On the Stability of Pendulum-type Motions in the Approximate Problem of Dynamics of a Lagrange Top with a Vibrating Suspension Point

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This paper addresses the motion of a Lagrange top in a homogeneous gravitational field under the assumption that the suspension point of the top undergoes high-frequency vibrations with small amplitude in three-dimensional space. The laws of motion of the suspension point are supposed to allow vertical relative equilibria of the top’s symmetry axis. Within the framework of an approximate autonomous system of differential equations of motion written in canonical Hamiltonian form, pendulum-type motions of the top are considered. For these motions, its symmetry axis performs oscillations of pendulum type near the lower, upper or inclined relative equilibrium positions, rotations or asymptotic motions. Integration of the equation of pendulum motion of the top is carried out in the whole range of the problem parameters. The question of their orbital linear stability with respect to spatial perturbations is considered on the isoenergetic level corresponding to the unperturbed motions. The stability evolution of oscillations and rotations of the Lagrange top depending on the ratios between the intensities of the vertical, horizontal longitudinal and horizontal transverse components of vibration is described.

Keywords: Lagrange’s top, high-frequency vibrations, pendulum-type motions, stability

1. Introduction

Pendulum-type motions of a heavy rigid body with a fixed point were first described in 1894 by B. K. Mlodzeevsky [1], who showed that such motions can occur about a horizontal principal axis of inertia if the center of mass of the body lies in the principal plane of inertia which is perpendicular to this axis. The stability of small pendulum-type oscillations of the body with a fixed point was analyzed in [2]. The problem of stability of planar and similar rotations of the
body with the center of mass on the principal axis of inertia was studied in [3]. Pendulum-type motions were investigated for bodies with mass geometry corresponding to the cases due to Kovalevskaya [4–7], Goryachev–Chaplygin [8, 9], and Bobylev–Steklov [10, 11]. The case of a dynamically symmetric body whose center of mass lies in the equatorial plane of the ellipsoid of inertia is examined in [12, 13].

The study of the influence of high-frequency vibrations on the properties of pendulum-type motions of the body and on their orbital stability is of interest from both a theoretical and an applied point of view. It is well known [14–16] that, in the approximate problem of the motion of a heavy rigid body in the case of high-frequency vibrations of its suspension point, one should consider, along with the gravitational field, an additional “vibrational” potential field (due to a collection of forces of the moving space) which changes considerably the pattern of motions of the body.

This paper is concerned with pendulum-type motions of a Lagrange top whose suspension point undergoes prescribed high-frequency periodic vibrations of small amplitude in three-dimensional space. Types of vibrations that allow vertical relative equilibria of the axis of dynamical symmetry of the body are considered. In this case, in an approximate autonomous system of equations of motion of the top there exist partial solutions which correspond to pendulum-type motions of its axis in the vertical plane — to oscillations near the lower, upper or inclined relative equilibria and to rotations and asymptotic motions. Integration of the equation of pendulum motion of the top is performed over the whole range of the system parameters. An analysis is made of their orbital linear stability with respect to spatial perturbations on the isoenergetic level corresponding to unperturbed motions. In particular, regions of orbital stability of oscillations of the axis of the top in a neighborhood of the upper and inclined equilibria are identified. A description is given of the evolution of the stability of oscillations and rotations of the Lagrange top as the relationships between the intensities of the vertical, horizontal longitudinal, and horizontal transverse components of vibrations are changed.

Previously, the dynamics of the Lagrange top with a vibrating point of suspension was investigated in quite a number of papers. For the case of vertical harmonic small-amplitude oscillations of the suspension point of the top, nonresonant and resonant periodic motions which are born from its regular precession were investigated in [17]. Motions of the top which are close to its regular precessions under vertical high-frequency vibrations of the suspension point are dealt with in [18–20]. The stability of a “sleeping” Lagrange top in the case of vertical harmonic oscillations of the suspension point with an arbitrary frequency and amplitude is investigated in [21]. In [22], within the framework of an approximate autonomous system obtained in [15, 16], Markeev considered the case of high-frequency vibrations which admits two cyclic coordinates (“vibrational symmetry”) in the system. He also solved the problem of existence and stability of a number of particular motions of the Lagrange top: stationary rotations about the vertical and inclined axes and regular precessions. In [23], the problem of the existence and stability of stationary rotations of the top for a wide range of types of high-frequency periodic vibrations of its suspension point in three-dimensional space is investigated.

2. Formulation of the problem

Let us consider the motion of the Lagrange top in a homogeneous gravitational field under the assumption that its suspension point $O$ undergoes a prescribed periodic motion in three-dimensional space.
We introduce a fixed coordinate system \( O_XYZ \) with origin at a fixed point \( O \) of the above space. We also introduce a translationally moving coordinate system \((Oxyz)\). Assume that the axes \( O_X \) and \( O_Y \) are directed vertically upward and that the axes of the coordinate system \( Oxyz \) are directed along the principal axes of inertia of the body for point \( O \), with \( OZ \) being the axis of its dynamical symmetry. We define the orientation of the coordinate system \( Oxyz \) in \( OXYZ \) using Euler’s angles \( \psi, \theta, \varphi \).

Let \( m \) be the mass of the body, \( l = OG \) the distance from the suspension point to the center of mass \( G \) of the top, and let \( A \) and \( C \) be, respectively, the equatorial and axial moments of inertia of the body relative to point \( O \). The radius vector \( O, O \) in the coordinate system \( O_XYZ \) will be given by the components \( O_x O = (u(t), v(t), w(t))^T \), assuming that the functions \( u(t), v(t) \) and \( w(t) \) are periodic with frequency \( \Omega \).

Assume that the greatest deviation \( h_s \) of the suspension point of the top from the point \( O \) is small in comparison with the reduced length \( L = A/ml \), and the frequency \( \Omega \) of its oscillations is large in comparison with the characteristic frequency \( \omega_s = \sqrt{g/L} \). Introduce a small parameter \( \varepsilon^2 = h_s/L \) \((0 < \varepsilon \ll 1)\) and set \( \omega_s/\Omega \sim \varepsilon^2, h_s/\Omega \sim 1 \).

Introduce generalized momenta \( P_\psi, P_\theta, P_\varphi \) conjugate to the angles \( \psi, \theta, \varphi \). Using the methods of perturbation theory under the assumptions made, one can find \cite{24} a canonical change of variables \( \psi, \theta, P_\psi, P_\theta \rightarrow \tilde{\psi}, \tilde{\theta}, \tilde{P}_\psi, \tilde{P}_\theta \), which reduces the Hamiltonian to an autonomous form up to terms of degree 4 in \( \varepsilon \). Neglecting terms \( O(\varepsilon^4) \), we write the approximate Hamiltonian in the form \cite{23}

\[
\tilde{H} = \frac{P_\psi^2}{2C} + \frac{\tilde{P}_\theta^2}{2A} + \frac{(\tilde{P}_\psi - P_x \cos \tilde{\theta})^2}{2A \sin^2 \tilde{\theta}} + \Pi(\tilde{\theta}, \tilde{\psi}),
\]

\[
\Pi(\tilde{\theta}, \tilde{\psi}) = \Pi_v + \Pi_\psi, \quad \Pi_v = -mg \cos \tilde{\psi} \sin \tilde{\theta},
\]

\[
\Pi_\psi = \frac{m^2}{2A} \left[ ((a_2 - a_1) \sin^2 \tilde{\psi} + a_2 \cos^2 \tilde{\psi}) \sin^2 \tilde{\theta} + (a_X Z \sin \tilde{\psi} - a_Y Z \cos \tilde{\psi}) \sin 2\tilde{\theta} - a_{XY} \sin 2\tilde{\psi} \right].
\]

The term \( \Pi_v \) is a vibrational potential \cite{14–16}. In the expression for it the following notation is used:

\[
a_1 = a_Y - a_X, \quad a_2 = a_Y - a_Z, \\
a_X = \langle u^2 \rangle, \quad a_Y = \langle v^2 \rangle, \quad a_Z = \langle w^2 \rangle, \quad a_{XY} = \langle uv \rangle, \quad a_{YZ} = \langle vw \rangle.
\]

Here, the angle brackets denote average (for a period) values of the functions appearing in them.

We note that both the initial nonautonomous and the approximate system with the Hamiltonian function (2.1) have a cyclic coordinate \( \varphi \), and the corresponding momentum \( P_\varphi \) is constant.

The approximate system describes the solutions of the initial system with error of order \( \varepsilon^{4-\gamma} \) for the coordinates and order \( \varepsilon^{3-\gamma} \) for momenta on the time interval \( t \sim \varepsilon^{2-\gamma} \) \cite{16}. Assuming \( \gamma = 5/2 \), we find that the solutions of the complete system on the time interval \( t \sim \varepsilon^{-1/2} \) are related to the solutions of the approximate system by

\[
\psi = \tilde{\psi} + O(\varepsilon^{3/2}), \quad \theta = \tilde{\theta} + O(\varepsilon^{3/2}), \\
P_\psi = \tilde{P}_\psi + O(\varepsilon^{1/2}), \quad P_\theta = \tilde{P}_\theta + O(\varepsilon^{1/2}).
\]

In what follows, we will study the motions of the Lagrange top staying within the framework of the approximate system with the Hamiltonian (2.1).
In this work we will consider cases of vibrations of the suspension point which allow vertical upper and lower relative equilibria of the symmetry axis of the top, which take place under the condition \( a_{XY} = a_{YZ} = a_{XZ} = 0 \). This case includes, in particular, the case of arbitrary (within the framework of the assumptions made) vibrations in the horizontal plane, the case of vibrations in the vertical plane (under the condition \( a_{XY} = 0 \)), and others.

It should be noted that in the case of “vibrational symmetry” \( a_X = a_Z \) for a different choice of axes of the moving coordinate system (when the axis \( OZ \) is vertical) the system has two cyclic coordinates. This case will not be dealt with in this study.

Introducing dimensionless parameters, momenta and time by the formulae

\[
\alpha = \frac{a_1 m^2 l^2}{A^2 \omega^2}, \quad \beta = \frac{a_2 m^2 l^2}{A^2 \omega^2}, \quad \gamma = \frac{A}{C},
\]

\[
\tilde{P}_\psi = A \omega_s P_1, \quad \tilde{P}_\theta = A \omega_s P_2, \quad P_\varphi = A \omega_s P_3, \quad \tau = \omega_s t
\]

and omitting the tildes, we rewrite the Hamiltonian in the form

\[
H = \frac{\gamma P_3^2}{2} + \frac{P_2^2}{2} + \frac{(P_1 - P_3 \cos \theta)^2}{2 \sin^2 \theta} - \cos \psi \sin \theta - \frac{1}{2}((\beta - \alpha) \sin^2 \psi + \beta \cos^2 \psi) \sin^2 \theta.
\] (2.2)

When \( P_3 = 0 \), the system with the Hamiltonian (2.2) has partial solutions for which the axis of dynamical symmetry of the top executes pendulum-type motions in the fixed vertical plane \( OXY \) or \( OYZ \). Further we will consider the motion of the top’s axis in the plane \( OXY \). We will call the vibrations of the suspension point of the top along the axes \( OX \), \( OY \) and \( OZ \) longitudinal, vertical and transverse, respectively.

For the motions we have \( \theta = \pi/2 \), \( P_2 = 0 \), and the change of the variables \( \psi \) and \( P_1 \) is described by canonical equations with the Hamiltonian function

\[
\dot{\hat{H}} = \frac{P_1^2}{2} - \cos \psi - \frac{\alpha}{2} \cos^2 \psi.
\] (2.3)

Here, \( \psi \) is the angle defining the position of the axis of the top relative to the lower vertical position.

We note that the Hamiltonian function (2.3) depends only on the parameter \( \alpha \), which characterizes the difference of intensities (defined by the root-mean-square values of velocities) of vertical and longitudinal vibrations. The parameter \( \beta \) defining the difference of intensities of vertical and transverse vibrations does not influence the pendulum-type motions of the top’s axis, but does influence their stability.

The goal of this paper is to describe and investigate the orbital (linear) stability of pendulum-type motions of the Lagrange top with a vibrating suspension point, which are given by a system of canonical equations with the Hamiltonian function (2.3), with respect to the variables \( \theta, \psi \) and \( P_1, P_2 \) at the fixed value \( P_3 = 0 \).
3. Pendulum-type motions of the system with the Hamiltonian (2.3)

3.1. Phase portraits

We first consider the motion of the system with one degree of freedom with the Hamiltonian (2.3). This is a conservative system with the potential energy

$$\Pi(\psi, \alpha) = -\cos \psi - \frac{\alpha}{2} \cos^2 \psi.$$ (3.1)

The system has a first integral (energy integral) of the form

$$h = \frac{P_1^2}{2} - \cos \psi - \frac{\alpha}{2} \cos^2 \psi = \text{const}.$$ (3.2)

Let us describe the behavior of the function $\Pi(\psi, \alpha)$ depending on the value of the parameter $\alpha$; by virtue of the evenness of this function in $\psi$, it suffices to consider the interval $\psi \in [0; \pi]$.

For $|\alpha| \leq 1$ the function (3.1) has the maximum point $\psi = 0$ and the minimum point $\psi = \pi$ (Fig. 1a). For $\alpha > 1$ the potential energy has two minimum points $\psi = 0$ and $\psi = \pi$, and the maximum point $\psi_* = \pi - \arccos(1/\alpha)$ (Fig. 1b). For $\alpha < -1$ the function (3.1) has two maximum points $\psi = 0$ and $\psi = \pi$, and the minimum point $\psi_* = \arccos(-1/\alpha)$ (Fig. 1c).

The minimum and maximum points of the potential energy correspond, respectively, to stable and unstable equilibrium points of the system with the Hamiltonian (2.3).

To define other motions of the system, we consider relation (3.2) and rewrite it as

$$P_1 = \pm \sqrt{2h + 2 \cos \psi + \alpha \cos^2 \psi}.$$ (3.3)

The trinomial radical expression (quadratic in the quantity $u = \cos \psi$) has two roots for $2\alpha h < 1$:

$$\psi_1 = \arccos u_1, \quad \psi_2 = \arccos u_2,$$

$$u_1 = \frac{1 - \sqrt{1 - 2\alpha h}}{\alpha}, \quad u_2 = \frac{1 + \sqrt{1 - 2\alpha h}}{\alpha}.$$ (3.3)

The condition of existence of the roots $u_1$ and $u_2$, together with the restrictions $|u_1| \leq 1$ and $|u_2| \leq 1$, divides the region of parameters $\alpha$ and $h$ into eight regions with qualitatively different motion patterns (Fig. 2). The parts of the hyperbola $2\alpha h = 1$ (with $\alpha > 0$ and $\alpha < -1$), the straight lines $h = -\alpha/2 \pm 1$ and the ray $\alpha = 0$, $h > -1$ serve as boundaries of the regions. $B_1 (-1, -1/2)$ and $B_2 (1, 1/2)$ denote the common points of the hyperbola and the straight lines (bifurcation points).

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**Fig. 1.** Potential energy for $|\alpha| \leq 1$, $\alpha > 1$ and $\alpha < -1$.
Qualitatively different phase portraits of the system with Hamiltonian (2.3) are shown in Fig. 3a–c for the cases $|\alpha| \leq -1$, $\alpha > 1$ and $\alpha < -1$, respectively.

The points of the lower boundary of region $\Gamma_1$ correspond to the stable lateral equilibrium point (Fig. 3c). For $\alpha \geq -1$ the straight line $h = -1 - \alpha/2$ corresponds to the stable lower equilibrium point (Figs. 3a, 3b), and for $\alpha < -1$ it corresponds to the unstable lower equilibrium point and to motion along the separatrix passing through the origin of coordinates (Fig. 3c). For $\alpha \leq 1$ the straight line $h = 1 - \alpha/2$ corresponds to the unstable upper equilibrium point and to an oscillation with the level set of the energy integral $h = 1 - \alpha/2$ near the lower equilibrium point. The upper boundary of region $\Gamma_2$ corresponds to the unstable lateral equilibrium point and to motion along the separatrix (Fig. 3b).

In region $\Gamma_0$ the motion is impossible.

In region $\Gamma_1$ the axis of the top executes oscillations in a neighborhood of stable lateral equilibria corresponding to closed curves in Fig. 3c, which enclose the points $(\pm \psi_*, 0)$. The range of possible values of the angle $\psi$ for these oscillations is given by the interval $\psi \in (0; \arccos(-1 - 2/\alpha))$.

In regions $\Gamma_2$ and $\Gamma_2$ the axis of the top executes oscillations near the lower equilibrium point. These oscillations correspond to the closed curves that enclose the origin of coordinates in Fig. 3a, to a part of analogous closed curves in Fig. 3b (up to the level set of the energy integral $h = 1 - \alpha/2$) and to closed curves in Fig. 3c which enclose three singular points of the system and separatrix trajectories. For $|\alpha| \leq 1$ the amplitude of oscillations can change from 0...
to π, for α > 1 the maximal value of the amplitude decreases to \( \arccos(1 - 2/\alpha) \), and for α < -1 the amplitude of oscillations can change from \( \arccos(-1 - 2/\alpha) \) to π.

Each point of region Γ₂₃ corresponds to motions of two types. Motions of the first type are oscillations near the lower equilibrium point. In Fig. 3b they correspond to the closed curves enclosing the oscillation trajectory with the level of energy \( h > 1 - \alpha/2 \). Motions of the second type are oscillations near the upper equilibrium point with the maximal amplitude \( \arccos(1/\alpha) \), they correspond to the closed curves enclosing the point \((±\pi, 0)\) in Fig. 3b.

In regions Γ₃₁, Γ₃₂ and Γ₃₃ the system executes rotations, they correspond to nonclosed curves in the phase portraits of Fig. 3.

3.2. Integration of pendulum-type motion

Let us integrate the equations of motion of the system with the Hamiltonian (2.3) in each of the regions in Fig. 2. Taking into account that, by virtue of (2.3), \( \dot{\psi} = P₁ \), and using (3.3), we obtain

\[
\frac{d\psi}{d\tau} = \pm \sqrt{2h + 2\cos \psi + \alpha \cos²\psi}.
\]

Again assuming that \( u = \cos \psi \), we rewrite this equation as

\[
\tau = \pm \int_{u₀}^{u} \frac{du}{\sqrt{f(u)}} \quad f(u) = \alpha(1 - u²)(u - u₁)(u - u₂).
\]  

The integral on the right-hand side of (3.4) is expressed in terms of elliptic functions. In what follows, we will use the following notation: \( F(u, k) \) is an elliptic integral of the first kind, \( sn(ξτ, k) \) and \( cn(ξτ, k) \) are an elliptic sine and an elliptic cosine, respectively, \( K(k) \) is a complete elliptic integral of the first kind, and \( k \) is the absolute value of the elliptic integral.

For values of the parameters \( \alpha \) and \( h \) from region Γ₁ the roots of the polynomial \( f(u) \) from (3.4) satisfy the condition \(-1 < u₂ < u₁ < 1\). For oscillation trajectories in a neighborhood of the lateral position we have \( u₂ ≤ u ≤ u₁ \). Assuming that \( u|_{τ=0} = u₂ \), and using the relations from [25], we obtain

\[
\tau = -\frac{F(u₄, k₁)}{ξ₁}, \quad u₄ = \arcsin \sqrt{\frac{(1 - u₁)(u₂ - u)}{(1 - u)(u₂ - u₁)}},
\]

\[
ξ₁ = \frac{1}{2} \sqrt{-\alpha(1 - u₁)(u₂ + 1)}, \quad k₁ = \sqrt{\frac{2(u₂ - u₁)}{(1 - u₁)(u₂ + 1)}}.
\]

Inverting the resulting function \( \tau(u) \) with respect to \( u \), we obtain

\[
u(τ) = \frac{(u₂ - u₁)sn²(ξ₁τ, k₁) - u₂(1 - u₁)}{(u₂ - u₁)sn²(ξ₁τ, k₁) - 1 + u₁}.
\]

The solution thus obtained is periodic with frequency \( \omega₁ = πξ₁/2K(k₁) \).

The motions in region Γ₂₁ are oscillations in a neighborhood of the lower equilibrium point, for which we have \( u₁ < -1 < u₂ ≤ u ≤ 1 \).
For oscillations near the lower equilibrium point we have
\[ u = \omega \text{ and its frequency is} \]
the upper position, \( u - 1 < 0 \).
\[ \xi_{21} = \frac{1}{2} \sqrt{\frac{\alpha (u - 1)}{2}}, \quad k_{21} = \sqrt{\frac{(u - 1)(u + 1)}{2(u - 1)}}. \]

This solution is periodic with frequency \( \omega_{21} = \pi \xi_{21}/2K(k_{21}) \).

The motions in region \( \Gamma_{22} \) are oscillations near the lower equilibrium point with \(-1 < u_2 < u < 1 < u_1 \). In this case, the solution with the initial condition \( u|_{\tau=0} = 1 \) has the form
\[ u(\tau) = \frac{u_1(u_1 - 1)\sin^2(\xi_{22}\tau, k_{22}) + u_2 - u_1}{(u_1 - 1)\sin^2(\xi_{22}\tau, k_{22}) + u_2 - u_1}, \]
\[ \xi_{22} = \frac{1}{2} \sqrt{-2\alpha(u_2 - u_1)}, \quad k_{22} = \sqrt{\frac{(u_1 + 1)(u_2 - 1)}{(u_1 - 1)(u_2 - 1)}}. \]

and its frequency is \( \omega_{22} = \pi \xi_{22}/2K(k_{22}) \).

In region \( \Gamma_{23} \) the roots of the polynomial \( f(u) \) satisfy the conditions \(-1 < u_1 < u_2 < 1 \).

For oscillations near the lower equilibrium point we have \( u \in [u_2; 1] \), and for oscillations near the upper position, \( u \in [-1; u_1] \). For oscillations of the first type with \( u|_{\tau=0} = 1 \) we obtain
\[ u(\tau) = \frac{(1 - u_2)\sin^2(\xi_{23}\tau, k_{23}) - u_2 - 1}{(u_2 - 1)\sin^2(\xi_{23}\tau, k_{23}) - u_2 - 1}, \]
\[ \xi_{23} = \frac{1}{2} \sqrt{\alpha(u_2 + 1)(1 - u_1)}, \quad k_{23} = \sqrt{\frac{(u_1 + 1)(u_2 - 1)}{(u_1 - 1)(u_2 + 1)}}. \]

For oscillations of the second type with \( u|_{\tau=0} = -1 \) we have
\[ u(\tau) = \frac{(u_1 + 1)\sin^2(\xi_{23}\tau, k_{23}) + u_1 - 1}{(u_1 + 1)\sin^2(\xi_{23}\tau, k_{23}) - u_1 + 1}. \]

The frequencies of both solutions are the same and equal to \( \omega_{23} = \pi \xi_{23}/2K(k_{23}) \).

For rotations in region \( \Gamma_{31} \) \( u \in [-1; 1] \) the polynomial \( f(u) \) has two real roots \( u = \pm 1 \), and the roots \( u_1 \) and \( u_2 \) are complex conjugate. Let \( u_{1,2} = a \pm ib \). Integrating with the initial condition \( u|_{\tau=0} = -1 \), we obtain [25]
\[ a = -\frac{1}{\alpha}, \quad b = \frac{\sqrt{2\alpha h - 1}}{\alpha}, \quad a_1 = \sqrt{(1 - a)^2 + b^2}, \quad b_1 = \sqrt{(1 + a)^2 + b^2}, \]
\[ u(\tau) = \frac{(a_1 + b_1)\cos(\xi_{31}\tau, k_{31}) + a_1 - b_1}{(b_1 - a_1)\cos(\xi_{31}\tau, k_{31}) - a_1 - b_1}, \]
\[ \xi_{31} = \sqrt{\alpha a_1 b_1}, \quad k_{31} = \sqrt{\frac{4 - (a_1 - b_1)^2}{4a_1 b_1}}. \]

The frequency of the resulting solution is \( \omega_{31} = \pi \xi_{31}/4K(k_{31}) \).
For rotations in region $\Gamma_{32}$ ($u \in [-1; 1]$), the roots of the polynomial $f(u)$ are real, with $u_1 < u_2 < -1 < 1$. Integrating (3.4) with the initial condition $u|_{\tau=0} = 1$, we obtain

$$u(\tau) = \frac{2u_1 \sin^2(\xi_{32} \tau, k_{32}) - u_1 - 1}{2 \sin^2(\xi_{32} \tau, k_{32}) - u_1 - 1},$$

$$\xi_{32} = \frac{1}{2} \sqrt{\alpha(u_2 - 1)(u_1 + 1)}$$

$$k_{32} = \sqrt{\frac{2(u_2 - u_1)}{(u_2 - 1)(u_1 + 1)}}.$$

The frequency of the solution is $\omega_{32} = \pi \xi_{32}/2K(k_{32})$.

For rotations in region $\Gamma_{33}$ we have $u_2 < -1 \leq u \leq 1 < u_1$. The solution with the initial condition $u|_{\tau=0} = 1$ can be represented as

$$u(\tau) = \frac{2u_1 \sin^2(\xi_{33} \tau, k_{33}) - u_1 - 1}{2 \sin^2(\xi_{33} \tau, k_{33}) - u_1 - 1},$$

$$\xi_{33} = \frac{1}{2} \sqrt{-\alpha(1 - u_2)(u_1 + 1)},$$

$$k_{33} = \sqrt{\frac{2(u_1 - u_2)}{(1 - u_2)(u_1 + 1)}}$$

with frequency $\omega_{33} = \pi \xi_{33}/2K(k_{33})$.

**Boundary curves.** We now integrate the equations of motion on the curves separating the regions considered. The points on the boundary of regions $\Gamma_{23}$ and $\Gamma_{31}$ correspond to motion along a separatrix (Fig. 3b), and the radicand $f(u)$ is a complete square. Integration is performed in elementary functions [27], and the dependence $\tau(\psi)$ has the form

$$\tau = \frac{1}{\sqrt{\alpha^2 - 1}} \ln \left( \sqrt{\frac{\alpha^2 - 1}{1 + \alpha \cos \psi}} \phi + \frac{\alpha}{1 + \alpha \cos \psi} \right), \quad \psi|_{\tau=0} = 0.$$

As $\tau \to \pm \infty$, this solution asymptotically tends to the point $\psi_* = \pi - \arccos(1/\alpha)$, which corresponds to the lateral equilibrium point of the top.

Consider the rectilinear boundary $h = 1 - \alpha/2$. For $\alpha < 1$, its points correspond to motions along separatrices (Figs. 3a, 3c), and for $\alpha > 1$, they correspond to oscillations near the lower equilibrium point (Fig. 3b). On this boundary the function $f(u)$ has the double root $u = -1$. Integrating, we find

$$u(\tau) = \frac{\mu^2 + (4\alpha - 6)\mu + 1}{\mu^2 + (4\alpha - 2)\mu + 1}, \quad \mu = e^{2\tau \sqrt{4 - \alpha}}, \quad u|_{\tau=0} = 1.$$

The function $u(\tau)$ in the region $\alpha > 1$ is a periodic function with frequency $2\sqrt{4 - \alpha}$, and in the region $\alpha < 1$ it asymptotically approaches the point $u = -1$, which corresponds to the upper equilibrium point. On the boundary of regions $\Gamma_1$ and $\Gamma_{22}$, which corresponds to motion along the separatrix (Fig. 3c), the radicand $f(u)$ has the double root $u = 1$. Integration gives

$$u(\tau) = \frac{\mu^2 + (4\alpha + 6)\mu + 1}{\mu^2 - (4\alpha + 2)\mu + 1}, \quad \mu = e^{2\tau \sqrt{1 - \alpha}}, \quad u|_{\tau=0} = u_1.$$

On this solution, the axis of the top asymptotically approaches the lower equilibrium point.
For the bifurcation point $B_2$ the function $f(u)$ is a complete square, so that integration yields

$$\psi = 2 \arctg \tau, \quad \psi|_{\tau=0} = 0.$$ 

The axis of the top on this solution asymptotically tends to the upper equilibrium point $\psi = \pi$.

On the boundary $\alpha = 0$, the equations of motion of the top’s axis (in this approximation) are identical with the equations of motion of a mathematical pendulum in a homogeneous gravitational field; the results of integration of these equations are presented, for example, in [26].

4. Perturbed motion

Let us introduce action-angle variables [26] for a system with the Hamiltonian (2.3) in each of the regions of oscillations and rotations. We introduce the angle variable by the formula $w = \omega \tau$, where $\omega$ is the frequency of pendulum-type motion, and introduce the action variable in the case of oscillations by the formula

$$I = \frac{2}{\pi} \int_{\psi_0}^{\hat{\psi}} P_1 d\psi = \frac{2}{\pi} \int_{u_0}^{\hat{u}} \sqrt{f_1(u)} du, \quad f_1(u) = \frac{\alpha(u - u_1)(u - u_2)}{1 - u^2},$$

where $\hat{\psi}$ is the maximal angle of deviation along the axis of the top from the lower vertical position, and in the case of rotations, by the formula

$$I = \frac{1}{\pi} \int_{0}^{\pi} P_1 d\psi = \frac{1}{\pi} \int_{1}^{-1} \sqrt{f_1(u)} du,$$

where $\omega$ is the frequency of the motion, which is calculated by formulae from Section 3.2.

The Hamiltonian of unperturbed motion in the action-angle variables can be written in the form [26]

$$\dot{H} = \omega I.$$

Consider a system with the Hamiltonian function (2.2). As unperturbed motion we take pendulum-type motions of the system with the Hamiltonian (2.3) as described in the previous section. Next, we rewrite the Hamiltonian (2.2) in the variables $I, w, P_2$ and $\theta$ and introduce perturbations $r = I - I_0$, $p_2 = P_2$, $q_2 = \theta - \pi/2$, where $I_0$ is the value of the action variable on the unperturbed motion, and expand the resulting Hamiltonian function as a series in the perturbations $r, p_2, q_2$.

$$H = H_2 + H_4 + \ldots.$$ (4.1)

Here, $H_{2k}$ is the form of degree $2k$ in the variables $q_2, p_2$ and $|r|^{1/2}$, and the form $H_2$ is

$$H_2 = \omega r + f_{20} q_2^2 + f_{02} p_2^2,$$

$$f_{20} = \frac{1}{2} \left( \cos \psi + (\beta - \alpha) \sin^2 \psi + \beta \cos^2 \psi + P_1^2 \right), \quad f_{02} = \frac{1}{2}.$$
In the coefficient $f_{20}$, the functions $\psi(I,w)$ and $P_1(I,w)$ are calculated on the unperturbed motion. The values of $\psi(I,w)$ are determined from the formulae obtained in Section 3.2, and the values of $P_1(I,w)$ are found using relation (3.3).

The problem of the orbital stability of pendulum-type motions of the body is equivalent to the problem of the stability of a canonical system with the Hamiltonian (4.1) with respect to the variables $r$, $q_2$, $p_2$. The criterion of stability of this system is equivalent to the criterion of stability of a reduced system with one degree of freedom, which describes the motion of the system on the isoenergetic level $H = 0$ corresponding to the unperturbed motion [24]. We take the quantity $w$ as a new independent variable. Let us solve the relation $H = 0$ for $r$ by rewriting it as

$$r = -K(q_2,p_2,w),$$

where

$$K = K_2 + K_4 + \ldots, \quad K_2 = f_{20}q_2^2 + f_0p_2^2. \tag{4.2}$$

The equations of motion of the system with the Hamiltonian (4.2) can be written as

$$\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}. \tag{4.3}$$

The problem of the orbital stability of pendulum-type motions of the Lagrange top with a vibrating suspension point (within the framework of the approximate autonomous system) reduces to the problem of stability of the equilibrium point of the nonautonomous Hamiltonian system (4.3) with one degree of freedom.

5. Linear stability analysis

The linearized system of equations of perturbed motion which corresponds to the Hamiltonian $K_2$ has the form

$$\frac{dp_2}{dw} = \frac{2}{\omega}f_{02}p_2, \quad \frac{dp_2}{dw} = -\frac{2}{\omega}f_{20}q_2. \tag{5.1}$$

The characteristic equation of this system can be written as

$$\rho^2 - 2\kappa\rho + 1 = 0, \tag{5.2}$$

where $2\kappa = x_{11}(T) + x_{22}(T)$, $T = 2\pi/\omega$ is a period of the solution $\psi(\tau)$. The functions $x_{11}(w)$ and $x_{22}(w)$ are the elements of the matrix $X(w) = ||x_{ij}(w)|| (i, j = 1, 2)$ of the fundamental solutions satisfying the initial conditions $X(w) = E_2$, where $E_2$ is the second-order unit matrix.

If $|\kappa| > 1$, then the equilibrium point of the complete system with the Hamiltonian (4.2) is unstable. If $|\kappa| < 1$, linear stability takes place. In the boundary case $\kappa = 1$, the characteristic equation has the multiple root $\rho = 1$ (first-order resonance), and in the boundary case $\kappa = -1$, the multiple root $\rho = -1$ (second-order resonance). A complete solution of the problem of stability of the solution of the system (5.1) in the case $|\kappa| \leq 1$ requires nonlinear analysis. This investigation is not carried out in this work.
5.1. Generating curves

For points \((\alpha, h)\) lying on the boundary curves (see Fig. 2) which correspond to stable equilibrium points in the unperturbed motion, the system (5.1) is autonomous. In this case, the conditions of stability of the lower equilibrium point \((h^* = -1 - \alpha/2, \alpha > 1)\) in the system (5.1) are given by the inequality \(\beta > -1\), those of the upper equilibrium point \((h^* = 1 - \alpha/2, \alpha > 1)\) are given by the inequality \(\beta > 1\), and those of the lateral equilibrium point \((h^* = 1/2, \alpha < -1)\) are given by the relation \(\alpha > \beta\). The frequencies of small linear oscillations which correspond to lateral, lower and upper stable equilibrium points of the system are calculated, respectively, by the formulae

\[
\Omega_1 = \sqrt{\frac{4\alpha(\alpha - \beta)}{\alpha^2 - 1}}, \quad \Omega_2 = \sqrt{\frac{\beta + 1}{\alpha + 1}}, \quad \Omega_3 = \sqrt{\frac{\beta - 1}{\alpha - 1}}.
\]

In the three-dimensional space of parameters \(\alpha, \beta, h\) near the surfaces \(h^* = h^*(\alpha)\), regions of instability (regions of parametric resonance) can appear. These regions are generated by the curves on the surfaces \(h^* = h^*(\alpha)\) such that the points of these curves satisfy the relations \(\Omega_j = n\) (first-order resonance) or \(\Omega_j = (2n - 1)/2\) (second-order resonance); here \(n\) is a natural number, and \(j = 1, 2, 3\).

The generating curves corresponding to lateral, lower and upper equilibria are shown in Figs. 4a, 4b and 4c in the plane of parameters \(\alpha, \beta\). Light gray denotes regions of instability of the corresponding equilibrium point. The solid lines show first-order resonance curves and the intermittent lines indicate second-order resonance curves. For the lower and lateral equilibrium points, a countable set of generating curves (Figs. 4a, 4b) emanates from the point \(\alpha = -1, \beta = -1\). For the upper equilibrium point, such a bundle emanates from the point \(\alpha = 1, \beta = 1\) (Fig. 4c). In Fig. 4, dark gray denotes regions in which there is a countable set of generating curves of first- and second-order resonances.

\[\text{Fig. 4. Generating curves for lateral, lower and upper equilibria.}\]

5.2. Results of stability analysis

To solve the question of stability of the pendulum-type motions under study, we perform a numerical integration of Eqs. (5.1) and construct the boundaries \(|\kappa| = 1\) of the regions of parametric resonance. Calculations have shown that in the three-dimensional parameter space two resonant surfaces (the boundaries of instability regions) emanate from the generating curves of first- and second-order resonances.
of first-order resonance at \( h > h_\ast \), while only one resonant surface emanates from the generating curves of second-order resonance, and there are no regions of parametric resonance.

In the space of the parameters \( \alpha, \beta, h \), the sections \( \beta = \text{const} \) were considered for different values of \( \beta \) and stability diagrams were obtained in the plane of parameters \( \alpha \) and \( h \). The following parameter range was explored: \( \alpha \in [-5; 5] \) and \( h \in [-3.5; 3.5] \).

A series of stability diagrams was constructed for values of \( \beta \) from the interval \( \beta \in [-2; 2] \); the case of oscillations in a neighborhood of the upper equilibrium point was considered for the values \( \beta \in [-2; 3] \). As is evident from Fig. 4, the cases \( \beta < -1, -1 < \beta < 1 \) and \( \beta > 1 \) are qualitatively different. When \( \beta < -1 \), the root-mean-square velocity of transverse vibrations of the suspension point exceeds the root-mean-square vertical velocity \( (a_Y - Ag/(ml) > a_Z) \), with increasing \( \beta \) the influence of the two components becomes equalized, and when \( \beta > 1 \) \( (a_Y + Ag/(ml) < a_Z) \) the vertical vibrations predominate over the transverse ones.

Stability diagrams are shown in Figs. 5–11 for the cases \( \beta = -2 \) (Fig. 5), \( \beta = -1 \) (Fig. 6), \( \beta = 0 \) (Fig. 7), \( \beta = 0.85 \) (Fig. 8), \( \beta = 1 \) (Fig. 9) and \( \beta = 2 \) (Fig. 10). For region \( \Gamma_2 \) Figs. 5–10 show the evolution of the stability of oscillations near the lower position; the picture of stability of oscillations in a neighborhood of the upper position is shown in Fig. 11 for the cases \( \beta = 2 \) and \( \beta = 3 \).

The boundaries of the region shown in Figs. 5–11 as heavy lines correspond to the boundaries in Fig. 2. The regions enclosed between them are divided into several regions of linear stability and regions of instability; the boundaries between them, shown by thin lines, are first-order resonance curves or vertical straight lines corresponding to the case of "vibration symmetry" \( \alpha = \beta \), which has been excluded from consideration. The second-order resonance curves are shown in Figs. 5–11 as intermittent lines.

Regions of linear stability in Figs. 5–11 are not colored, light gray indicates regions of instability, and dark gray indicates regions containing a countable set of resonance curves. The boundaries of the dark gray regions are drawn along one of the first- or second-order resonance curves. These regions include a part of the boundaries of the bifurcation diagram (see Fig. 2), whose points correspond to asymptotic motions of the system with the Hamiltonian \( (2.3) \).

We describe the evolution of the stability properties of each type of motions.
5.2.1. Stability of oscillations in a neighborhood of the lateral equilibrium point

Let us consider oscillations in a neighborhood of the lateral equilibrium point in region $\Gamma_1$. These oscillations exist and are unstable under the condition $\beta < \alpha < -1 \ (a_Z > a_X > a_Y + + Ag/(ml))$, and can be stable for $\alpha < -1, \alpha < \beta$, when the parameter $a_X$, which characterizes the intensity of longitudinal vibrations, is the largest among the three quantities $a_X, a_Z, a_Y + + Ag/(ml)$.

The generating curves for first- and second-order resonance surfaces on the lower boundary of region $\Gamma_1$ are shown in Fig. 4a. In the sections $\beta = \text{const}$ with $\beta < -1$ there exist two generating points of first-order resonance and two generating points of second-order resonance. When $\beta > -1$, a countable set of generating points arises in a neighborhood of the bifurcation point $B_1$. 

Fig. 6. Diagram of stability for $\beta = -1$.

Fig. 7. Diagram of stability for $\beta = 0$. 
Consider the case $\beta < -1 \ (a_X > a_Z > a_Y + Ag/(ml))$. In the corresponding sections of region $\Gamma_1$ (see Fig. 5) there are two regions $\gamma_{11}, \gamma_{12}$ of linear stability and three regions of instability. One of the instability regions lies to the right of the vertical boundary straight line $\alpha = \beta$. The boundaries of the other instability region emanate from the generating point on the lower boundary of region $\Gamma_1$ and end on the upper boundary at points $S_1$ and $S_3$. The boundary of the third instability region (adjacent to the upper boundary of region $\Gamma_1$) emanates from point $S_3$. Second-order resonance curves also pass through points $S_1$ and $S_3$.

As the value of $\beta$ increases, points $S_1$ and $S_3$ approach the bifurcation point $B_1$, and when $\beta = -1$, they merge with it, and region $\gamma_{11}$ disappears (Fig. 6). In this case, the only boundary curve separating the region into the subregion $\gamma_{12}$ of linear stability and a subregion of instability emanates from the angular point $B_1$. 
Consider the case $\beta > -1 \ (a_X > a_Y + Ag/(ml) > a_Z)$. As $\beta$ passes through $\beta = -1$, on the lower boundary of region $\Gamma_1$ in a neighborhood of the bifurcation point $B_1$ there arises a countable set of generating points from which regions of parametric resonance alternating with regions of linear stability emanate. These regions of stability and instability make up the dark gray zone $\delta_1$ in a neighborhood of the upper boundary of region $\Gamma_1$. When $-1 < \beta < 0$, with increasing $\beta$, the first generating point of the region of parametric resonance moves along the lower boundary of region $\Gamma_1$ towards a decrease in $\alpha$, and when $\beta = 0$, it moves indefinitely away (together with the instability region and region $\gamma_{12}$). The stability region $\gamma_{13}$ shown in Figs. 7–10 lies to the left of the following region of parametric resonance. When $\beta \geq 0$, it occupies a considerable part of region $\Gamma_1$ (except for region $\delta_1$).

The region $\delta_1$, which is adjacent to the upper boundary of region $\Gamma_1$ (Figs. 7–10), contains a countable set of alternating regions of linear stability and regions of instability.

We note that, as $\alpha$ increases, the boundary of region $\gamma_{12}$ runs along the lower boundary of region $\Gamma_1$, and the boundaries of region $\gamma_{13}$ and the resonance curves from region $\delta_1$ run along the upper boundary of $\Gamma_1$.

Thus, in the case $\beta \leq -1$ the region of stability of oscillations in a neighborhood of lateral relative equilibria is bounded above by the level of energy $h$ (dependent on the amplitude of
oscillations), or the oscillations are always unstable in the region of existence. In the case \( \beta > -1 \), for each value \( \alpha < -1 \) there is a countable set of intervals of change of \( h \) (up to the upper boundary of region \( \Gamma_1 \)), where both stability and instability of the motions under study take place.

### 5.2.2. Stability of oscillations near the lower equilibrium point

We now consider oscillations near the lower equilibrium point which occur in regions \( \Gamma_{21}, \Gamma_{22} \) and \( \Gamma_{23} \) (see Fig. 2).

When \( \alpha < -1 \) (\( a_Y + Ag/(ml) < a_X \)), the points of the common lower rectilinear boundary of these regions correspond to asymptotic motions of the unperturbed system; when \( \alpha > -1 \), in the case \( \beta < -1 \) (\( a_X < a_Y + Ag/(ml) < a_Z \)), these points correspond to the unstable lower equilibrium, and in the case \( \beta > -1 \) (\( a_Y + Ag/(ml) > a_X, a_Y + Ag/(ml) > a_Z \)) they correspond to the stable lower equilibrium.

Let first \( \beta < -1 \) (see Fig. 5), then on the whole lower boundary of the regions we have instability. Calculations show that the bifurcation points \( S_3 \) and \( S_4 \) described in Section 5.2.1 are generating points of two regions \( \gamma_{22} \) and \( \gamma_{21} \) of linear stability of oscillations of the system near the lower equilibrium point. The first of them, which arises in a neighborhood of point \( S_3 \), lies in the region \( \alpha < \beta < -1 \), its right boundary is the straight line segment \( \alpha = \beta \), the angular point \( S_2 \) lies on the intersection of this straight line with the upper boundary of region \( \Gamma_{22} \).

The second region of stability exists for \( \alpha > \beta \), and its upper boundary is almost horizontal. Region \( \gamma_{21} \) passes through the whole region of oscillations under consideration. The width of the region depends on the value of the parameter \( \beta \). When \( \beta = -2 \), this region is rather narrow (Fig. 5), with growing \( \beta \) (but with \( \beta < -1 \)) region \( \gamma_{21} \) becomes increasingly wider, and its lower boundary approaches gradually the common lower boundary of regions \( \Gamma_{21} \) and \( \Gamma_{22} \).

Two second-order resonance curves pass through the bifurcation points \( S_1, S_2 \) and \( S_4 \) inside the above-mentioned stability regions. Outside regions \( \gamma_{21} \) and \( \gamma_{22} \), the oscillations are unstable in the part of the plane of the parameters \( \alpha \) and \( h \) in the case \( \beta < -1 \).

In the section \( \beta = -1 \) the points \( S_1 \) and \( S_3 \), merging with each other, merge with point \( B_1 \). The instability region near and to the left of the point takes the shape shown in the insert of Fig. 6; with further increase in \( \beta \) this region decreases and then disappears when \( \beta > 0 \). The lower boundary of region \( \gamma_{21} \) with \( \beta = -1 \) coincides with the lower boundaries of regions \( \Gamma_{21} \) and \( \Gamma_{22} \) (which correspond to the stable lower equilibrium of the system). With further increase in \( \beta \) the qualitative shape of region \( \gamma_{21} \) remains the same, but the angular point \( S_1 \) moves down and to the right along the lower boundary straight line of regions \( \Gamma_{21} \) and \( \Gamma_{22} \).

Thus, if the relation \( \alpha > \beta > -1 \) (\( a_Y + Ag/(ml) > a_Z > a_X \)) is satisfied, the oscillations in a neighborhood of the lower equilibrium are stable up to some (almost fixed) level of energy.

Let us consider the evolution of the stability properties in the part of the region lying to the left of the straight line \( \alpha = \beta \) (the case \( a_Y + Ag/(ml) > a_X > a_Z \)). As \( \beta \) passes through \( \beta = -1 \), in a neighborhood of the bifurcation point \( B_1 \) on the lower boundary of the region there arises a countable set of generating points of first- and second-order resonances (see Fig. 4b), and in the region of oscillations under study, in addition to the stability region \( \gamma_{22} \), there arises a countable set of regions of linear stability, which alternate with instability regions. These regions lie in the dark gray zone \( \delta_{21} \) (see Fig. 7).

As the parameter \( \beta \) increases, the zone \( \delta_{21} \) expands (together with the stability and instability regions located in it). When \( \beta \approx 0.047719 \), the upper boundary of the stability region \( \gamma_{23} \)
approaches at a tangent the upper boundary of region $\Gamma_{22}$ and has with the latter a common point $S_4$. In the part of the parameter region under study, point $S_4$ appears when $\beta \approx 0.841551$.

Between the stability regions $\gamma_{22}$ and $\gamma_{23}$ there is an instability region which arises on the lower boundary straight line and has a common segment $S_2 S_4$ with the upper boundary straight line; this region becomes narrow as $\beta$ grows (see Figs. 8–10).

For values of $\beta$ from the interval $0.047719 < \beta < 1.132590$ the stability region $\gamma_{23}$ fills a large part of region $\Gamma_{22}$. Other resonance curves generated by the points lying on the lower boundary of region $\Gamma_{22}$ converge also to the angular point $S_4$. Thus, near the upper boundary of regions $\Gamma_{21}$, $\Gamma_{22}$ and $\Gamma_{23}$ to the left of point $S_4$ there appears a thin region $\delta_{22}$ with a countable set of resonance curves and stability and instability regions. In the diagrams presented, this region has discernible dimensions only in Fig. 10. As the value of $\beta$ increases, point $S_4$ moves along the upper boundary of regions $\Gamma_{21}$, $\Gamma_{22}$ and $\Gamma_{23}$. When $\beta = 1.132590$, point $S_4$ merges with point $S_2$. With further increase in $\beta$ regions $\delta_{21}$ and $\delta_{22}$ expand (Fig. 10).

5.2.3. Stability of oscillations in a neighborhood of the upper equilibrium point

We now consider oscillations in a neighborhood of the upper equilibrium point which occur in region $\Gamma_{23}$. These oscillations exist and are unstable when the conditions $\alpha > 1$ and $\beta \leq 1$ are satisfied (i.e., when $a_Y < a_Y - Ag/(ml) \leq a_Z$).

In the case $\alpha > 1$ and $\beta > 1$, i.e., when the quantity $a_Y - Ag/(ml)$ has the largest value among the three quantities $a_Y - Ag/(ml)$, $a_X$ and $a_Z$, oscillations can be both stable and unstable. Diagrams of stability of these motions are shown in Fig. 11 for the cases $\beta = 2, 3$. As $\beta$ changes, the stability pattern is qualitatively the same. In the case $\alpha > \beta$ ($a_Z > a_X$), when the transverse vibrations are more intense than the longitudinal ones, the region of existence of the oscillations breaks down into an instability subregion and the subregion $\gamma'_{21}$ of linear stability. These two subregions are divided by the boundary curve emanating from point $S_5$.

Thus, in this case the stability of the upper oscillations is observed for all values of $\alpha$, up to a certain level of energy $h(\alpha)$, which defines the amplitude of oscillations.

In the case $\alpha < \beta$ ($a_Z < a_X$), when the longitudinal oscillations are more intense than the transverse ones, the stability pattern is different. The region to the left of the boundary $\alpha = \beta$ contains a countable set of stability and instability regions, where the first region $\gamma'_{22}$ of linear stability adjacent to this boundary is shown. All first- and second-order resonance curves, starting at different points of the lower boundary of the region, converge to one point $S_2$, at which the upper boundary of the region and the straight line $\alpha = \beta$ intersect. In this case, for each $1 < \alpha < \beta$ there is a countable set of intervals of stability and instability in the whole corresponding range of change of the parameter $h$.

5.2.4. Stability of rotations

Let us consider the regions of rotations $\Gamma_{31}$, $\Gamma_{32}$ and $\Gamma_{33}$. Calculations show that, in the region given by the conditions $\alpha < \beta \leq 0$ ($a_Y \leq a_Z < a_X$), the rotations are unstable (see Figs. 5–7). If $\beta \leq 0$ and $\alpha > \beta$ ($a_Z > a_X, a_Z \geq a_Y$), then the rotations are unstable up to a certain level of energy (which is almost independent of the value of the parameter $\alpha$), and are linearly stable on a large level of energy (region $\gamma_{31}$).

As $\beta$ passes through $\beta \approx 0.047719$, a qualitative rearrangement of the stability diagram occurs in the region of rotations. The instability region with $\alpha < \beta$ and the stability region with $\alpha > \beta$ become bounded from above by the (common) boundary above which we have,
respectively, the region \( \gamma_{32} \) of linear stability and a region of instability. Simultaneously, the point \( S_4 \) (see Section 5.2.2) appears on the lower boundary of the region of rotations, to the right of this point there is a curvilinear triangle (with the side \( S_2S_4 \)) of the instability region; to the left, towards a decrease in \( \alpha \) (and also on the other side of the boundary), a countable set of resonance curves and regions of stability and instability (which form a narrow region \( \delta_{31} \)) emanates from point \( S_4 \).

With increasing \( \beta \) the point \( S_4 \) moves along the lower boundary of the region of rotations to the right towards \( S_2 \). In this case, the stability region increases when \( \alpha < \beta \) and decreases when \( \alpha > \beta \), and disappears at \( \beta \approx 1.132590 \) (when points \( S_4 \) and \( S_2 \) merge). As \( \beta \) increases further, the rotations are always unstable when \( \alpha > \beta \) and always stable when \( \alpha < \beta \), except for the points of zone \( \delta_{31} \) with alternating regions of stability and instability (see Fig. 11).

### 5.2.5. Summary

The results presented in this paper on the stability of pendulum-type motions of the Lagrange top pertain to a wide spectrum of vibrations of the suspension point. Each particular law of vibration of the suspension point of the top corresponds to one of the stability diagrams described above (or to one that does not qualitatively differ from it).

For example, when the suspension point of the top undergoes horizontal longitudinal motion (case \( \beta = 0 \), \( \alpha < 0 \)) or transverse motion (case \( \alpha = 0 \), \( \beta < 0 \)), we have stability results presented in Fig. 7 (left half-plane) and in Figs. 5 and 6 (ordinate axis), respectively.

If the suspension point of the top moves in the horizontal plane \( OXZ \) along an elliptic trajectory given by the law \( X = a \cos \Omega t \), \( Z = b \sin \Omega t \), then in the cases \( a > b \) (with the ellipse elongated in the longitudinal direction) and \( a < b \) (with the ellipse elongated in the transverse direction) we have \( \alpha < \beta < 0 \) and \( \beta < \alpha < 0 \) and the corresponding parts of the left half-plane in Figs. 5 and 6.

When the suspension point moves in the plane \( OXY \), which coincides with the plane of pendulum-type motions of the top’s axis, according to the law \( X = a \cos \Omega t \), \( Y = b \sin \Omega t \) in the cases \( b > a \) (with the ellipse elongated along the vertical), \( b = a \) (a circular trajectory) or \( b < a \) (with the ellipse elongated in the longitudinal direction) we obtain, respectively, \( 0 < \alpha < \beta \), \( 0 = \alpha < \beta \) or \( \alpha < 0 < \beta \). In these cases, the stability diagrams are represented by the corresponding parts in Figs. 8–10, and for the case \( 0 < \alpha < \beta \) also by a part of Fig. 11.

Similarly, for the cases of an elliptic trajectory of the suspension point in the vertical plane \( OYZ \) perpendicular to the plane of pendulum-type motions, for cases where the ellipse is elongated along the vertical (\( 0 < \beta < \alpha \)), we have the corresponding parts of the stability diagram in Figs. 8–10 and a part of diagrams in Fig. 11. The cases where the ellipse is elongated in the transverse direction (\( \beta < 0 < \alpha \)) or is a circle (\( \beta = 0 < \alpha \)) correspond to a part of diagrams in Figs. 5–7.

We point out one feature of the results obtained. The shape of pendulum-type motion is defined by the parameters \( \alpha \) and \( h \), and the stability pattern depends also on the parameter \( \beta \). As is evident from the diagrams, the stability properties of pendulum-type motions change in most cases to the opposite as \( \beta \) crosses the straight line \( \alpha = \beta \).

Therefore, if \( \beta \) is close to \( \alpha \), then, by choosing a value of \( \beta \) (i.e., by increasing and decreasing the intensity of the transverse component of vibrations) one can almost always achieve stabilization of the pendulum-type motions under study.
References


