NONLINEAR PHYSICS AND MECHANICS

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Qualitative Analysis of the Dynamics of a Trailed Wheeled Vehicle with Periodic Excitation

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This article examines the dynamics of the movement of a wheeled vehicle consisting of two links (trolleys). The trolleys are articulated by a frame. One wheel pair is fixed on each link. Periodic excitation is created in the system due to the movement of a pair of masses along the axis of the first trolley. The center of mass of the second link coincides with the geometric center of the wheelset. The center of mass of the first link can be shifted along the axis relative to the geometric center of the wheelset. The movement of point masses does not change the center of mass of the trolley itself. Based on the joint solution of the Lagrange equations of motion with undetermined multipliers and time derivatives of nonholonomic coupling equations, a reduced system of differential equations is obtained, which is generally nonautonomous. A qualitative analysis of the dynamics of the system is carried out in the absence of periodic excitation and in the presence of periodic excitation. The article proves the boundedness of the solutions of the system under study, which gives the boundedness of the linear and angular velocities of the driving link of the articulated wheeled vehicle. Based on the numerical solution of the equations of motion, graphs of the desired mechanical parameters and the trajectory of motion are constructed. In the case of an unbiased center of mass, the solutions of the system can be periodic, quasi-periodic and asymptotic. In the case of a displaced center of mass, the system has asymptotic dynamics and the mobile transport system goes into rectilinear uniform motion.

Keywords: trailed wheeled vehicle, nonholonomic problem, qualitative analysis, periodic excitation, time-dependent dynamic system, stability

1. Introduction

The dynamics of mobile wheeled vehicles is an area of research that is of great practical interest at the present time in connection with the development of new technologies and robotics.

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A sufficient number of works are devoted to the study of the movement of wheeled vehicles. Moreover, modifications of wheeled vehicles can be very diverse [1–6].

The first works go back to Rocard [7] and Stücker [8, 9]. An important role in the development of this field of dynamics of rigid bodies and their systems was played by the works of Lobas and Martynenko [10, 11]. Earlier works differ mainly in the use of analytical and graphical research methods. Especially noteworthy is the work [12], in which the methods of geometric mechanics are applied to the problem of a two-link wheeled vehicle.

The development of computational tools led to the spread of numerical methods [13, 14], which made it possible to study more and more complex models. A few years ago, the field of dynamics of wheeled vehicles could have been considered to be largely unexplored. However, the appearance of various works in recent years, on the one hand, increased the interest of scientists in this topic and the appearance of new wheel models, on the other hand, contributed to the development and clearer understanding of research methods.

This work is devoted to the qualitative analysis of the dynamics of a mobile articulated wheeled vehicle consisting of two trolleys, each of which is attached to one wheel pair. The trolleys are connected to each other by a rigid frame. On the first trolley, a pair of moving masses oscillates along the axis according to the periodic law. At the same time, the movement of point masses does not change the center of mass of the trolley, which in general does not coincide with the geometric center of the wheelset, but is located on the axis of the trolley. Thus, the motion of the masses creates periodic excitation in the mechanical system under study.

There has been previous work on the dynamics of the movement of multi-link wheeled vehicles [1–3, 15, 16], including the studies by the author [17, 18]. In [17], the dynamics of a multi-link wheeled vehicle moving with constant moments of inertia was studied. In [18], the focus was on the dynamics of a multi-link wheeled vehicle with a controlled driving trolley (the law of motion of the driving trolley was set).

The system dealt with in this paper can serve as some analogue of a roller-racer, which is in fact a model of a bundle of two Chaplygin sleighs, in which a knife edge is replaced with wheel pairs. An analysis of the dynamics of the Chaplygin sleigh and a roller-racer is made in [4, 19–23]. In the classical model of the inertial motion of the roller-racer [23], it is shown that the motion of a roller-racer in absolute space is asymptotic. In [4, 20] the design of a roller-racer with a given angle between the platforms as a function of time is considered. It was also proved that the increase in velocity in this model is unlimited, as in the model of the Chaplygin sleigh [19]. In the latter work an acceleration (unlimited velocity increase) was found in the special case. But in the rolling model of the roller-racer with viscous friction there is no constant acceleration and all trajectories of the reduced system asymptotically tend to a periodic solution. Thus, the following question arose: Is it possible to achieve an unlimited increase in the speed of an articulated wheeled vehicle by adding periodic excitation to the system, for example, due to vibrations of moving masses?

The same paper presents a qualitative study of the dynamics of a two-link articulated wheeled vehicle with periodic excitation, which sets the variable moment of inertia of the driving link. The possibility of an unbounded increase in angular or linear velocity in the system due to the movement of point masses is analyzed. Mathematically, the problem is reduced to the joint solution of differential equations written in Lagrangian form and derivatives of nonholonomic coupling equations. Then the resulting system of equations of motion is reduced by means of certain substitutions to a more compact three-dimensional system of equations in new variables. It is shown that the solutions of this system are bounded. This implies boundedness of the linear and angular velocities of the trolleys. This is true in general for all trajectories both as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$. The system exhibits slightly different dynamics with an unbiased center of mass of the driving link and with a displaced center of mass of the driving link. However, in the former case, the movement of the system remains curved, but in the latter case, the movement becomes

rectilinear and uniform over time. Mathematical reasoning and conclusions are confirmed by a graphic representation of the trajectories of motion and the desired mechanical parameters.

2. Mathematical model of a trailed wheeled vehicle

Consider a system of two trolleys moving on a plane. Each trolley has a wheelpair fastened to it. The links of the system are connected at the geometric centers O_1 and O_2 of the wheel pairs by a metal frame of length l . Moreover, the second link is rigidly connected to the frame, which can freely rotate relative to the fixing point with the first link O_1 . Suppose that the center of mass of the second trolley is at the geometric center O_2 of the wheelset, and that the center of mass of the first trolley in general may not coincide with the geometric center of the wheelset and is displaced along the axis of the trolley by a distance b .

Along the axis of the driving trolley, a pair of point masses m_p move according to the periodic law, moving away from the center of mass over a distance of $\delta \sin \Omega t$ and $-\delta \sin \Omega t$. Moreover, the movement of the masses does not change the location of the center of mass of the trolleys.

Fig. 1. Construction of the mobile wheeled vehicle

We introduce two coordinate systems on the plane (Fig. 1):

- a fixed coordinate system Oxy ;
- a moving coordinate system $O_1x_1y_1$, attached to the geometric center of the wheelset of the first trolley O_1 , and the axis O_1x_1 coincides with the axis of symmetry of the trolley. The moving masses move along the O_1x_1 axis according to the given laws $b \pm \delta \sin \Omega t$.

To describe the dynamics of the system under consideration, we introduce the following notation:

ρ_1 is the radius vector of the center of mass of the first trolley in the moving coordinate system;

ρ_2 is the radius vector of the center of mass of the second trolley in the moving coordinate system;

r_-, r_+ is the radius vectors of the moving masses in the moving coordinate system;

$v = (v_1, v_2)$ is the velocity of the wheelset of the first driving trolley;

ω is the angular velocity of the driving trolley;

u_i are the velocities of the centers of mass of the trolleys;

u_-, u_+ are the the velocities of the moving masses.

The orientation of the trolleys in the fixed coordinate system will be specified using the angles ψ , θ (Fig. 1): the angle ψ is the angle between the axis of the first trolley and the positive direction of the Ox axis, the angle θ is the angle between the axis of the second link and the positive direction of the Ox axis. Then the angle between the axes of the links will be, respectively, equal to $\varphi = \theta - \psi$. Moreover, if the axis of the second trolley makes a complete turn relative to the axis of the driving trolley counterclockwise, then the angle φ takes positive values, otherwise the angle φ takes negative values. The following expressions hold:

$$\dot{\psi} = \omega, \quad \dot{\theta} = \omega + \dot{\varphi}.$$

Next, we consider the movement of a wheeled vehicle in the moving coordinate system. Referred to the axes of the moving coordinate system $O_1x_1y_1$, the coordinates of the radius vectors are

$$\begin{aligned} \boldsymbol{\rho}_1 &= (b, 0), \\ \boldsymbol{\rho}_2 &= (-l \cos \varphi, -l \sin \varphi), \\ \mathbf{r}_+ &= (b + \delta \sin \Omega t, 0), \\ \mathbf{r}_- &= (b - \delta \sin \Omega t, 0). \end{aligned}$$

To abbreviate some of our forthcoming formulas, we define the vector product of the vertical vector $\mathbf{e}_z = (0, 0, 1)$ and the two-dimensional vector $\mathbf{a} = (a_1, a_2)$ as follows:

$$\mathbf{e}_z \times \mathbf{a} = (-a_2, a_1).$$

The velocities of the centers of mass will be determined by the formulas:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v} + \omega \mathbf{e}_z \times \boldsymbol{\rho}_1, \\ \mathbf{u}_2 &= \mathbf{v} + (\omega + \dot{\varphi}) \mathbf{e}_z \times \boldsymbol{\rho}_2. \end{aligned}$$

The velocities of the moving masses will be determined in accordance with the formulas:

$$\mathbf{u}_{\pm} = \mathbf{v} + \dot{\mathbf{r}}_{\pm} + \omega \mathbf{e}_z \times \mathbf{r}_{\pm}.$$

Nonholonomic constraints are imposed on the wheels on each of the links, which show whether the speed vector is codirected with the longitudinal axis of the trolley:

$$\begin{aligned} v_2 &= 0, \\ v_1 \sin \varphi - v_2 \cos \varphi + l(\omega + \dot{\varphi}) &= 0. \end{aligned} \tag{2.1}$$

Equations (2.1) are nonholonomic coupling equations.

The kinetic energy of the trolleys is represented as

$$\begin{aligned} T_1 &= \frac{1}{2} I_1 \omega^2 + \frac{1}{2} m_1 \mathbf{u}_1^2, \\ T_2 &= \frac{1}{2} I_2 (\omega + \dot{\varphi})^2 + \frac{1}{2} m_2 \mathbf{u}_2^2, \end{aligned}$$

and the kinetic energy of pairs of the moving masses as

$$T_p = \frac{1}{2} m_p ((\mathbf{u}_-)^2 + (\mathbf{u}_+)^2),$$

where m_k is the weight of the k th trolley, m_p is the weight of the moving mass, and I_k is the moment of inertia of the k th trolley about their centers of mass.

Thus, in view of the second nonholonomic coupling equation, the kinetic energy of the entire system takes the form

$$\begin{aligned} T &= T_1 + T_2 + T_p, \\ T &= \frac{1}{2}(J_1 + d^2 \sin^2 \Omega t) \omega^2 + \frac{1}{2} J_2 (\omega + \dot{\varphi})^2 + \frac{1}{2} M (v_1^2 + v_2^2) + \\ &+ M_1 b v_2 \omega + M_2 l (v_1 \sin \varphi - v_2 \cos \varphi) (\omega + \dot{\varphi}) + m_p \delta^2 \Omega^2 \cos^2 \Omega t, \end{aligned} \quad (2.2)$$

where

$$d = \sqrt{2m_p} \delta;$$

$J_1 = I_1 + m_1 b^2 + 2m_p b^2$ is the moment of inertia of the first trolley with the moving mass about O_1 ;

$J_2 = I_2 + m_2 l^2$ is the moment of inertia of the second trolley about O_1 ;

$M_1 = m_1 + 2m_p$ is the total mass of the first trolley with the moving masses;

$M_2 = m_2$ is the mass of the second trolley;

$M = M_1 + M_2$ is the mass of the entire wheeled vehicle.

We see that the kinetic energy of the system changes over time. For further modeling, the formulation of the equations of motion in Lagrangian form is required.

Equations of motion. The equations of motion with the kinetic energy (2.2) for the system under consideration are generally written in the form [24]:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial v_1} \right) - \omega \frac{\partial T}{\partial v_2} &= \lambda_1 \frac{\partial f_1}{\partial v_1} + \lambda_2 \frac{\partial f_2}{\partial v_1}, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial v_2} \right) + \omega \frac{\partial T}{\partial v_1} &= \lambda_1 \frac{\partial f_1}{\partial v_2} + \lambda_2 \frac{\partial f_2}{\partial v_2}, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \omega} \right) + v_1 \frac{\partial T}{\partial v_2} - v_2 \frac{\partial T}{\partial v_1} &= \lambda_1 \frac{\partial f_1}{\partial \omega} + \lambda_2 \frac{\partial f_2}{\partial \omega}, \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\varphi}} \right) - \frac{\partial T}{\partial \varphi} &= \lambda_1 \frac{\partial f_1}{\partial \dot{\varphi}} + \lambda_2 \frac{\partial f_2}{\partial \dot{\varphi}}, \end{aligned} \quad (2.3)$$

where the conditions f_1 and f_2 are the coupling equations defined by Eqs. (2.1).

To determine the required values, it is necessary to jointly solve the system of equations (2.3) and the time derivatives of the relations (2.1). The resulting differential equations will be solved for the time derivatives $\dot{v}_1$, $\dot{v}_2$, $\dot{\omega}$, and $\dot{\varphi}$. Moreover, the coupling equations will be integrals of the resulting system. From the coupling equations we have

$$v_2 = 0, \quad \dot{\varphi} = -\frac{v_1}{l} \sin \varphi - \omega.$$

Therefore, the resulting system can be reduced to a system that is solved for the time derivatives $\dot{v}_1$, $\dot{\omega}$, and $\dot{\varphi}$. We get a closed system in the variables v_1 , ω , and φ .

A finite system of equations solved for unknown functions v_1 , ω and φ will take the form

$$\begin{aligned} \dot{\varphi} &= -\frac{v_1}{l} \sin \varphi - \omega, \\ \dot{\omega} &= -\frac{\omega(M_1 b v_1 + \Omega d^2 \sin 2\Omega t)}{\tilde{I}_1}, \\ \dot{v}_1 &= \frac{M_1 b l^2 \omega^2 - (J_2 - 2M_2 l^2) \sin \varphi \cos \varphi v_1 \left(-\frac{v_1}{l} \sin \varphi - \omega\right)}{\Delta}, \\ \tilde{I}_1 &= J_1 + d^2 \sin^2 \Omega t, \\ \Delta &= (J_2 - 2M_2 l^2) \sin^2 \varphi + M l^2. \end{aligned} \quad (2.4)$$

The equations of motion (2.4) determine the phase flow on a 3-dimensional space $\mathcal{M}^3 = \{(\varphi, \omega, v_1) \mid \varphi \in (-\pi, \pi]\}$.

The trajectory of movement. To determine the trajectory of the driving trolley, it is also necessary to add the equations

$$\dot{x} = v_1 \cos \psi, \quad \dot{y} = v_1 \sin \psi, \quad \dot{\psi} = \omega$$

to the system (2.4).

The system (2.4) has a rather complex structure. Of particular interest is the study of the system for the existence of noncompact trajectories in the system. In other words, the question arises: is it possible to accelerate this mechanical system due to the movement of moving masses and to find out how the variable moment of inertia of the driving trolley affects the trajectory of movement?

3. The reduced system and its symmetries

For further analysis of the dynamics, we rewrite the last two equations of the system (2.4) in the form

$$\begin{aligned} \dot{\omega} \tilde{I}_1 + \tilde{I}_1 \omega &= -M_1 b v_1 \omega, \\ v_1 \sqrt{\Delta} + v_1 \frac{\dot{\Delta}}{2\sqrt{\Delta}} &= \frac{M_1 b l^2 \omega^2}{\sqrt{\Delta}}. \end{aligned}$$

We will make the following change of variables:

$$P = \omega \tilde{I}_1, \quad Q = v_1 \sqrt{\Delta}.$$

The reduced system has the final form

$$\begin{aligned} \dot{\varphi} &= -\frac{Q}{l\sqrt{\Delta}} \sin \varphi - \frac{P}{\tilde{I}_1}, \\ \dot{P} &= -\frac{M_1 b P Q}{\tilde{I}_1 \sqrt{\Delta}}, \\ \dot{Q} &= \frac{M_1 b l^2 P^2}{\tilde{I}_1^2 \sqrt{\Delta}}. \end{aligned} \tag{3.1}$$

Symmetries and involutions of equations. Equations (3.1) are invariant under the following change of variables:

- the symmetry

$$t \rightarrow t, \quad \varphi \rightarrow -\varphi, \quad P \rightarrow -P, \quad Q \rightarrow Q.$$

- the involutions

$$\begin{aligned} t \rightarrow -t, \quad \varphi \rightarrow -\varphi, \quad P \rightarrow P, \quad Q \rightarrow -Q, \\ t \rightarrow -t, \quad \varphi \rightarrow \varphi, \quad P \rightarrow -P, \quad Q \rightarrow -Q. \end{aligned}$$

Thus, the system (3.1) is reversible.

Consider the projection of the phase space on the plane (Q, P) . The set of points lying on a straight line $P = 0$ form an invariant manifold. The phase trajectories do not intersect the invariant manifold. The sign of the function $P(t)$ does not qualitatively affect the dynamics of the system.

Next, we first consider the dynamics of the reduced system (3.1) in the absence of periodic excitation caused by the movement of masses ($d = 0$), and then we consider the dynamics of a reduced system with periodic excitation ($d \neq 0$).

It is worth noting that in the case

$$J_2 = 2M_2l^2 \quad (3.2)$$

the entire mass of the second trolley is concentrated in the center of the second wheelset.

Proposition. *The system of equations of motion (3.1) is equivalent to the system of equations of motion for the Chaplygin sleigh under condition (3.2).*

There has been a fairly large amount of research devoted to a qualitative analysis of the equations of motion for the Chaplygin sleigh, for example, [19, 20, 25, 26]. Therefore, we will further assume that

$$J_2 \neq 2M_2l^2.$$

4. Qualitative analysis of the system in the absence of periodic excitation

In the absence of periodic excitation $\tilde{I}_1 = J_1$ the system (3.1) is autonomous. We consider separately the case of a displaced center of mass and an unbiased center of mass.

Case $b = 0$. This case is quite trivial. Therefore, we will describe here only the main results of a qualitative analysis of the system. In this case

$$P(t) = P_0, \quad Q(t) = Q_0,$$

and the system (3.1) is reduced to a single differential equation with respect to the angle φ :

$$\dot{\varphi} = -\frac{Q_0}{l\sqrt{\Delta}} \sin \varphi - \frac{P_0}{J_1} \quad (4.1)$$

the solution of which can be found analytically or numerically and can be presented graphically.

For $Q_0 = 0$ it is true that $\dot{\varphi} = -\omega$ and $\theta = \text{const}$. This suggests the following. If the first link with an unbiased center of mass and in the absence of periodic excitation is given only angular velocity, it will continue to rotate at a constant angular velocity, and the second link will not change its location relative to the fixed coordinate system.

For $Q_0 \neq 0$ we introduce the notation

$$A = \frac{P_0 l}{J_1 Q_0}.$$

If the condition

$$|A| > \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$$

is satisfied, Eq. (4.1) has no stationary points. Numeric experiments show that in this case there are periodic and quasi-periodic modes in the system. The first trolley moves with a constant angular velocity and a variable linear velocity along a curved trajectory.

If the condition

$$|A| < \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$$

is satisfied, Eq. (4.1) has two stationary points belonging to the interval $(-\pi, \pi]$:

$$\begin{aligned}\varphi_1^* &= -\operatorname{sign}(A) \arcsin \left(\sqrt{\frac{Ml^2 A^2}{1 - (J_2 - 2M_2 l^2) A^2}} \right), \\ \varphi_2^* &= -\operatorname{sign}(A) \left(\pi - \arcsin \left(\sqrt{\frac{Ml^2 A^2}{1 - (J_2 - 2M_2 l^2) A^2}} \right) \right).\end{aligned}$$

The point φ_1^* is a stable stationary point and the point φ_2^* is an unstable stationary point. Numerical experiments have also shown that

$$\begin{aligned}\lim_{t \rightarrow +\infty} \varphi(t) &= \varphi_1^*, \\ \lim_{t \rightarrow +\infty} v_1(t) &= Q_0 \sqrt{\frac{1 - (J_2 - 2M_2 l^2) A^2}{Ml^2}}.\end{aligned}$$

If the condition

$$|A| = \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$$

is satisfied, two points of rest merge into one on the interval $(-\pi, \pi]$:

$$\varphi^* = -\operatorname{sign}(A) \frac{\pi}{2}.$$

Numerical experiments have shown that

$$\begin{aligned}\lim_{t \rightarrow +\infty} \varphi(t) &= \varphi^*, \\ \lim_{t \rightarrow +\infty} v_1(t) &= Q_0 |A|.\end{aligned}$$

Thus, for

$$|A| \leq \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$$

the functions $v_1(t)$ and $\varphi(t)$ exhibit asymptotic behavior. The driving trolley will tend to move uniformly in a circle.

In the case $d = 0$, $b = 0$ graphs of the functions v_1 and ω and the trajectory of motion are plotted in Fig. 2. Figure 2a illustrates the case $|A| < \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$. Figure 2b illustrates the case $|A| > \frac{1}{\sqrt{J_2 + (M_1 - M_2)l^2}}$.

Case $b \neq 0$. On a given level set of the energy integral

$$\frac{l^2}{J_1} P^2 + Q^2 = H^2$$

the system (3.1) can be reduced to an autonomous system of two differential equations

$$\begin{aligned}\dot{\varphi} &= -\frac{\sqrt{H^2 - \frac{l^2}{J_1} P^2}}{l\sqrt{\Delta}} \sin \varphi - \frac{P}{J_1}, \\ \dot{P} &= -\frac{M_1 b P \sqrt{H^2 - \frac{l^2}{J_1} P^2}}{J_1 \sqrt{\Delta}}\end{aligned}\tag{4.2}$$

Fig. 2. Trajectory of motion and graphs of functions $\varphi(t)$, $v_1(t)$ for the following values of constants: (a) $J_2 = 12$, $l = 2$, $M_1 = 1.5$, $M_2 = 0.5$, $\omega(0) = -8$, $v_1(0) = 20$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = \frac{\pi}{2}$, (b) $J_2 = 18$, $l = 5$, $M_1 = 1.8$, $M_2 = 0.2$, $\omega(0) = 5$, $v_1(0) = 20$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = \frac{\pi}{2}$

with equilibrium positions on the interval $\varphi \in (-\pi, \pi]$

$$\begin{aligned} \varphi &= 0, & P &= 0, \\ \varphi &= \pi, & P &= 0. \end{aligned}$$

The equilibrium position $\varphi = 0$, $P = 0$ is asymptotically stable, since it corresponds to the negative real eigenvalues

$$\lambda_1 = -\frac{M_1 b H}{J_1 \sqrt{M l^2}}, \quad \lambda_2 = -\frac{H}{l \sqrt{M l^2}}, \quad H > 0$$

of the matrix of the linearized system.

The equilibrium position $\varphi = \pi$, $P = 0$ is unstable, since the following characteristic numbers

$$\lambda_1 = -\frac{M_1 b H}{J_1 \sqrt{M l^2}}, \quad \lambda_2 = \frac{H}{l \sqrt{M l^2}}, \quad H > 0$$

of the matrix of the linearized system correspond to it.

Thus, the functions P and Q are bounded and for $t \rightarrow +\infty$ the systems (3.1) and (2.4) also exhibit asymptotic behavior:

$$\begin{aligned}\lim_{t \rightarrow +\infty} \varphi(t) &= 0, \\ \lim_{t \rightarrow +\infty} \omega(t) &= 0, \\ \lim_{t \rightarrow +\infty} v_1(t) &= \frac{H}{\sqrt{Ml^2}}.\end{aligned}$$

Figure 3 shows the trajectories of the first trolley in the case at hand.

Fig. 3. Trajectory of motion for the following values of constants: (a) $J_1 = 10.25$, $J_2 = 9$, $\Omega = 2$, $b = 0.5$, $l = 2$, $M_1 = 1$, $M_2 = 1.5$, $\omega(0) = 5$, $v_1(0) = -20$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = \frac{\pi}{4}$, (b) $J_1 = 10.25$, $J_2 = 9$, $\Omega = 2$, $b = 0.5$, $l = 2$, $M_1 = 1$, $M_2 = 1.5$, $\omega(0) = 8$, $v_1(0) = 20$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = \frac{\pi}{4}$

A detailed study of the movement of a wheeled vehicle in the absence of moving masses is presented in [3] for two trailers and in [17] for three trailers.

5. Qualitative analysis of the system with periodic excitation

In this case, the system of equations (3.1) is nonautonomous and is characterized by more complex dynamics. Let's consider some important aspects in the dynamics of the reduced system (3.1).

To begin with, let's consider the question of the possible unlimited increase of the functions $P(t)$ and $Q(t)$. Their unlimited increase will entail an increase in the angular velocity ω and linear velocity v_1 of the driving trolley, respectively.

Theorem 1. *The functions $P(t)$ and $Q(t)$ that are the solution of the system of differential equations (3.1) are bounded.*

Proof. If $b = 0$, then the functions $P(t) = P_0$, $Q(t) = Q_0$ are obviously bounded.

Now we assume that $b \neq 0$. As mentioned earlier, due to this symmetry and the fact that $P = 0$ is an invariant manifold, the sign of the function $P(t)$ does not qualitatively affect the dynamics of the system. We investigate the behavior of the system for $Q(0) > 0$ and $Q(0) < 0$.

Let the condition $Q(0) > 0$ be satisfied. Since for any $t \in [0, +\infty)$ the derivative $\dot{Q}(t)$ is nonnegative ($\dot{Q}(t) \geq 0$), the function $Q(t)$ does not decrease. From the initial condition $Q(0) > 0$ it follows that $Q(t) > 0$ at any time.

The second equation of the system (3.1) can also be represented as

$$(\ln |P|)' = -\frac{\widetilde{M}_1 b Q}{\widetilde{I}_1 \sqrt{\Delta}}.$$

Thus, the condition

$$(\ln |P|)' < -k_0 < 0$$

is satisfied, which means that the function $\ln |P(t)|$ decreases indefinitely and

$$0 < |P(t)| < |P(0)|e^{-k_0 t}.$$

Then $|P(t)| \rightarrow 0$ as $t \rightarrow +\infty$.

For the function $Q(t)$, the following holds:

$$Q(t) = Q(0) + \int_0^t \frac{M_1 b l^2 P^2(\tau)}{\widetilde{I}_1^2(\tau) \sqrt{\Delta(\tau)}} d\tau < Q(0) + \int_0^t \frac{M_1 b l^2 P^2(0) e^{-2k_0 \tau}}{\widetilde{I}_1^2(\tau) \sqrt{\Delta(\tau)}} d\tau.$$

Since the function $F(\tau) = \frac{1}{\widetilde{I}_1^2(\tau) \sqrt{\Delta(\tau)}}$ is bounded on the interval $[0, +\infty)$ and takes only positive values, the integral

$$\int_0^t \frac{M_1 b l^2 P^2(0) e^{-2k_0 \tau}}{\widetilde{I}_1^2(\tau) \sqrt{\Delta(\tau)}} d\tau$$

increases monotonically and converges to some positive number C^2 as $t \rightarrow +\infty$.

Therefore, the function $Q(t)$ increases monotonically and is bounded for any t . It is obvious that it behaves asymptotically as $t \rightarrow +\infty$.

Fig. 4. Projections of phase curves on the plane (Q, P)

Now assume that $Q(0) < 0$. Starting from some point in time t^* , the function Q will take positive values due to the nonnegativity of its time derivative ($\dot{Q} > 0$). Therefore, the function $\ln |P|$ will first monotonically increase, and as soon as the function Q begins to take positive values, the function $\ln |P|$ will begin to monotonically decrease. Figure 4 shows the approximate behavior of projections of phase curves on the plane (Q, P) depending on the initial conditions $Q(0) < 0$, $P(0) < 0$ and $Q(0) < 0$, $P(0) > 0$.

Fig. 5. Trajectory of motion and graphs of the functions $\omega(t)$ and $v_1(t)$ for the following values of constants: (a) $J_1 = 6$, $J_2 = 8.6$, $d = 1$, $\Omega = 2$, $l = 3$, $M_1 = 0.3$, $M_2 = 0.2$, $\omega(0) = 3$, $v_1(0) = 20$, $\varphi(0) = \frac{3\pi}{4}$, $\psi(0) = -\frac{\pi}{4}$, (b) $J_1 = 10$, $J_2 = 11$, $d = 0.5$, $\Omega = 2$, $l = 3$, $M_1 = 0.5$, $M_2 = 0.5$, $\omega(0) = 4$, $v_1(0) = 10$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = -\frac{\pi}{2}$, (c) $J_1 = 10$, $J_2 = 13.2$, $d = 2$, $\Omega = 2$, $l = 4$, $M_1 = 1.5$, $M_2 = 0.2$, $\omega(0) = 5$, $v_1(0) = 30$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = -\frac{\pi}{2}$

Therefore, starting from a certain time instant, the qualitative behavior of the function $Q(t)$ will not depend on the sign of its initial condition. $\square$

If the functions $P(t)$ and $Q(t)$ are bounded, then the functions $\omega(t)$ and $v_1(t)$ are bounded too.

Similarly to the previous section, we consider the case of an unbiased center of mass of the driving trolley and the case of an offset center of mass of the driving trolley.

Case $b = 0$. The functions $P(t)$ and $Q(t)$ are constant functions

$$P(t) = P_0, \quad Q(t) = Q_0,$$

and the system (3.1) is reduced to a single nonautonomous equation with periodic coefficients

$$\dot{\varphi} = -\frac{Q_0}{l\sqrt{\Delta}} \sin \varphi - \frac{P_0}{\tilde{I}_1}. \quad (5.1)$$

For $Q_0 = 0$ the angular velocity ω is periodic in time and $\theta = \text{const}$. Similarly to movement without moving masses, when only the angular velocity is imparted to the driving link, the second link will remain motionless, and the first will rotate about its axis with a periodically changing angular velocity.

If only a rectilinear translational motion is imparted to the driving link, that is, $P_0 = 0$, then Eq. (5.1) will have an asymptotically stable rest point $\varphi = 0$. Thus, the second link will take a stable position, in which the axes of the links will be on the same straight line, and the system will tend to straight-line motion with velocity

$$v_1(t) = \frac{Q_0}{\sqrt{Ml^2}} = \frac{v_1(0)\sqrt{(J_2 - 2M_2l^2)\sin^2 \varphi(0) + Ml^2}}{\sqrt{Ml^2}}.$$

The velocity limit value of the system in this case depends on the initial position of the second trolley relative to the first trolley for $J_2 - 2M_2l^2 \neq 0$. The system can achieve the highest speed when $\varphi(0) = \pm\frac{\pi}{2}$ for $J_2 - 2M_2l^2 > 0$ and when $\varphi(0) = 0, \pm\pi$ for $J_2 - 2M_2l^2 < 0$.

In the general case, for $P_0 \neq 0, Q_0 \neq 0$, equation (5.1) has no stationary points and can be solved numerically. Numerical experiments show that the function φ , which is the solution of the differential equation (5.1), is quasi-periodic.

Figure 5 shows the driving trolley trajectories and graphs of the functions $\omega(t)$ and $v_1(t)$ that are derived from the functions $P(t)$ and $Q(t)$, respectively, for different values of numerical parameters.

Numerical experiments confirm the asymptotic dynamics of the system in this case. No chaotic dynamics have been found in numerical experiments for Eq. (5.1).

Case $b \neq 0$. In this case, the system (3.1) is a nonautonomous system with periodic coefficients with a period $T = \frac{\pi}{|\Omega|}$. Figure 6 shows the trajectories of the driving trolley and graphs of the desired mechanical parameters. In this numerical example, starting from a certain point in time, the system goes into rectilinear uniform motion. Other numerical examples show similar dynamics.

6. Conclusion

The dynamics of an articulated wheeled vehicle consisting of two trolleys connected by a frame has been studied and demonstrated in numerical experiments. The system was shown to undergo periodic excitation due to the motion of a pair of masses on the first trolley, which did not change the center of mass of the trolley itself. A system of differential equations of motion was obtained and reduced to a system of equations in newly introduced variables. A qualitative analysis of the reduced system was carried out.

Fig. 6. Trajectory of motion and graphs of the functions $\varphi(t)$, $\omega(t)$ and $v_1(t)$ for the following values of constants: $J_1 = 11.28$, $J_2 = 12.25$, $d = 0.5$, $\Omega = 1$, $b = 0.8$, $l = 2.5$, $M_1 = 2$, $M_2 = 1$, $\omega(0) = 5$, $v_1(0) = -20$, $\varphi(0) = \frac{\pi}{2}$, $\psi(0) = \frac{\pi}{4}$

The study has shown that the linear and angular velocities of the driving link are always bounded regardless of the initial conditions. The desired mechanical parameters demonstrate both periodic, quasi-periodic and asymptotic dynamics in the case of an unbiased center of mass of the driving trolley, and only asymptotic behavior in the case of a displaced center of mass of the driving link. In the case of an offset center of mass of the driving trolley the system goes into uniform rectilinear motion. There are no chaotic fluctuations in the system. By suitably choosing the system parameters, one can achieve the necessary motion mode.

Graphs of motion trajectories and mechanical parameters have been constructed using the Maple software package.

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