MATHEMATICAL PROBLEMS OF NONLINEARITY

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On Nonlinear Oscillations and Stability of Coupled Pendulums in the Case of a Multiple Resonance

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The points of suspension of two identical pendulums moving in a homogeneous gravitational field are located on a horizontal beam performing harmonic oscillations of small amplitude along a fixed horizontal straight line passing through the points of suspension of the pendulums. The pendulums are connected to each other by a spring of low stiffness. It is assumed that the partial frequency of small oscillations of each pendulum is exactly equal to the frequency of horizontal oscillations of the beam. This implies that a multiple resonance occurs in this problem, when the frequency of external periodic action on the system is equal simultaneously to two its frequencies of small (linear) natural oscillations. This paper solves the nonlinear problem of the existence and stability of periodic motions of pendulums with a period equal to the period of oscillations of the beam. The study uses the classical methods due to Lyapunov and Poincaré, KAM (Kolmogorov, Arnold and Moser) theory, and algorithms of computer algebra.

The existence and uniqueness of the periodic motion of pendulums are shown, its analytic representation as a series is obtained, and its stability is investigated. For sufficiently small oscillation amplitudes of the beam, depending on the value of the dimensionless parameter which characterizes the stiffness of the spring connecting the pendulums, the found periodic motion is either Lyapunov unstable or stable for most (in the sense of Lebesgue measure) initial conditions or formally stable (stable in an arbitrarily large, but finite, nonlinear approximation).

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1. Introduction

Consider a system of two identical pendulums moving in a homogeneous gravitational field. Each of the pendulums is a weightless rod of length ℓ . Point masses weighing mg each are fastened to the ends A_1 and A_2 of these rods. The distance between the suspension points O_1 and O_2 of the pendulums is constant and equal to δ . The suspension points themselves are located on a horizontal beam which performs harmonic oscillations of small amplitude a along a horizontal straight line (the point O shown in Fig. 1 is a fixed point of this straight line). The pendulums are connected to each other by a linear elastic spring of low stiffness k . The distances O_1B_1 and O_2B_2 of the points of attachment of the spring to the pendulums from their suspension points are the same and equal to b . In an unstressed state the length of the spring is equal to the distance δ between the suspension points. It is assumed that the oscillation frequency ω of the beam is exactly equal to the partial frequency $\sqrt{g/\ell}$ of small linear oscillations of each pendulum. The motion of the pendulums occurs in a fixed vertical plane passing through points O_1 and O_2 .

Fig. 1. Coupled pendulums suspended from a beam that oscillates along a horizontal straight line ($O_1A_1 = O_2A_2 = \ell$, $O_1B_1 = O_2B_2 = b$, $O_1O_2 = \delta$, $OO_1 = a \cos \omega t$, $\omega = \sqrt{g/\ell}$).

Let φ_1 and φ_2 be the angles made by the rods O_1A_1 and O_2A_2 with the vertical. For the kinetic and potential energies of the system of two pendulums we have the following expressions (the dot denotes differentiation with respect to time t):

$$\begin{aligned} T &= T_2 + T_1 + T_0, & \Pi &= -mg\ell(\cos \varphi_1 + \cos \varphi_2) + \frac{1}{2}k\Delta^2, \\ T_2 &= \frac{1}{2}m\ell^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2), & T_1 &= -ma\ell\omega \sin \omega t (\cos \varphi_1 \dot{\varphi}_1 + \cos \varphi_2 \dot{\varphi}_2), \\ T_0 &= m\omega^2 a^2 \sin^2 \omega t, \\ \Delta &= \sqrt{[b(\cos \varphi_2 - \cos \varphi_1)]^2 + [\delta + b(\sin \varphi_2 - \sin \varphi_1)]^2} - \delta. \end{aligned} \quad (1.1)$$

The system of differential equations given by the Lagrangian function $L = T - \Pi$ admits, for any parameter values, a family of partial solutions in which, for any t , the pendulums deviate from the vertical by equal angles $\varphi_1 \equiv \varphi_2 = \varphi(t)$, where $\varphi(t)$ satisfies the second-order differential equation:

$$\frac{d^2 \varphi}{dt^2} + \frac{g}{\ell} \sin \varphi = \frac{a}{\ell} \omega^2 \cos \omega t \cos \varphi. \quad (1.2)$$

This equation describes the motion of a mathematical pendulum in the presence of harmonic horizontal oscillations of its suspension point with amplitude a and frequency ω .

The case $\omega = \sqrt{g/\ell}$ studied in this paper corresponds to the multiple $1 : 1 : 1$ resonance, which has the property that the frequency of external periodic action on the system coincides with each of its two frequencies of small natural oscillations.

Nonlinear resonant oscillations of coupled pendulums without external periodic actions were investigated earlier in [1–3]. Forced oscillations at the $1 : 1 : 1$ resonance were considered in the report [4], which contains an analysis of the problem of the existence and linear stability of possible types of periodic motions. Below we consider one of these motions, namely, that for which $\varphi_1(t) \equiv \varphi_2(t)$. A procedure for constructing periodic motion is presented, its analytic representation is obtained and the nonlinear problem of stability is considered. The investigation relies on the classical and modern methods of nonlinear dynamics [5–11].

A systematic study of the problem of nonlinear oscillations of Hamiltonian systems in the presence of resonance goes back to the turn of the 20th century [12–14]. A short bibliography of works on the theory and applications of resonant problems in mechanics and physics can be found, for example, in [15]. References [16–19] address several model problems which are described by an autonomous Hamiltonian system at the $1 : 1$ resonance and are important for astrophysics and celestial mechanics. In [15, 20] the $1 : 1$ resonance was considered in the problem of the motion of a satellite, a rigid body, relative to the center of mass in a circular orbit. Nonlinear oscillations of a dynamically symmetric satellite at the $1 : 1 : 1$ resonance in an elliptic orbit of small eccentricity were examined in [21–23]. The problem of the stability of an autonomous Hamiltonian system in the degenerate (transcendental) resonant case was studied in [24]. Multiple resonances, different from the $1 : 1 : 1$ resonance, in a nonautonomous Hamiltonian system with two degrees of freedom were investigated in the recent series of publications [25–32]. General theoretical results obtained in these works were applied in investigating the motion of a satellite, a rigid body, relative to the center of mass and in analyzing the stability of Lagrangian solutions to the planar bounded elliptic three-body problem.

2. The Hamiltonian function and its expansion as a series

As generalized momenta corresponding to generalized coordinates φ_i , we take the dimensionless variables p_{φ_i} defined by the equations

$$p_{\varphi_i} = \frac{1}{m\ell\sqrt{g\ell}} \frac{\partial L}{\partial \dot{\varphi}_i} \quad (i = 1, 2).$$

From these equations we obtain expressions for generalized velocities:

$$\dot{\varphi}_i = \omega \left(p_{\varphi_i} + \frac{a}{\ell} \sin \omega t \cos \varphi_i \right) \quad (i = 1, 2). \quad (2.1)$$

Instead of time t , as an independent variable we take the dimensionless variable

$$\tau = \omega t. \quad (2.2)$$

The canonically conjugate variables φ_i, p_{φ_i} ($i = 1, 2$) correspond to the Hamiltonian function H' which is calculated from the formula

$$H' = \frac{1}{mg\ell} (T_2 - T_0 + \Pi), \quad (2.3)$$

on the right-hand side of which the variables $\dot{\varphi}_i$ should be replaced with their expressions (2.1). This yields

$$H' = \sum_{i=1}^2 \left[\frac{1}{2} p_{\varphi_i}^2 - \cos \varphi_i + \frac{a}{\ell} \sin \tau \cos \varphi_i p_{\varphi_i} + \frac{1}{2} \left(\frac{a}{\ell} \right)^2 \sin^2 \tau \cos^2 \varphi_i \right] + \frac{1}{2} \beta \left\{ \sqrt{(\cos \varphi_2 - \cos \varphi_1)^2 + [\gamma + (\sin \varphi_2 - \sin \varphi_1)]^2} - \gamma \right\}^2, \quad (2.4)$$

$$\beta = \frac{kb^2}{mgl}, \quad \gamma = \frac{\delta}{b}. \quad (2.5)$$

For ease of the calculations presented below to investigate the motions in which $\varphi_1 = \varphi_2$, it is worthwhile to make a canonical transformation (with a valence equal to $1/2$) $\varphi_i, p_{\varphi_i} \rightarrow \tilde{\varphi}_i, \tilde{p}_{\varphi_i}$ using the formulae

$$\begin{aligned} \tilde{\varphi}_1 &= \frac{1}{2}(\varphi_1 + \varphi_2), & \tilde{\varphi}_2 &= \frac{1}{2}(\varphi_1 - \varphi_2), \\ \tilde{p}_{\varphi_1} &= \frac{1}{2}(p_{\varphi_1} + p_{\varphi_2}), & \tilde{p}_{\varphi_2} &= \frac{1}{2}(p_{\varphi_1} - p_{\varphi_2}). \end{aligned} \quad (2.6)$$

The new variables correspond to a Hamiltonian function of the form

$$\begin{aligned} \tilde{H} &= \frac{1}{2}(\tilde{p}_{\varphi_1}^2 + \tilde{p}_{\varphi_2}^2) - \cos \tilde{\varphi}_1 \cos \tilde{\varphi}_2 + \\ &+ \frac{1}{4} \beta \left[\sqrt{4 \sin^2 \tilde{\varphi}_1 \sin^2 \tilde{\varphi}_2 + (\gamma - 2 \cos \tilde{\varphi}_1 \sin \tilde{\varphi}_2)^2} - \gamma \right]^2 + \\ &+ \frac{a}{\ell} \sin \tau (\tilde{p}_{\varphi_1} \cos \tilde{\varphi}_1 \cos \tilde{\varphi}_2 - \tilde{p}_{\varphi_2} \sin \tilde{\varphi}_1 \sin \tilde{\varphi}_2) + \frac{1}{4} \left(\frac{a}{\ell} \right)^2 \sin^2 \tau \cos 2\tilde{\varphi}_1 \cos 2\tilde{\varphi}_2. \end{aligned} \quad (2.7)$$

The motions under study (for which $\varphi_1(t) \equiv \varphi_2(t)$) are given by the equation $\tilde{\varphi}_2 = \tilde{p}_{\varphi_2} = 0$, and $\tilde{\varphi}_1(\omega t)$ is a solution to Eq. (1.2).

Let us introduce a small parameter ε by setting

$$a = \varepsilon^3 \ell, \quad \beta = \varepsilon^2 s \quad (0 < \varepsilon \ll 1). \quad (2.8)$$

and make a canonical transformation $\tilde{\varphi}_i, \tilde{p}_{\varphi_i} \rightarrow Q_i, P_i$ (with a valence equal to ε^{-2}):

$$\tilde{\varphi}_1 = \varepsilon Q_1, \quad \tilde{\varphi}_2 = \varepsilon Q_2, \quad \tilde{p}_{\varphi_1} = \varepsilon P_1, \quad \tilde{p}_{\varphi_2} = \varepsilon P_2. \quad (2.9)$$

The new Hamiltonian function G can be represented by a convergent series in powers of ε (the terms that do not depend on Q_1, Q_2, P_1, P_2 have been discarded):

$$G = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \varepsilon^3 G_3 + \varepsilon^4 G_4 + \varepsilon^5 G_5 + \dots \quad (2.10)$$

The coefficients of this series are polynomials in Q_1, Q_2, P_1, P_2 and are 2π -periodic in τ . For the first six of them one can obtain the following expressions:

$$G_0 = \frac{1}{2} \sum_{i=1}^2 (Q_i^2 + P_i^2), \quad G_1 = 0, \quad (2.11)$$

$$G_2 = -\frac{1}{24} Q_1^4 - \frac{1}{4} Q_1^2 Q_2^2 - \frac{1}{24} Q_2^4 + \sin \tau P_1 + s Q_2^2, \quad G_3 = 0, \quad (2.12)$$

$$\begin{aligned} G_4 &= \frac{1}{720} (Q_1^6 + Q_2^6) + \frac{1}{48} Q_1^2 Q_2^2 (Q_1^2 + Q_2^2) - \frac{1}{2} \sin \tau [P_1 (Q_1^2 + Q_2^2) + 2Q_1 Q_2 P_2] - \\ &- \frac{1}{3} s Q_2^2 (3Q_1^2 + Q_2^2), \quad G_5 = -\frac{2s}{\gamma} Q_1^2 Q_2^3. \end{aligned} \quad (2.13)$$

The motion under study corresponds to the periodic solution $Q_i = Q_i^*(\tau)$, $P_i = P_i^*(\tau)$ ($i = 1, 2$) of canonical equations with the Hamiltonian function (2.10). In this case, $Q_2^* = P_2^* \equiv 0$ and Q_1^* satisfies the second-order differential equation

$$\frac{d^2 Q_1^*}{d\tau^2} + Q_1^* = \varepsilon^2 \left(\cos \tau + \frac{1}{6} Q_1^{*3} \right) - \varepsilon^4 \left(\frac{1}{2} \cos \tau Q_1^{*2} + \frac{1}{120} Q_1^{*5} \right) + O(\varepsilon^6). \quad (2.14)$$

If the function Q_1^* is found, then P_1^* is calculated according to the equation

$$P_1^* = \frac{dQ_1^*}{d\tau} - \varepsilon^2 \sin \tau + O(\varepsilon^4). \quad (2.15)$$

3. Construction of a periodic solution

The periodic solution $Q_1^*(\tau)$ to Eq. (2.14) can be found using an algorithm based on the Poincaré method for quasi-linear systems with one degree of freedom [6]. Set

$$Q_1^* = c_1 \sin \tau + c_2 \cos \tau + \varepsilon^2 f_2(\tau) + \varepsilon^4 f_4(\tau) + \dots, \quad (3.1)$$

where c_1, c_2 are constants and $f_2, f_4, \dots$ are 2π -periodic functions which need to be found. Substituting this expression for Q_1^* into Eq. (2.14) and equating the coefficients at ε^2 and ε^4 in series expansions of its left- and right-hand sides, we obtain

$$\frac{d^2 f_2}{d\tau^2} + f_2 = \cos \tau + \frac{1}{6} (c_1 \sin \tau + c_2 \cos \tau)^3, \quad (3.2)$$

$$\frac{d^2 f_4}{d\tau^2} + f_4 = -\frac{1}{2} (c_1 \sin \tau + c_2 \cos \tau)^2 \left[\cos \tau - f_2 + \frac{1}{60} (c_1 \sin \tau + c_2 \cos \tau)^3 \right]. \quad (3.3)$$

In order for Eq. (3.2) to admit a periodic solution, it is necessary and sufficient that the coefficients at $\sin \tau$ and $\cos \tau$ on its right-hand side be equal. This condition allows a unique definition of the constants c_1 and c_2 . We find

$$c_1 = 0, \quad c_2 = -2. \quad (3.4)$$

For this choice of c_1 and c_2 the system of equations (3.2)–(3.3) can be written as

$$\frac{d^2 f_2}{d\tau^2} + f_2 = -\frac{1}{3} \cos 3\tau, \quad (3.5)$$

$$\frac{d^2 f_4}{d\tau^2} + f_4 = 2 \cos^2 \tau \left(f_2 - \cos \tau + \frac{2}{15} \cos^3 \tau \right). \quad (3.6)$$

From (3.5) we have

$$f_2 = c_3 \sin \tau + c_4 \cos \tau + \frac{1}{24} \cos 3\tau, \quad (3.7)$$

where c_3 and c_4 are constants. For such a function f_2 the right-hand side of Eq. (3.6) can be written as

$$\frac{1}{80} [40c_3 \sin \tau + (120c_4 - 105) \cos \tau + A_3 \sin 3\tau + B_3 \cos 3\tau + A_5 \sin 5\tau + B_5 \cos 5\tau].$$

We do not present the expressions for the constant coefficients A_3, B_3, A_5, B_5 since they will not be required in what follows. From the condition that there are no harmonics $\sin \tau$ and $\cos \tau$ in the last expression, we obtain

$$c_3 = 0, \quad c_4 = \frac{7}{8}. \quad (3.8)$$

From (3.1), (3.4), (3.7) and (3.10) we have the following expression for the periodic solution $Q_1^*(\tau)$ to equation (2.14) calculated with an error of order ε^4 :

$$Q_1^* = -2 \cos \tau + \frac{1}{24} \varepsilon^2 (21 \cos \tau + \cos 3\tau) + O(\varepsilon^4). \quad (3.9)$$

The function $P_1^*(\tau)$ corresponding to this solution is found from (2.15) and (3.9):

$$P_1^* = 2 \sin \tau - \frac{1}{8} \varepsilon^2 (15 \sin \tau + \sin 3\tau) + O(\varepsilon^4). \quad (3.10)$$

We note that in the terms through degree three in ε the constructed periodic solution does not depend on the parameter γ contained in the fifth-degree terms of the expansion of the Hamiltonian function (2.10).

4. The Hamiltonian function of the perturbed motion and its preliminary transformation

To investigate the stability of the pendulums' motion described by the found periodic solution $Q_1 = Q_1^*(\tau)$, $P_1 = P_1^*(\tau)$, $Q_2 = 0$, $P_2 = 0$, we introduce perturbations q_j, p_j ($j = 1, 2$) by setting

$$Q_1 = Q_1^* + q_1, \quad Q_2 = q_2, \quad P_1 = P_1^* + p_1, \quad P_2 = p_2. \quad (4.1)$$

The equations of the perturbed motion are given by the Hamiltonian function Φ , which is calculated by the formula

$$\Phi = G + q_1 \frac{dP_1^*}{d\tau} - p_1 \frac{dQ_1^*}{d\tau}, \quad (4.2)$$

where G is the function (2.10), in which Q_j, P_j ($j = 1, 2$) have been replaced by their expressions (4.1).

The function Φ is analytic in the variables q_j, p_j ($j = 1, 2$) and 2π -periodic in τ . Its expansion as a series contains no first-degree terms in q_j, p_j ($j = 1, 2$). Neglecting terms that are insignificant for the equations of motion and do not depend on q_j, p_j ($j = 1, 2$), one can write this expansion as

$$\Phi = \Phi_2 + \Phi_3 + \Phi_4 + O((q_1^2 + p_1^2 + q_2^2 + p_2^2)^{5/2}), \quad (4.3)$$

$$\Phi_2 = \frac{1}{2} (q_1^2 + p_1^2 + q_2^2 + p_2^2) + \varepsilon^2 [-\cos^2 \tau q_1^2 + (s - \cos^2 \tau) q_2^2] + \varepsilon^4 \Phi_2^*, \quad (4.4)$$

$$\Phi_3 = \varepsilon^2 \left(\frac{1}{3} \cos \tau q_1^3 + \cos \tau q_1 q_2^2 \right) + \varepsilon^4 \Phi_3^*, \quad (4.5)$$

$$\Phi_4 = \varepsilon^2 \left[-\frac{1}{24} (q_1^4 + q_2^4) - \frac{1}{4} q_1^2 q_2^2 \right] + \varepsilon^4 \Phi_4^*. \quad (4.6)$$

Here Φ_k^* ($k = 2, 3, 4$) is the form of degree k in q_1, q_2, p_1, p_2 with coefficients 2π -periodic in τ .

For further analysis it is worthwhile to make the transformation $q_j, p_j \rightarrow U_j, V_j$ ($j = 1, 2$), which eliminates in the form (4.4) its first four terms independent of the parameter ε . This transformation is a rotation by angle τ in the planes $q_1 p_1$ and $q_2 p_2$, is canonical and given by

$$q_j = U_j \cos \tau + V_j \sin \tau, \quad p_j = -U_j \sin \tau + V_j \cos \tau \quad (j = 1, 2). \quad (4.7)$$

The new Hamiltonian function F is obtained from the function (4.3) by discarding the first four terms from the form Φ_2 and by subsequently replacing q_j, p_j with their expressions (4.7). The function F is of order two in ε .

The function F can be simplified by making a canonical, near-identity change of variables $U_j, V_j \rightarrow u_j, v_j$ which eliminates the independent variable τ from the terms of order ε^2 in the expansion of F as a series in powers of ε . The explicit form of the change of variables $U_j, V_j \rightarrow u_j, v_j$ will not be required in what follows and is therefore not presented. The new variables u_j, v_j correspond to the Hamiltonian function Γ of the form

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + O((u_1^2 + v_1^2 + u_2^2 + v_2^2)^{5/2}), \quad (4.8)$$

$$\Gamma_2 = \varepsilon^2 \left(-\frac{3}{8}u_1^2 - \frac{1}{8}v_1^2 + \frac{4s-3}{8}u_2^2 + \frac{4s-1}{8}v_2^2 \right) + \varepsilon^4 \Gamma_2^*, \quad (4.9)$$

$$\Gamma_3 = \varepsilon^2 \left(\frac{3}{8}u_2^2 u_1 + \frac{1}{8}v_2^2 u_1 + \frac{1}{4}u_2 v_2 v_1 + \frac{1}{8}u_1 v_1^2 + \frac{1}{8}u_1^3 \right) + \varepsilon^4 \Gamma_3^*, \quad (4.10)$$

$$\Gamma_4 = \varepsilon^2 \left[-\frac{1}{32}(u_1^2 v_1^2 + 3u_1^2 u_2^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + 3v_1^2 v_2^2 + u_2^2 v_2^2) - \frac{1}{64}(u_1^4 + u_2^4 + v_1^4 + v_2^4) - \frac{1}{8}u_1 u_2 v_1 v_2 \right] + \varepsilon^4 \Gamma_4^*, \quad (4.11)$$

where Γ_k^* ($k = 2, 3, 4$) is the form of degree k in u_1, u_2, v_1, v_2 with coefficients 2π -periodic in τ . Note that $\Gamma_k - \varepsilon^4 \Gamma_k^*$ ($k = 2, 3, 4$) are averaged (over τ) sets of terms of order ε^2 in the functions (4.4)–(4.6), in which the transformation (4.7) was made and then U_j, V_j were replaced with u_j, v_j .

5. Analysis of the linear problem

The characteristic equation of the linearized system of equations of the perturbed motion is reciprocal, as is that of any linear system of Hamiltonian differential equations with periodic coefficients, and its roots (multipliers) are continuous in ε [5, 6]. It immediately follows from this fact and from (4.9) that, if the parameters $(4s-3)$ and $(4s-1)$ have opposite signs, then at sufficiently small ε the characteristic equation has a root whose absolute value is larger than 1. Therefore [5], in the interval

$$\frac{1}{4} < s < \frac{3}{4} \quad (5.1)$$

the periodic motion of the pendulums is Lyapunov unstable (not only by virtue of the linearized equations of the perturbed motion, but also in a complete nonlinear system with the Hamiltonian function (4.8)).

If

$$0 \leq s \leq \frac{1}{4} \quad \text{or} \quad s \geq \frac{3}{4}, \quad (5.2)$$

then, at small ε , all multipliers have absolute values equal to 1, and a rigorous solution to the problem of stability requires an additional (nonlinear) analysis of the equations of the perturbed motion.

For values of s from the intervals (5.2), the characteristic exponents defined by linear equations with the Hamiltonian function (4.9) will be (for small ε) purely imaginary quantities $\pm i\lambda_1, \pm i\lambda_2$, where i is an imaginary unit and λ_1, λ_2 are real numbers of any sign. If λ_1, λ_2 are such that

$$k_1\lambda_1 + k_2\lambda_2 = N, \quad (5.3)$$

where N, k_1, k_2 are integer numbers such that $|k_1| + |k_2| = k$, then the system exhibits a resonance of order k . In the stability problem, the most important resonances are those of lower orders (through order four).

To investigate the nonlinear problem, we first need to make a canonical transformation reducing the quadratic part Γ_2 of expansion (4.8) to normal coordinates. For this we make a linear change of variables $u_1, u_2, v_1, v_2 \rightarrow \xi_1, \xi_2, \eta_1, \eta_2$ by the formulae

$$\begin{aligned} u_1 &= -\alpha\xi_1, & u_2 &= -\beta\xi_2, & v_1 &= \alpha^{-1}\eta_1, & v_2 &= \beta^{-1}\eta_2, \\ \alpha &= \frac{1}{\sqrt[4]{3}}, & \beta &= \sqrt[4]{\frac{4s-1}{4s-3}}. \end{aligned} \quad (5.4)$$

This transformation is canonical with a valence equal to -1 . It transforms the function Γ_2 to a function $\tilde{\Gamma}_2$ of the form

$$\tilde{\Gamma}_2 = \varepsilon^2 \left[\frac{1}{2} \omega_1 (\xi_1^2 + \eta_1^2) - \frac{1}{2} \sigma \omega_2 (\xi_2^2 + \eta_2^2) \right] + \varepsilon^4 \tilde{\Gamma}_2^*, \quad (5.5)$$

where

$$\omega_1 = \frac{\sqrt{3}}{4} \simeq 0.433013, \quad \omega_2 = \frac{1}{4} \sqrt{(4s-1)(4s-3)}, \quad (5.6)$$

and $\sigma = -1$ in the region $0 < s < 1/4$ and $\sigma = 1$ in the region $s > 3/4$. In the interval $0 < s < 1/4$ the parameter ω_2 decreases monotonically from the value $\sqrt{3}/4$ to zero, and in the interval $s > 3/4$ it increases monotonically. Graphs of the function $\omega_2(s)$ are presented in Figs. 2 and 3.

If we eliminate the value $s = 1$ which lies in the region $s > 3/4$ and for which $\omega_1 = \omega_2 = \sqrt{3}/4$, then for the other values of s inside the regions (5.2) of linear stability at small ε no first- or second-order resonances are possible. For such parameter values, we use a linear real canonical transformation $\xi_1, \xi_2, \eta_1, \eta_2 \rightarrow x_1, x_2, X_1, X_2$, 2π -periodic in τ and analytic in ε , which can be constructed, for example, by the Deprit–Hori method [8, 33]. Then the Hamiltonian function of the perturbed motion reduces to the form

$$H = H_2 + H_3 + H_4 + O((x_1^2 + X_1^2 + x_2^2 + X_2^2)^{5/2}), \quad (5.7)$$

where

$$H_2 = \frac{1}{2} \lambda_1 (x_1^2 + X_1^2) + \frac{1}{2} \lambda_2 (x_2^2 + X_2^2), \quad (5.8)$$

$$\lambda_1 = \varepsilon^2 [\omega_1 + O(\varepsilon^2)], \quad \lambda_2 = \varepsilon^2 [-\sigma \omega_2 + O(\varepsilon^2)], \quad (5.9)$$

and the third- and fourth-degree terms H_3 and H_4 in the expansion (5.7) are given by the equations

$$H_3 = \varepsilon^2 \left[\frac{3^{3/4}}{24\beta^2} x_1 X_2^2 + \frac{3^{1/4}}{24} (x_1^3 + 3x_1 X_1^2 + 6x_2 X_1 X_2) + \frac{3^{3/4}\beta^2}{8} x_1 x_2^2 \right] + \varepsilon^4 H_3^*,$$

Fig. 2. Dependence of the frequency ω_2 on s at $0 \leq s \leq 1/4$. Fig. 3. Dependence of the frequency ω_2 on s at $s \geq 3/4$.

$$H_4 = \varepsilon^2 \left[\frac{1}{64\beta^4} X_2^4 + \frac{\sqrt{3}}{96\beta^2} (x_1^2 + 9X_1^2) X_2^2 + \frac{1}{192} (x_1^4 + 6x_1^2 X_1^2 + 24x_1 x_2 X_1 X_2 + \right. \\ \left. + 9X_1^4 + 6x_2^2 X_2^2) + \frac{\sqrt{3}\beta^2}{32} (x_1^2 + X_1^2) x_2^2 + \frac{\beta^4}{64} x_2^4 \right] + \varepsilon^4 H_4^*.$$

Here, H_3^* and H_4^* are forms of degrees three and four in x_1, x_2, X_1, X_2 with coefficients 2π -periodic in τ .

6. On possible resonances

Taking into account expressions (5.9) and (5.6) for λ_1, λ_2 and assuming that the dimensionless parameter s (which characterizes the stiffness of the spring connecting the pendulums) cannot be arbitrarily large, we find that at small ε the resonances (5.3) inside the regions (5.2) of linear stability take only place when the integer N on the right-hand side of relation (5.3) is zero. If the order of the resonance does not exceed 4, then in the plane $s\varepsilon$ at small ε there exist

only seven resonant curves. In Figs. 2 and 3, they emanate from the corresponding generating points $(s_i, 0)$ of the abscissa axes.

In the region $0 < s < 1/4$, two resonances (see Fig. 2) are possible. For these resonances:

$$1) \omega_1 = 2\omega_2, s_1 = \frac{1}{2} - \frac{1}{8}\sqrt{7} \simeq 0.169281; \quad 2) \omega_1 = 3\omega_2, s_2 = \frac{1}{2} - \frac{1}{6}\sqrt{3} \simeq 0.211325.$$

And in the region $s > 3/4$ five resonances (Fig. 3) take place for which:

$$3) \omega_1 = 3\omega_2, s_3 = \frac{1}{2} - \frac{1}{6}\sqrt{3} \simeq 0.788675; \quad 4) \omega_1 = 2\omega_2, s_4 = \frac{1}{2} + \frac{1}{8}\sqrt{7} \simeq 0.830719;$$

$$5) \omega_1 = \omega_2, s_5 = 1; \quad 6) \omega_2 = 2\omega_1, s_6 = \frac{1}{2} + \frac{1}{4}\sqrt{13} \simeq 1.401388;$$

$$7) \omega_2 = 3\omega_1, s_7 = \frac{1}{2} + \frac{1}{2}\sqrt{7} \simeq 1.822876.$$

The value of the parameter s on the i th resonant curve differs from its value s_i at the generating point by values of order ε^2 .

7. A nonlinear problem inside the regions (5.2).

The nonresonant case

Let the parameter s lie inside the regions (5.2) and suppose that its value is not resonant ($s \neq s_i$ ($i = 1, 2, \dots, 7$)). Using a real near-identity 2π -periodic (in τ) canonical Birkhoff transformation [7] $x_1, x_2, X_1, X_2 \rightarrow y_1, y_2, Y_1, Y_2$, we reduce the Hamiltonian function (5.7) to its normal form through terms of degree four in y_1, y_2, Y_1, Y_2 . In the symplectic polar coordinates r_j, ψ_j ,

$$y_j = \sqrt{2r_j} \sin \psi_j, \quad Y_j = \sqrt{2r_j} \cos \psi_j \quad (j = 1, 2) \quad (7.1)$$

the normalized Hamiltonian function can be written as

$$H = \lambda_1 r_1 + \lambda_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + O((r_1 + r_2)^{5/2}), \quad (7.2)$$

where λ_1, λ_2 are given by equations (5.9). But finding constant coefficients c_{ij} requires rather cumbersome calculations. These calculations have been performed using methods of computer algebra, and the following expressions have been obtained for the coefficients c_{ij} :

$$\begin{aligned} c_{20} &= \varepsilon^2 \left[-\frac{1}{96} + O(\varepsilon^2) \right], \quad c_{11} = \varepsilon^2 \left[-\sigma \frac{\sqrt{3}(8s-3)^2}{1536\omega_2(\omega_1^2 - 4\omega_2^2)} + O(\varepsilon^2) \right], \\ c_{02} &= \varepsilon^2 \left[\frac{(8s-3)(1280s^3 - 1376s^2 + 264s - 9)}{24576\omega_2^2(\omega_1^2 - 4\omega_2^2)} + O(\varepsilon^2) \right]. \end{aligned} \quad (7.3)$$

In addition to considering the Lyapunov stability, we will also analyze the stability for most (in the sense of Lebesgue measure) initial conditions $u_j(0), v_j(0)$ ($j = 1, 2$) and the formal stability.

On the stability for most initial conditions (see [9, 34–37]). If the inequality

$$c_{11}^2 - 4c_{20}c_{02} \neq 0 \quad (7.4)$$

is satisfied, then most trajectories $u_j(\tau), v_j(\tau)$ ($j = 1, 2$) of the system with the Hamiltonian function (4.8) which satisfy the condition $u_1^2(0) + u_2^2(0) + v_1^2(0) + v_2^2(0) < \mu$ at the initial time

$\tau = 0$ remain for all $\tau \geq 0$ in a small neighborhood of the unperturbed motion $u_j = v_j = 0$ ($j = 1, 2$). The set of initial conditions corresponding to the above-mentioned “majority” has a relative measure that is not smaller than $1 - O(\mu^{1/4})$.

However, the trajectories starting in an addition to the set of “most” initial conditions which has a relative measure of order $\mu^{1/4}$ can, generally speaking, go arbitrarily far away from the initial position (this phenomenon is known as “Arnold diffusion”). But it was shown in [38–40] that for almost any system with a Hamiltonian function of the form (4.8) there exist positive numbers δ_1 and δ_2 such that the difference of the quantity $u_1^2(\tau) + u_2^2(\tau) + v_1^2(\tau) + v_2^2(\tau)$ from its small initial value does not exceed a quantity of order μ^{δ_1} for all values of τ satisfying the inequality

$$0 < \tau < \exp(\mu^{-\delta_2}).$$

This implies that, if there does exist “Arnold diffusion”, it occurs exponentially slowly.

Based on the formulae (7.3) and (5.6), the left-hand side of inequality (7.4) can be rewritten as

$$c_{11}^2 - 4c_{20}c_{02} = \varepsilon^4 [d + O(\varepsilon^2)], \quad (7.5)$$

where the parameter d does not depend on σ and has the same expression in both regions $0 < s < 1/4$ and $s > 3/4$:

$$d = -\frac{(8s - 3)(81920s^5 - 169984s^4 + 110336s^3 - 22944s^2 + 360s + 243)}{9437184\omega_2^2(\omega_1^2 - 4\omega_2^2)^2}. \quad (7.6)$$

When ε is small, the inequality $d \neq 0$ is [9, 37] a sufficient stability condition for most initial conditions.

1. Analysis shows that in the region $0 < s < 1/4$ the function $d(s)$ defined by equation (7.6) is positive for all values of s (recall that the resonant values s_i ($i = 1, 2, \dots, 7$) are not considered in this section). Figure 4 shows a graph of the function $d(s)$ in the interval $1/8 < s < 1/4$. The curve $d(s)$ has two vertical asymptotes: $s = s_1$ (resonance $\omega_1 = 2\omega_2$) and $s = 1/4$ (the boundary of the region); the point $s = 0.216774$ is a point of local minimum of the function $d(s)$, at this point $d = 2.644152$. We also note that $d(0) = 1/768 \simeq 0.001302$ (Fig. 4 does not show the point with coordinates 0 and $d(0)$).

Thus, if s belongs to the interval $0 < s < 1/4$ (such that $s \neq s_1, s \neq s_2$), then for sufficiently small ε the periodic motion of the pendulums is stable for most initial conditions.

2. The behavior of the function $d(s)$ with $s > 3/4$ is more complex (see Figs. 5 and 6). The graph of the function $d(s)$ has two vertical asymptotes: $s = 3/4$ (the boundary of the region) and $s = s_4$ (resonance $\omega_1 = 2\omega_2$). In the interval $3/4 < s < s_4$ the function $d(s)$ is positive and has the minimal value $d = 0.441034$ at the point $s = 0.770815$ (Fig. 5); when $s_4 < s < 1$, the function decreases monotonically from infinitely large values to the value $d(1) = 115/186624 \simeq 0.000616$. When $1 < s < s_* = 1.004417$, the function $d(s)$ is positive, and when $s > s_*$, it is negative (Fig. 6); when $s = s_*$, the function $d(s)$ vanishes. The point $s = 1.279761$ is a minimum point at which $d = -0.006942$. As s increases, the graph of the function $d(s)$ approaches from below the horizontal asymptote $d = -5/1152 \simeq -0.004340$.

It follows from the analysis of the function $d(s)$ that in the region $s > 3/4$ outside the resonant points s_i ($i = 3, \dots, 7$) and the point s_* , at which $d = 0$, stability takes place at small ε for most initial conditions.

On the formal stability (see [41–48]). The concept of formal stability was introduced by J. Moser for systems possessing a sign-definite formal integral which is a (possibly divergent)

Fig. 4. Graph of the function $d(s)$ for $1/8 < s < 1/4$. Fig. 5. Graph of the function $d(s)$ for $3/4 < s < 1$. Fig. 6. Graph of the function $d(s)$ for $s > 1/4$.

power series. This concept is very important in investigating the stability on a finite (but very large) time interval. The presence of formal stability implies that the Lyapunov instability (if there is any in the problem of interest) does not occur if terms up to an arbitrarily high (but finite) order in the coordinates and momenta of the perturbed motion are taken into account in the expansion of the Hamiltonian function. In the presence of formal stability, even if there exist trajectories going far away from the unperturbed motion, the motion along them is extremely slow. The corresponding estimates were obtained by J. Moser [42, 43], C. L. Siegel [45], J. Glimm [46], N. Nekhoroshev [39, 40] and others (see [9, 48]). Formal stability is quite sufficient to solve the question of stability in most physical and technical problems described by Hamiltonian differential equations.

1. In the region $0 < s < 1/4$ the quantities λ_j ($j = 1, 2$) from (5.9) at small ε have the same signs (they are positive). Therefore [41], in this region, for all s (including the resonant values s_1 and s_2) formal stability takes place if ε is sufficiently small.

2. In contrast, in the region $s > 3/4$ the quantities (5.9) with small ε have different signs. Here, to solve the question of formal stability, we use the results of [47]. We assume that there are no resonances up to order four (i.e., $s \neq s_i, i = 3, \dots, 7$).

Consider the expression

$$D = c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2 \quad (\sigma_j = |\lambda_j|, j = 1, 2). \quad (7.7)$$

If the value of (7.7) does not vanish for $\sigma_2 = r\sigma_1$, where r is a rational number, then the unperturbed motion is formally stable [47].

Calculations show that, when $\sigma_2 = r\sigma_1$, the expression for D with an error of order ε^4 can be represented as

$$D = \varepsilon^2 \frac{243P}{16384\omega_2^2(3 - 64\omega_2^2)Q}, \quad (7.8)$$

where P and Q are polynomials of the form

$$\begin{aligned} P &= 16r^{12} + 184r^{10} + 577r^8 - 1508r^6 + 290r^4 + 108r^2 - 15, \\ Q &= 8192s^6 - 24576s^5 + 38016s^4 - 41984s^3 + 28416s^2 - 8928s + 783. \end{aligned} \quad (7.9)$$

But it is easy to see that the polynomial P with integer-valued coefficients has no rational roots $r = \ell/m$, where ℓ and m are coprime integers. Indeed, otherwise (see, e.g., [49]) ℓ and m must be the denominators of the numbers 15 and 16, respectively, and a simple enumeration of ℓ and m shows that the numbers $r = \ell/m$ are not the roots of the polynomial $P(r)$.

Thus, for all $s > 3/4$ (except perhaps for the values $s = s_i$ ($i = 3, \dots, 7$)) which have not been considered so far) the motion under study is formally stable.

8. A nonlinear problem in the region $s > 3/4$. Resonant cases

Resonance $\lambda_1 + 2\lambda_2 = 0$ ($s = s_4 + O(\varepsilon^2)$). Using the Birkhoff transformation, the Hamiltonian function (5.7) can be reduced to the following normal form:

$$H = \lambda_1 r_1 + \lambda_2 r_2 + \chi \sqrt{r_1} r_2 \sin(\psi_1 + 2\psi_2) + O((r_1 + r_2)^2), \quad (8.1)$$

where

$$\chi = \varepsilon^2 \frac{(1 + \sqrt{7})}{2 \cdot 12^{3/4}} + O(\varepsilon^4).$$

For small ε the parameter χ is nonzero. Therefore [11], instability takes place.

Resonance $2\lambda_1 + \lambda_2 = 0$ ($s = s_6 + O(\varepsilon^2)$). Since the variables x_2, X_2 appear in the term H_3 of expansion (5.7) in powers whose sum is equal to two, the form H_3 contains no resonant terms and can be completely eliminated using the Birkhoff transformation. The Hamiltonian function normalized through terms of degree four in y_1, y_2, Y_1, Y_2 will be given by an equation of the form (7.2) in which

$$\begin{aligned} c_{20} &= \varepsilon^2 \left[-\frac{1}{96} + O(\varepsilon^2) \right], & c_{11} &= \varepsilon^2 \left[\frac{53 + 4\sqrt{13}}{2160} + O(\varepsilon^2) \right], \\ c_{02} &= \varepsilon^2 \left[-\frac{2051 + 328\sqrt{13}}{17280} + O(\varepsilon^2) \right]. \end{aligned} \quad (8.2)$$

For the left-hand side of inequality (7.4) we obtain the expression (7.5), in which

$$d = -\frac{80227 + 13064\sqrt{13}}{18662400} \simeq -0.006823. \quad (8.3)$$

It follows from (7.4), (8.2) and (8.3) that for sufficiently small ε the form, quadratic (in r_1, r_2), in equation (7.2) will be sign-definite. Therefore [46], the motion under study is formally stable. It will also be stable for most initial conditions, since inequality (7.4) is obviously satisfied for small ε .

Resonances $\lambda_1 + 3\lambda_2 = 0$ ($s = s_3 + O(\varepsilon^2)$) and $3\lambda_1 + \lambda_2 = 0$ ($s = s_7 + O(\varepsilon^2)$). These four-order resonances are investigated similarly to the above-mentioned third-order resonance $2\lambda_1 + \lambda_2 = 0$. Here one can also completely eliminate the third-degree terms H_3 in (5.7) and, similarly to the third-order resonance, the presence of commensurabilities $\lambda_1 + 3\lambda_2 = 0$ and $3\lambda_1 + \lambda_2 = 0$ does not lead to a change in the structure of normalized terms of degree four in y_1, y_2, Y_1, Y_2 . They are given by Eq. (7.2).

For the resonance $\lambda_1 + 3\lambda_2 = 0$ we have

$$\begin{aligned} c_{20} &= \varepsilon^2 \left[-\frac{1}{96} + O(\varepsilon^2) \right], & c_{11} &= \varepsilon^2 \left[-\frac{19 + 8\sqrt{3}}{40} + O(\varepsilon^2) \right], \\ c_{02} &= \varepsilon^2 \left[-\frac{413 + 256\sqrt{3}}{480} + O(\varepsilon^2) \right], & d &= \frac{17843 + 9664\sqrt{3}}{57600} \simeq 0.600374, \end{aligned}$$

and for the resonance $3\lambda_1 + \lambda_2 = 0$

$$\begin{aligned} c_{20} &= \varepsilon^2 \left[-\frac{1}{96} + O(\varepsilon^2) \right], & c_{11} &= \varepsilon^2 \left[\frac{113 + 8\sqrt{7}}{7560} + O(\varepsilon^2) \right], \\ c_{02} &= \varepsilon^2 \left[-\frac{3347 + 512\sqrt{7}}{30240} + O(\varepsilon^2) \right], & d &= -\frac{1001437 + 154048\sqrt{7}}{228614400} \simeq -0.006163. \end{aligned}$$

In both resonant cases, both formal stability and stability for most initial conditions take place.

9. Conclusion

The above analysis shows that the resonant periodic motion of the pendulums with small values of the amplitude of horizontal oscillations of their suspension points is Lyapunov unstable if the dimensionless parameter s , which characterizes the stiffness of the spring connecting the pendulums, lies in the interval $1/4 < s < 3/4$.

In the interval $0 < s < 1/4$ the motion is stable for most (in the sense of Lebesgue measure) initial conditions and is also formally stable.

If $s > 3/4$, formal stability takes place for all s except for one value, $s = s_4 = 1/2 + \sqrt{7}/8 \simeq 1.401388$, which corresponds to the third-order resonance; for this value of s the motion under study is Lyapunov unstable. If $s > 3/4$ and $s \neq s_4$, $s \neq s_* = 1.004417$, then the motion is stable for most initial conditions.

The stability at the boundary points of the intervals (5.2) and at the point $s = 1$, which lies inside the second interval, has not been investigated. We note that, when $s = 0$, we arrive at the problem of the stability of resonant periodic oscillations of one mathematical pendulum under horizontal oscillations of its suspension point; it is well known (see the monograph [50, Chapter 4]) that these oscillations are Lyapunov stable. The problem of the stability of the above-mentioned periodic motions of coupled pendulums in the cases $s = 1/4, 3/4$ and 1 are a subject for a separate study.

Conflict of Interest

The authors declare that they have no conflict of interest.

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