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Characteristics of Chaotic Regimes in a Space-distributed Gyroklystron Model with Delayed Feedback

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Within the framework of the nonstationary model with nonfixed field structure, we investigate the model of a 3-mm band gyrokystron with delayed feedback. It is shown that both chaotic and hyperchaotic generation regimes are possible in this system. The chaotic regime due to a Feigenbaum period-doubling cascade is characterized by one positive Lyapunov exponent. Further transition to hyperchaos is determined by the appearance of another positive exponent in the Lyapunov spectrum. The correlation dimension of the corresponding attractors reaches values of about 3.5.

Keywords: chaos, hyperchaos, Lyapunov exponents, gyrokystron

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Introduction

Among the various types of gyroamplifiers, gyroklystrons have received the greatest research effort. They are characterized by a combination of high efficiency and amplification coefficient [38, 44]. It is of interest to obtain *chaotic* generation regimes using them as a basis. As in the case with usual klystrons [17, 25], this can be achieved by introducing external delayed feedback.

The dynamics of gyroklystrons with delayed feedback was examined earlier only within the framework of approximation of a fixed field structure [1, 18], which can be applied only in the case of relatively high Q factors of the input and output resonators. Meanwhile, to broaden the band of chaotic generation, it is appropriate to use low- Q resonators and thus apply the *spatial time-domain* approach. One of its versions is based on the description of the field evolution by a parabolic equation.

The statistical methods of generation analysis are the main tools for determining its chaotic properties: power spectrum, autocorrelation function, correlation dimension of an attractor in phase space, characteristic Lyapunov exponents etc. [8, 11, 36]. The spectrum of Lyapunov exponents, namely, its signature, is the most informative for differentiation of periodic and quasi-periodic behavior, and various types of chaos: the number of positive exponents in the spectrum determines the dimension of the unstable manifold of a chaotic attractor, and their absence determines the regularity of the regime.

The total number of Lyapunov exponents of a dynamical system is equal to the dimension of phase space. In the case of a distributed system the phase space is infinite-dimensional, there are infinitely many exponents, and there is a high probability that *hyperchaos* will arise. The detection of this more “developed” type of chaotic behavior in the system under discussion provides important additional information on the dynamics of the gyroklystron, which was missing from the previous “simpler” models.

Calculation of Lyapunov exponents in distributed systems, including electronic ones, is a fairly complicated problem. To date, such an analysis, which has allowed hyperchaotic generation regimes to be identified, has been carried out for gyrotrons [10] and backward-wave tubes [7, 37]. Also, low- and high-dimensional chaos was detected, for example, in electronic traveling-wave systems [39, 40], laser-based delayed-feedback systems [29], ionization waves [4], and in some examples of classical plasmic and electronic systems [26].

In Section 1 we formulate the spatial time-domain model of a gyroklystron with delayed feedback. In Section 2 we perform a numerical investigation of the characteristic features of generation regimes in the system for different values of the control parameter. In Section 3, for two specific parameter values, using realizations possessing qualitative characteristics of chaotic and hyperchaotic behavior, we carry out a quantitative analysis of Lyapunov exponents, which confirms the assumption on the pattern of observed dynamics. In doing so, we apply a refined method of calculating Lyapunov exponents from time series, which is the least laborious for the systems considered and provides a fairly high accuracy. In addition, this approach simulates analysis of experimental time realizations, which is of particular importance both for applications and from a fundamental point of view, specifically, for verification of the applicability of the mathematical model used.

1. The model and the main equations

Assume that the axisymmetric space of interaction of the gyroklystron with length z_k whose profile is given by the function $r(z)$ includes an input and an output resonator, which are sepa-

rated by a drift space that is overcritical for the operating mode (Fig. 1a). Assume also that the weakly relativistic helical electron beam interacts in resonators with a wave at frequency ω close to cut-off frequency ω_c and gyrofrequency $\omega_H = eH_0/mc\gamma_0$, where H_0 is the guiding magnetic field, c is the velocity of light, e and m are the charge and the mass of particles, respectively, and γ_0 is the initial relativistic mass factor of electrons. Excitation of input radiation and extraction of radiation from the output resonator are performed by diffraction, which corresponds to quite a number of experimentally realized systems [3, 45, 49]. To form a feedback loop, a part of output radiation with transmission coefficient R and delay time t_0 is entered to the amplifier input. Let us choose cut-off frequency as a carrier and represent the field in the operating region as $\vec{E} = \text{Re} [A(z, t) \vec{E}_\perp(r) \exp(i\omega_c t - im\varphi)]$, where $A(z, t)$ is the slowly varying complex amplitude of the field, the function $\vec{E}_\perp(r)$ describes the radial structure of the operating mode, and φ is the azimuth angle. Under such conditions, electron-wave interaction can be described by a system of equations that include an inhomogeneous equation of parabolic type for the evolution of the complex amplitude of the field along with averaged equations of electron motion [22]:

$$\begin{aligned} i \frac{\partial^2 a}{\partial Z^2} + \frac{\partial a}{\partial \tau} + (\sigma(Z) + i\delta(Z)) a &= \frac{I_0}{2\pi} \int_0^{2\pi} p d\theta_0, \\ \frac{\partial p}{\partial Z} + \frac{g^2}{4} \frac{\partial p}{\partial \tau} + ip(\Delta_H - 1 + |p|^2) &= -a. \end{aligned} \quad (1.1)$$

Fig. 1. Scheme of the gyrokystron model (a) and the amplitude characteristic of the gyrokystron without a feedback loop (the dotted line indicates the load characteristic of the feedback loop for $R = 0.52$) (b).

The following normalized variables have been used in the system of equations (1.1):

$$\begin{aligned} \tau &= \frac{\omega_c t \beta_{\perp 0}^4}{8\beta_{\parallel 0}^2}, & Z &= \frac{\beta_{\parallel 0}^2 \omega_c z}{2\beta_{\parallel 0} c}, & p &= \frac{(p_x + ip_y) e^{-i\omega_c t + i(m-1)\varphi}}{p_{\perp 0}}, \\ a &= \frac{eAJ_{m-1}(R_0\omega_c/c)}{mc\omega_c\gamma_0\beta_{\perp 0}^3}, & I_0 &= 16 \frac{eI_b}{mc^3} \frac{\beta_{\parallel 0}}{\beta_{\perp 0}^6 \gamma_0} \frac{J_{m-1}^2(R_0\omega_c/c)}{(\nu_n^2 - m^2)J_m^2(\nu_n)}, \end{aligned}$$

where I_b is the current of the electron beam, J_m is the Bessel function, ν_n is the n th root of the equation $J'_m(\nu) = 0$, R_0 is the radius of injection of the helical electron beam, $g = \beta_{\perp 0}/\beta_{\parallel 0}$ is the pitch factor, $V_{\perp 0} = \beta_{\perp 0}c$ and $V_{\parallel 0} = \beta_{\parallel 0}c$ are the initial values of the transverse and longitudinal velocities of electrons, $\Delta_H = 2(\omega_c - \omega_H)/\omega_c\beta_{\perp 0}^2$ is the parameter of mismatch between the cut-off frequency of the operating mode and the unperturbed value of gyrofrequency. The function $\delta(Z) = 8\beta_{\parallel 0}^2(\omega_c - \omega_c(Z))/\beta_{\perp 0}^4\omega_c$, $\omega_c(Z) = c\nu_n/r(Z)$ describes the profile of the electrodynamic

system, and the function $\sigma(Z)$ defines the Ohmic losses related to the finite conductivity of the walls of the resonators and absorbers in the drift space.

We assume that at the entrance into the interaction space the electrons are uniformly distributed over cyclotron rotation phases $p(Z=0) = \exp(i\theta_0)$, $\theta_0 = [0, 2\pi)$. At the exit of the system in the section $Z = L$, where $L = \beta_{\perp 0}^2 \omega_c z_k / 2\beta_{\parallel 0} c$ is the normalized length of the resonator, the radiation boundary condition [24] is imposed:

$$a(L, \tau) + \frac{1}{\sqrt{\pi i}} \int_0^\tau \frac{1}{\sqrt{\tau - \tau'}} \frac{\partial a(L, \tau)}{\partial Z} d\tau' = 0. \quad (1.2)$$

To take account of the feedback loop, Eqs. (1.1) and (1.2) are supplemented with the boundary condition with the external signal entered to the input (see [23]):

$$a(0, \tau) - \frac{1}{\sqrt{\pi i}} \int_0^\tau \frac{1}{\sqrt{\tau - \tau'}} \frac{\partial a(0, \tau)}{\partial Z} d\tau' = R \cdot a(Z = L, \tau - \tau_0), \quad (1.3)$$

where $\tau_0 = \omega_c t_0 \beta_{\perp 0}^4 / 8\beta_{\parallel 0}^2$ is the normalized delay time of the signal.

In the normalizations used the electron efficiency can be written as

$$\eta = \frac{g^2}{1 + g^2} \eta_{\perp}, \quad \eta_{\perp} = 1 - \frac{1}{2\pi} \int_0^{2\pi} |p|^2 d\theta_0, \quad (1.4)$$

where $\eta_{\perp}$ is the so-called transverse efficiency.

In what follows, we consider the dynamics of a two-cavity gyrokystron with an operating frequency of 93 GHz, which is experimentally investigated in [49]. We assume that the helical electron beam with pitch factor 1.3, injection radius 2.7 mm, energy 70 keV and current 15 A interacts with the mode TE_{02} in the input and output resonators on the fundamental cyclotron harmonic. In this case, the values of the normalized parameters are $I_0 = 0.034$ and $L \approx 15$. We use the normalized value of delay $\tau_0 = 20$, which corresponds to the physical time it takes for the signal to pass along the feedback loop (about 1.4 ns). To broaden the generation band, we reduce the Q factor of the output resonator, Q , to values close to the minimal diffraction Q factor $Q \approx 130$ [46].

For the above-mentioned parameter values, we perform a numerical simulation of the boundary-value problem (1.1)–(1.3) while varying the transmission coefficient R , which we choose as the control parameter. As simulations show, the self-excitation of the system occurs at $R = 0.2$. An increase in the transmission coefficient to $R \approx 0.52$ transfers the load characteristic $a = a_0/R$, where a_0 is the amplitude of the input signal, to the area of maximal decrease of the amplitude characteristic (Fig. 1), which must lead to periodic self-modulation [13, 35]. As the transmission coefficient increases further, the dynamics of the system becomes more complicated. Special features of this dynamics will be considered in the next section.

2. Dynamics of a gyrokystron with delayed feedback

From numerical simulations we have obtained time series of absolute values of the slow complex amplitude of generation at the output of the amplifier $|a_i| = |a(i \cdot \Delta\tau)|$, where $\Delta\tau = 0.21125$ is the time interval between the counts. Figure 2 gives examples of such series for $R = 0.67$ and $R = 0.8$, which correspond to distinctly irregular behavior.

Fig. 2. Time dependence of the amplitude of the output signal: (a) in the chaotic regime at $R = 0.67$, (b) in the hyperchaotic regime at $R = 0.8$ (b).

Next, we consider the structure of the attractors. Despite the spatial extension of the system under study, even having only a time series at a localized point of space, one can reconstruct the phase portrait. For systems with delayed feedback one usually shows its projection onto the plane of values of the variable at instants that differ by the delay period. A similar approach is applied to recover multidimensional phase space from the scalar time realization of an unknown system [8, 11, 36]. According to this approach, we represent the phase space as a space of vectors

$$\{|a(\tau)|, |a(\tau + \tau_1)|, |a(\tau + 2\tau_1)|, \dots, |a(\tau + (d-1)\tau_1)|\} = \mathbf{a}, \quad (2.1)$$

where τ_1 is the delay time, which must be such that the components of the vector $\mathbf{a}$ in the space of these variables are sufficiently independent. Such τ_1 is in correlation with the time scale of natural oscillations of the system. We have estimated time τ_1 from the autocorrelation function. Figure 3 represents two variations of this function, which correspond to the realizations in Fig. 2. The delay $\tau_1 = 70 \cdot \Delta\tau$ shown in the enlarged fragments corresponds to a significant decrease of the functions. It turns out that such a choice is also optimal for further calculations of the Lyapunov exponents.

Figures 4a–4g show two-dimensional projections of phase portraits for several characteristic values of R starting from the threshold of self-oscillations: a) fixed point for the absolute value of the amplitude (P0); b) limit cycle (P1); c)–d) doubling of the limit cycle (P2–P4); e) complex structure of attractor (C); f) periodic structure (P3); g) again the complex structure (HC).

In order to ascertain the properties of an attractor that is more complex than the limit cycle, we can construct a Poincaré section for it. We have considered the section formed by the intersection with the plane $|a(\tau)| = |a(\tau + \tau_1)|$, which is the most convenient in our case. For regular modifications of the attractor we have obtained in the section fixed points (Figs. 4h and 4i), a cycle of period 2 (Fig. 4j), 4 (Fig. 4k), 3 (Fig. 4m) as predicted. The section in Fig. 4l is a distinctly fractal set and possesses features of a chaotic attractor of Feigenbaum type. The set in Fig. 4n is also fractal, and, compared with the case in Fig. 4l, it is “fuzzier”.

Fig. 3. Autocorrelation function for a wide range (left) and a narrow range (right) of time scales for the cases $R = 0.67$ (a) and $R = 0.8$ (b).

Constructions of the Poincaré section are suitable to illustrate the transformation of dynamical regimes for various values of the control parameter in the form of a bifurcation diagram (Fig. 4o). It is seen in the diagram that after the onset of self-oscillations the increase in the transfer ratio leads to a cascade of period-doubling bifurcations and to a transition to the regime of chaotic self-modulation according to the Feigenbaum scenario at $R \approx 0.65$.¹ With further increase in the transfer ratio in the interval of values up to $R \approx 0.72$ we have a chaotic crown of the bifurcation tree studded with many windows of periodicity, when self-modulation assumes a regular pattern. One of the most pronounced windows with $R \approx 0.71$ is depicted in Fig. 4o. The phase portrait and the Poincaré section with $R = 0.711$ that correspond to period 3 (basic to this window) are shown in Figs. 4f and 4m.

We note that the described alternation of regimes (Feigenbaum cascade) is also typical of the model of a usual klystron [38]. When $R \geq 0.72$, the model under study obviously exhibits regimes of chaos that is more “developed” than the Feigenbaum chaos, or hyperchaos. The crown

¹It should be kept in mind that formally it is because a description within the framework of slow amplitudes $|a|$ is used that such a scenario is possible. For the initial model, a sequence of torus-doubling bifurcations will take place instead, which is finite in the general case, in contrast to the infinite Feigenbaum cascade [2, 19, 30]. In fact, the difference between the two situations can hardly be noticed on the scale of resolution by the parameter we consider here.

Fig. 4. Projection of the phase portrait recovered by the method of delay with interval $\tau_1 = 70\Delta\tau$, at $R = 0.52$ (a), $R = 0.61$ (b), $R = 0.63$ (c), $R = 0.64$ (d), $R = 0.67$ (e), $R = 0.711$ (f), $R = 0.8$ (g). The panels of the figure show attractor (P0) corresponding to the absence of modulation, attractors of period 1 (P1), 2 (P2), 3 (P3), 4 (P4), chaotic attractor (C) and hyperchaotic attractor (HC). (h–n) is the Poincaré section formed by the intersection of these attractors with the plane $|a(\tau)| = |a(\tau + \tau_1)|$. (o) is a bifurcation diagram.

of the tree becomes homogeneous, it lacks large-scale windows. The attractor that is typical of this area, with $R = 0.8$, looks like having no pronounced structure (fuzzy) as compared with the Feigenbaum attractor with $R = 0.67$. If we return to the autocorrelation functions (see Fig. 3), we can notice that their envelope on large times falls down much faster for $R = 0.8$.

Considerable changes in the topology of the attractor must affect its correlation dimension [36]. Its calculation also allows an estimate of the minimal dimension d of the recovered phase space in which the detected attractors have no self-intersections, i.e., the embedding dimension.

To determine the embedding dimension, the correlation dimension D_2 of the attractor was calculated depending on d . Figure 5 shows graphs for the correlation integral $C(\delta) = \frac{1}{N(N-1)} \sum_{n,m=1}^N \theta(\delta - \|\mathbf{a}_n - \mathbf{a}_m\|)$ (here, θ is the Heaviside step function), and the inclination angle of these graphs is D_2 . This figure also shows the dependences of D_2 on d , which is saturated to the value ~ 2.3 in the case $R = 0.67$ and to the value ~ 3.5 in the case $R = 0.8$. This takes place at a space dimension, d , of at least 4–5, which gives an estimate of the embedding dimension from below. The value $2D_2 + 1$ (~ 6 and ~ 8 , respectively) gives an estimate from above for the two regimes.

The fact that the correlation dimension exceeds the value 3 at $R = 0.8$ points towards an increase in the dimension of the unstable manifold of the attractor and suggests that it becomes a hyperchaotic attractor. However, this is established with certainty only by calculating the spectrum of Lyapunov exponents.

Fig. 5. Correlation integral (left) for the dimension of the recovered space, $d = 1$ (triangles), 2 (rhombi), 3 (skew crosses), 4 (squares), 5 (straight crosses), 6 (circles), 7 (asterisks). Dependence of the correlation dimension on the dimension of space d (right). Control parameter $R = 0.67$ (a) and $R = 0.8$ (b).

3. Calculation of the spectrum of Lyapunov exponents

We recall that the Lyapunov exponents have the meaning of average velocities of exponential stretching and compression in different orthogonal directions of the phase volume along the reference trajectory belonging to the attractor, and are arranged in the spectrum in decreasing order. The state of this phase volume can be described by the basis from perturbation vectors relative to the reference trajectory. There are several main methods to keep track of the dynamics of the above vectors and to calculate the Lyapunov exponents using these vectors. The Benettin algorithm [9] is the best-known of them. To apply it, the system must be capable of being written in the form of a mathematical model that can be solved at least numerically, for example, in the form of ordinary differential equations. Model equations are solved together with a set of several (according to the number of sought-for exponents) linearized systems of equations in variations, each of which allows one to directly calculate the dynamics of its perturbation vector and to compute a particular Lyapunov exponent.

If it is difficult to perform a numerical simulation of the equations in variations, one can make use of another modification of the Benettin algorithm. It involves solving the system under study in the steady-state regime along with a set of the same systems, but with perturbed initial conditions. Differences between the state vector of the system, located on the attractor, and the state vectors of the associated close systems will determine the sought-for perturbation vectors.

In situations where calculation of close trajectories is very laborious or impossible (for example, in a natural experiment), the stability analysis of complex and chaotic attractors can

be restricted to a single system located on the attractor or even to one realization of only one scalar variable of the system. In such cases, to define the perturbation vectors, one chooses satellite trajectories from fragments of the reference trajectory itself on the attractor (which is reconstructed if necessary).

To calculate Lyapunov exponents from time series, the second modification of the Benettin algorithm [15, 16, 47] can be easily adapted (see also [11, 12]). However, this approach has a disadvantage, which is that it requires an optimal choice of the size of the small neighborhood from which the satellite trajectories start: a too large neighborhood does not necessarily satisfy the linear approximation, and a too small neighborhood can generate an increase in the machine error. Moreover, the probability of statistical error involved in this method is high, since at a fixed instant of time each single perturbation vector is determined from only one close trajectory.

A method that is more reliable and does not use linearized equations, i.e., can be applied to time series, is that based on calculating covariance matrices [42, 50] (see also the review [11, 12]). This method implies that, for the basis of the perturbation vectors, the linear evolution operator is evaluated from the statistical pattern of dynamics of a large number of satellite trajectories.

The specificity of the calculation of the integral term in the system (1.1) as a sum over “large particles” with a level of nonlinearity corresponding to formation of phase bunches leads to errors that decrease very slowly as the number of “large particle” increases. This effect is partially neutralized by using the solutions of the initial equations on neighboring trajectories near the reference trajectory² rather than using linearized equations. Moreover, the solution of a set of linearized systems, in addition to the system (1.1), for application of an exact (in the sense of a differential limit) Benettin algorithm, as well as the solution of a large number of “close” systems and application of the second (difference) modification of the algorithm is an exceptionally laborious problem. For the model under consideration, analysis of time realizations corresponding to attractors appears to be the most suitable method of calculating Lyapunov exponents.

Given its higher accuracy and in the context of saving machine time, the method of covariance matrices seems to be the most attractive since it is suitable for relatively short time series, and it is this method that will be used in what follows. Despite of its advantages, it is not very widely known, so we describe it in more detail.

The Lyapunov exponents Λ_i can be defined as follows:

$$\Lambda_i = \lim_{M \rightarrow \infty} S_M^i, \quad S_M^i = \frac{1}{MT} \sum_{m=1}^M \ln \left\| \mathbf{b}^{(T)}(\tau = mT) \mathbf{e}_i(\tau = mT) \right\|. \quad (3.1)$$

It is also useful to consider the spectrum of local exponents averaged over a small time interval $T_0 = M_0 T$:

$$\lambda_i(\tau = MT) = \frac{1}{M_0 T} \sum_{m=M-M_0}^M \ln \left\| \mathbf{b}^{(T)}(\tau = mT) \mathbf{e}_i(\tau = mT) \right\|. \quad (3.2)$$

Here, $\mathbf{e}_i$ are the perturbation vectors, which at the initial instant are taken to be arbitrary, but orthogonal to each other, and then, after each multiplication by $\mathbf{b}^{(T)}$ and calculation of the subsequent contribution to the sums (3.1) and (3.2), are subjected to orthogonalization and normalization, for example, using the Gram–Schmidt algorithm. Accordingly, T is the period of orthogonalization, and $\mathbf{b}^{(T)}$ is the matrix of the linear operator of evolution in time T .

²This was, for example, a motivation for using this modification of the Benettin method in [7, 37].

The linear operator $\mathbf{b}^{(\Delta T)}(\tau)$ of the evolution of perturbation in small time ΔT can be found as follows. From the time series $|a(\tau)|$ for the current point one selects other points located at a distance not greater than ε from this point in the reconstructed state space (2.1), but not too close to it in time. For each such chosen point $|a(\tau')|$ one forms the vectors $\mathbf{y} = \mathbf{a}(\tau') - \mathbf{a}(\tau)$ and $\mathbf{z} = \mathbf{a}(\tau' + \Delta T) - \mathbf{a}(\tau + \Delta T)$. For each pair of these vectors there is a transformation matrix: $\mathbf{z} = \mathbf{b}^{(\Delta T)}\mathbf{y}$. In the described algorithm, one searches for some approximation of it. The application of the least square method for it leads to the necessity of solving the matrix equation $\mathbf{b}\mathbf{v} = \mathbf{c}$, where $\mathbf{v}$ and $\mathbf{c}$ are covariance matrices whose (j, k) th elements are calculated as averaged over all found fragments of close trajectories of the product of the components of the vectors $\mathbf{y}$ and $\mathbf{z}$: $v^{j,k} = \langle y^j y^k \rangle$, $c^{j,k} = \langle z^j z^k \rangle$. By finding the covariance matrices, one can easily solve the matrix equation: $\mathbf{b} = \mathbf{c}\mathbf{v}^{-1}$.

The above procedure was applied to time series of length $\sim 2 \cdot 10^5 \Delta\tau$, which is of order 10^3 of periods of natural oscillations. The recovered attractors corresponding to the series were embedded in the space of dimension $d = 5$, which was sufficient for a correct choice of close trajectories and hence for a satisfactory estimate of the transformation matrix $\mathbf{b}$. The number of the Lyapunov exponents to be estimated and, accordingly, perturbation vectors to be calculated (it must not exceed the dimension of phase space) was chosen to be equal to the embedding dimension³. The matrix $\mathbf{b}$ was estimated for period $\Delta T = \tau_1$. On such an interval the linear approximation holds well. The orthogonalization of the vectors $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ from (3.1) and (3.2) was performed with period $T = 5 \cdot \Delta T$, which approximately corresponds to one circuit of the trajectory on the attractor. The transformation matrix for period T was calculated as the product: $\mathbf{b}^{(T)}(\tau) = \mathbf{b}^{(\Delta T)}(\tau) \cdot \mathbf{b}^{(\Delta T)}(\tau + \Delta T) \cdot \mathbf{b}^{(\Delta T)}(\tau + 2\Delta T) \cdot \mathbf{b}^{(\Delta T)}(\tau + 3\Delta T) \cdot \mathbf{b}^{(\Delta T)}(\tau + 4\Delta T)$. The radius of the neighborhood in which the search for close trajectories was performed was taken to be 3% of the typical size of the attractor so that it could be increased in rare cases successively to 9%.

Under the above conditions, the following estimate of the Lyapunov exponents was obtained:

$$\begin{aligned} \Lambda_1 \Delta\tau &= 0.0011, & \Lambda_2 \Delta\tau &= 0.0000, \\ \Lambda_3 \Delta\tau &= -0.0013, & \Lambda_4 \Delta\tau &= -0.0029, & \Lambda_5 \Delta\tau &= -0.0092 \end{aligned} \quad (3.3)$$

for $R = 0.67$ and

$$\begin{aligned} \Lambda_1 \Delta\tau &= 0.0061, & \Lambda_2 \Delta\tau &= 0.0026, & \Lambda_3 \Delta\tau &= 0.0000, \\ \Lambda_4 \Delta\tau &= -0.0033, & \Lambda_5 \Delta\tau &= -0.0098 \end{aligned} \quad (3.4)$$

for $R = 0.8$. The accumulating sums from (3.1) converge rather fast to these values, and the local exponents (3.2) (Fig. 6) vary relative to them. These results allow the conclusion that, indeed, the case $R = 0.67$ corresponds to the regime of chaos generation, and $R = 0.8$, since the two exponents are positive, corresponds to the regime of hyperchaos.

It should be noted that in such a model, as in a number of other space-distributed systems (see, e.g., [7, 10, 37]), two Lyapunov exponents must be equal to zero (except for the situation where there are no oscillations). Also, one zero exponent corresponds to perturbations like a shift along the phase trajectory, and the other corresponds to a shift of the argument of the complex amplitude of the field. Since the work was performed with the time series representing real amplitudes, the second zero exponent does, of course, not manifest itself.

As regards the accuracy of the estimate of exponents, it should be noted that the fluctuations of the local values over a fairly wide range and even the change of their sign are a normal phenomenon even when an accurate calculation of the linear operator is performed.

³Thus, all vectors in the described procedure had d components and the matrices had dimension $d \times d$.

Fig. 6. Spread of local values of Lyapunov exponents λ (3.2), estimated for period $T_0 = 10T$, and time dependences of the averaging of the sum S from the formula (3.1): (a) $R = 0.67$, (b) $R = 0.8$.

A convenient “reference point” is the result of searching for a zero exponent. In our case, the accuracy of its estimate was $\sim 10^{-4}$. The dissipativity condition [36] is satisfied, namely, that the sum of all exponents takes a negative value. At the same time, the results obtained do not contradict the Kaplan–Yorke formula [11, 36] for calculation of the Lyapunov dimension $D_\lambda = k + \left(\sum_{i=1}^k \Lambda_i \right) / |\Lambda_{k+1}|$, where k is the minimal number of the largest exponents whose sum is positive. The Lyapunov dimension reflects the geometric properties of the attractor along with other fractal dimensions and must approximately correspond in value to the correlation dimension. In our case, the values of the correlation dimension 2.3 at $R = 0.67$ and 3.5 at $R = 0.8$ are indeed close to the values of the Lyapunov dimension, which are, respectively, 2.8 and 4.5.

Conclusion

In this paper, we have investigated chaotic generation regimes in the spatial time-domain gyroklystron model with delayed feedback. To establish the dynamical properties of the system, we carried out analysis of time realizations for the amplitude of the envelope of radiation at the output of the amplifier. Based on this analysis, we drew conclusions on the existence of both chaotic regimes (demonstrated earlier by the well-known models) and hyperchaotic regimes of generation. The correlation dimension of the attractors by which these regimes are characterized exceeds the values 2 and 3, respectively. The signature of the spectrum of Lyapunov exponents was treated as the main instability criterion in the analysis made. The detected chaos and hyperchaos exhibit positive values of one and two largest exponents, respectively.

Generally speaking, Lyapunov exponents in distributed systems can in many cases be searched for by using the classical Benettin algorithm, which involves calculating the perturbation vectors with the aid of equations in variations or “close systems” (see [5–7, 10, 26–28, 31, 37] for examples). Calculation of the exponents from time series, however, is the most readily accessible approach for the problem considered. The covariance matrix method used for this provides a high accuracy and shows reliable results even for relatively short time series. Given the greater statistical significance even for local estimates of exponents (i.e., for the linear operator of perturbation evolution), this method is more efficient than the Benettin algorithm adapted for time series. In addition, since the method allows a straightforward calculation of the linear evolution operator, it can be convenient, for example, to calculate covariant Lyapunov vectors [20, 21, 32, 41] and to apply the algorithm for verifying the structural stability (roughness) of the chaotic regime [33, 34].

The question to be elucidated is whether the method applied for calculation of exponents from the space-localized time realization is adequate for investigating a space-extended system. Indeed, as is well known, the distributed system has infinitely many characteristic exponents. In practice, however, even using the Benettin algorithms specially adapted for space-time dynamics, although they offer great potential, one usually defines several largest exponents [5, 6, 31]. The approach used allows an estimate of only a limited number of largest exponents, but the most essential ones. This number is equal to the embedding dimension of the attractor reconstructed by time realization. Attempts to obtain a more complete spectrum in the method considered leads only to the appearance of spurious exponents [43, 48]. This is due to the spatial reduction (implied by our approach) of the infinite-dimensional phase space of the initial distributed system (time realization at a fixed point is considered) and due to a natural partial loss of information on it. Nevertheless, the spectrum of exponents obtained is more than sufficient to identify the types of dynamical regimes.

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