, $\psi(\omega) = O$, $f^2(U) \subset U$ and $\psi(\gamma \cap U) \subset OX_1$. Let $A_f = W_\sigma^u \cup \omega \cup f(\omega)$ and $l = W_\sigma^u \cup \psi^{-1}(OX_1) \cup f(\psi^{-1}(OX_1))$. Then l is a smooth curve which contains A_f , for which the points $\omega, f(\omega)$ are interior and $f(l) \subset l$. Let $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| \leq \frac{1}{2}\}$. Define the diffeomorphism $\tilde{\varphi} : \Pi_1 \rightarrow \mathbb{R}^2$ by

$$\tilde{\varphi}(x_1, x_2) = \left(-\frac{11}{10}x_1 + x_1^3, -\frac{x_2}{2} \right).$$

By construction, $\tilde{\varphi}(\Pi_1) \subset \text{int } \Pi_1$. The diffeomorphism $\tilde{\varphi}$ has the saddle O and the periodic sink orbit $\{P_0, \tilde{\varphi}(P_0)\}$ where $P_0(-x_0, 0)$, $\tilde{\varphi}(P_0) = (x_0, 0)$, $x_0 \in (0, 1/2)$. Let $\Pi_2 = \tilde{\varphi}(\Pi_1)$. Pick a closed neighborhood V of the set A_f and pick a diffeomorphism $\beta : V \rightarrow \Pi_1$ such that $f(V) \subset \text{int } V$, $\beta(l \cap V) = OX_1 \cap \Pi_1$, $\beta(f(V)) = \Pi_2$, $\beta(\omega) = P_0$ and $\beta(f(\omega)) = \tilde{\varphi}(P_0)$ (Fig. 13). Let $\tilde{f} = \beta f \beta^{-1} : \Pi_1 \rightarrow \Pi_2$. Then the family of maps $\chi_t : \Pi_2 \rightarrow \mathbb{R}^2$ is well-defined on the set Π_2 by

$$\chi_t = (1-t)\tilde{f} + t\tilde{\varphi}.$$

Fig. 13. Illustration to Lemma 3.

By construction, $\chi_t(\Pi_2) \subset \text{int } \Pi_2$ for every $t \in [0, 1]$. Notice that the origin of coordinates is the fixed saddle for the diffeomorphism χ_t and the points $P_0, \tilde{\varphi}(P_0)$ form the sink orbit. The isotopy $\xi_t = \tilde{f}^{-1}\chi_t|_{\Pi_2}$ joins the identity map and the diffeomorphism $\tilde{f}^{-1}\tilde{\varphi}$ and $\xi_t(\Pi_2) \subset \text{int } \Pi_1$.

By Proposition 5, there is an isotopy $\Xi_t : \Pi_1 \rightarrow \Pi_1$ which coincides with ξ_t on Π_2 and which is the identity on $\partial\Pi_1$. Let

$$\tilde{h}_t = \tilde{f}\Xi_t.$$

Notice that $\tilde{h}_1 = \tilde{\varphi}$ on Π_2 . Let $\Pi_3 = \tilde{\varphi}(\Pi_2)$. Define the arc $\eta_t : \Pi_3 \rightarrow \mathbb{R}^2$ by

$$\eta_t(x_1, x_2) = \left(-x_1 \left(1 + \frac{1}{10}(1 - 2t) \right) + x_1^3, -\frac{x_2}{2} \right).$$

By construction, $\eta_t(\Pi_3) \subset \text{int } \Pi_3$ for every $t \in [0, 1]$ and the isotopy $\zeta_t = \tilde{\varphi}^{-1}\eta_t$ joins the identity map with the diffeomorphism $\tilde{\varphi}^{-1}\eta_1$ and $\zeta_t(\Pi_3) \subset \text{int } \Pi_2$. By Proposition 5, there is an isotopy $\theta_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which coincides with ζ_t on Π_3 and which is the identity outside Π_2 . Let

$$\tilde{\Theta}_t = \tilde{\varphi}\theta_t.$$

Then the desired arc $H_{\varphi, \varphi_\sigma^u, t}$ is the product of the arcs $h_t, \Theta_t : M^2 \rightarrow M^2$ where h_t coincides with f outside V , $h_t(z) = \beta^{-1}(\tilde{h}_t(\beta(z)))$ for $z \in V$ and β_t coincides with h_1 outside $h_1(V)$, $\Theta_t(z) = \beta^{-1}(\tilde{\Theta}_t(\beta(z)))$ for $z \in h_1(V)$. $\square$

References

- [1] Newhouse, S., Palis, J., and Takens, F., Stable Arcs of Diffeomorphisms, *Bull. Amer. Math. Soc.*, 1976, vol. 82, no. 3, pp. 499–502.
- [2] Newhouse, S. and Peixoto, M. M., There Is a Simple Arc Joining Any Two Morse–Smale Flows, in *Trois études en dynamique qualitative*, Astérisque, vol. 31, Paris: Soc. Math. France, 1976, pp. 15–41.
- [3] Nozdrinova, E. V., Rotation Number As a Complete Topological Invariant of a Simple Isotopic Class of Rough Transformations of a Circle, *Russian J. Nonlinear Dyn.*, 2018, vol. 14, no. 4, pp. 543–551.
- [4] Blanchard, P. R., Invariants of the NPT Isotopy Classes of Morse–Smale Diffeomorphisms of Surfaces, *Duke Math. J.*, 1980, vol. 47, no. 1, pp. 33–46.
- [5] Grines, V., Medvedev, T., and Pochinka, O., *Dynamical Systems on 2- and 3-Manifolds*, Dev. Math., vol. 46, New York: Springer, 2016.
- [6] von Kerékjártó, B., Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche, *Math. Ann.*, 1919, vol. 80, no. 1, pp. 36–38.
- [7] Newhouse, S., Palis, J., and Takens, F., Bifurcations and Stability of Families of Diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, 1983, No. 57, pp. 5–71.
- [8] Milnor, J., *Lectures on the h-Cobordism Theorem*, Princeton, N.J.: Princeton Univ. Press, 1965.
- [9] Katok, A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia Math. Appl., vol. 54, Cambridge: Cambridge Univ. Press, 1995.
- [10] Rolfsen, D., *Knots and Links*, Math. Lect. Ser., No. 7, Berkeley, Calif.: Publish or Perish, Inc., 1976.
- [11] Banyaga, A., On the Structure of the Group of Equivariant Diffeomorphisms, *Topology*, 1977, vol. 16, no. 3, pp. 279–283.