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On Global Trajectory Tracking Control for an Omnidirectional Mobile Robot with a Displaced Center of Mass

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This paper addresses the trajectory tracking control design of an omnidirectional mobile robot with a center of mass displaced from the geometrical center of the robot platform. Due to the high maneuverability provided by omniwheels, such robots are widely used in industry to transport loads in narrow spaces. As a rule, the center of mass of the load does not coincide with the geometric center of the robot platform. This makes the trajectory tracking control problem of a robot with a displaced center of mass relevant. In this paper, two controllers are constructed that solve the problem of global trajectory tracking control of the robot. The controllers are designed based on the Lyapunov function method. The main difficulty in applying the Lyapunov function method for the trajectory tracking control problem of the robot is that the time derivative of the Lyapunov function is not definite negative, but only semidefinite negative. Moreover, the LaSalle invariance principle is not applicable in this case since the closed-loop system is a nonautonomous system of differential equations. In this paper, it is shown that the quasi-invariance principle for nonautonomous systems of differential equations is much convenient for the asymptotic stability analysis of the closed-loop system. Firstly, we construct an unbounded state feedback controller such as proportional-derivative term plus feedforward. As a result, the global uniform asymptotic stability property of the origin of the closed-loop system has been proved. Secondly, we find that, if the damping forces of the robot are large enough, then the saturated position output feedback controller solves the global trajectory tracking control problem without velocity measurements. The effectiveness of the proposed controllers has been verified through simulation tests. Namely, a comparative analysis of the bounded controller obtained and the well-known “PD+” like control scheme is carried out. It is shown that the approach proposed in this paper saves energy for control inputs.

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Besides, a comparative analysis of the bounded controller and its analogue constructed earlier in a cylindrical phase space is carried out. It is shown that the controller provides lower values for the root mean square error of the position and velocity of the closed-loop system.

Keywords: omnidirectional mobile robot, displaced mass center, global trajectory tracking control, output position feedback, Lyapunov function method

1. Introduction

Omnidirectional mobile robots known for their maneuverability are used in many areas of the human activity [22]. The most common are the models of three or four-wheeled mobile robots. Many different approaches in the trajectory tracking control design for such models of omnidirectional mobile robots have been proposed in the literature. In [14, 17], kinematical models of three-wheeled mobile robots were considered and position and trajectory tracking control problems were solved taking into account the actuator saturation. In [1], the dynamics of a four-wheeled omnidirectional robot was considered and an algorithm of computing the controller for program motion implementation was proposed. Algorithm for trajectory generation for a four-wheeled omnirobot was proposed in [23]. In [15], the experimental tuning of the proportional integral differential (PID) controller was considered to track the trajectory of a four-wheeled omnimobile robot.

The equations of motion of an omnidirectional mobile robot with three orthogonal universal wheels have been derived by many researchers (see, for example, [10, 18, 20, 30]). In an orthogonal universal wheel the rollers are aligned around its rim in such a way that the wheel can roll orthogonally to its driving direction [9]. In [18], a trajectory linearization control method with proportional integral (PI) terms was used to achieve an exponential stability of the closed-loop system. A kinematic control method based on feedback linearization was proposed in [13] in order to achieve simultaneous tracking and stabilization. The computed torque method with proportional differential (PD) terms was applied by [28] to solve the trajectory tracking control problem of the robot.

An adaptive backstepping based controller was proposed in [12] for trajectory tracking and stabilization of an omnidirectional wheeled mobile robot with parameter variations and uncertainties. In [11], the sliding mode tracking control design was proposed. In [3, 4] both the position stabilization and trajectory tracking control problems were solved using Lyapunov function method. In [24], the passivity-based controller based on “PD+” approach was proposed to solve the global trajectory tracking control problem using only position measurements. Many researchers applied the model-predictive control design to solve the trajectory tracking problem [8, 26, 29].

Note that in all the above-mentioned papers, a dynamic model was considered wherein the mass center of the robot coincides with the geometric center of the platform. In practice, such a requirement is difficult to achieve due to the operating conditions of the robot. For example, such robots are used to move various loads in narrow spaces. Therefore, the study of the dynamics and the trajectory tracking control of the robot with a displaced mass center are problems of current interest. In [16], the kinematic model of an omniwheeled spherical robot was considered taking into account the displacement of the center of mass inside the moving platform.

In this paper, we address the global trajectory tracking control problem of an omnidirectional mobile robot with a displaced mass center. The equations of motion for such a robot

model were obtained in [19]. To date, a nonlinear PI controller has been proposed in [5] to solve the position stabilization problem. The problem of trajectory tracking control design for such a robot model was considered in the previous papers of the authors [6, 7] wherein the output position feedback controllers were constructed taking into account such features as periodicity of the model equations in the angular coordinate of the robot as well as the presence of delay in the control feedback.

The main contributions of this paper are as follows. Firstly, we propose a novel controller inspired by the so-called PD+ approach to solve the global trajectory tracking control problem for the nonlinear model of an omnidirectional mobile robot with a displaced mass center. Secondly, we propose a stability analysis that provides the global uniform asymptotic stability property of the closed-loop system using a Lyapunov function with a semidefinite time derivative on the basis of the theory of limiting equations. Thirdly, we propose a bounded position output feedback controller which solves the global trajectory tracking control problem without velocity measurements.

Throughout this paper the following mathematical notations are used. $E \in \mathbb{R}^{n \times n}$ is the identity matrix. Superscript T denotes a transpose operation. Symbol $\|\cdot\|$ denotes the Euclidean vector norm.

2. Formulation of the Problem

Consider a dynamical model of a mobile robot (see, Fig. 1) with three omnidirectional wheels and a displaced center of mass, represented in [19]. We use the following notation and assumptions. The world coordinate frame $OXYZ$ is fixed on the ground plane OXY . OZ is the axis up. Let C be the center of an equilateral triangle $C_1C_2C_3$, at the vertices of which the centers of the robot wheels are situated. $Cx_1y_1z_1$ is a body coordinate frame fixed at the moving platform of the robot. The axis Cx_1 is parallel to the axis of rotation of the first wheel; the axis Cz_1 is vertical; the axes Cy_1 , Cz_1 and Cx_1 form a right-handed coordinate frame. The planes OXY and Cx_1y_1 are parallel.

Neglecting the mass and dimensions of the rollers, consider the robot model as a mechanical system of four absolutely rigid bodies, the position of which is determined by six independent parameters x , y , ψ , φ_1 , φ_2 and φ_3 , where x and y denote the platform center location in the world frame $OXYZ$; ψ is the platform orientation angle measured from the axis OX ; φ_j is the

Fig. 1. Robot's model.

angle of rotation of the j th wheel relative to its axis of rotation ($j = 1, 2, 3$). Denote the positive direction (counterclockwise) of the i th wheel rotation by the vector n_i , $i = 1, 2, 3$. The mass center of the system C_0 is shifted relative to the center C of the platform by distance d . The angle between the axis Cx_1 and the vector CC_0 is denoted by α . Denote by r the radius of the wheels. The distances from the center of the platform to the center C_j ($j = 1, 2, 3$) of each wheel are equal to a . Denote the masses of the platform and the wheel by m_0 and m_1 , respectively; let ρ_0 and ρ_1 be the radii of inertia of the platform and the wheel, respectively, relative to the vertical axis passing through their centers of mass; denote the radius of inertia of the wheel relative to its axis of rotation by ρ_3 . Let us introduce the following mass-inertia parameters.

$$\begin{aligned} m &= m_0 + 3m_1 \left(1 + \frac{\rho_3^2}{2r^2} \right), \\ m_s &= m_0 + 3m_1, \quad m_2 = m - m_s = \frac{3m_1\rho_3^2}{2r^2}, \\ I_s &= m_0(d^2 + \rho_0^2) + 3m_1 \left(\rho_1^2 + a^2 \left(1 + \frac{2\rho_3^2}{r^2} \right) \right). \end{aligned}$$

Three equations of nonholonomic constraints can be obtained which uniquely determine the projections $\dot{x}$, $\dot{y}$ of the velocity vector of the platform center and its angular velocity $\dot{\psi}$ through the angular velocities of the wheels $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ as follows [19]:

$$\begin{aligned} \dot{x} &= \frac{2r}{3} \sum_{i=1}^3 \sin \left(\psi + (i-1) \frac{2\pi}{3} \right) \dot{\varphi}_i, \\ \dot{y} &= -\frac{2r}{3} \sum_{i=1}^3 \cos \left(\psi + (i-1) \frac{2\pi}{3} \right) \dot{\varphi}_i, \\ \dot{\psi} &= -\frac{r}{3a} \sum_{i=1}^3 \dot{\varphi}_i. \end{aligned} \tag{2.1}$$

The velocity vector of the platform center C is such as $V = (V_x, V_y, \Omega)^T$, where V_x and V_y are the components of the V in the body frame $Cx_1y_1z_1$; and Ω is the platform angular velocity.

The kinematic equations which express the relationship between the generalized velocities and pseudo-coordinates of the robot are given by [19]:

$$\begin{aligned} \dot{x} &= V_x \cos \psi - V_y \sin \psi, \\ \dot{y} &= V_x \sin \psi + V_y \cos \psi, \\ \dot{\psi} &= \Omega, \\ \dot{\varphi}_1 &= \frac{1}{r}(V_y + a\Omega), \quad \dot{\varphi}_2 = \frac{1}{2r}(\sqrt{3}V_x + V_y - 2a\Omega), \\ \dot{\varphi}_3 &= \frac{1}{2r}(-\sqrt{3}V_x + V_y - 2a\Omega). \end{aligned} \tag{2.2}$$

Using (2.2), we obtain

$$\begin{aligned} V_x &= \dot{x} \cos \psi + \dot{y} \sin \psi, \\ V_y &= -\dot{x} \sin \psi + \dot{y} \cos \psi, \\ \Omega &= \dot{\psi}. \end{aligned} \tag{2.3}$$

The robot dynamic equations can be expressed in the variables V_x , V_y and Ω as follows [19]:

$$\begin{aligned} m\dot{V}_x - m_0d \sin \alpha \dot{\Omega} - m_s \Omega V_y - m_0d \cos \alpha \Omega^2 &= \frac{\sqrt{3}}{2r}(M_2 - M_3), \\ m\dot{V}_y + m_0d \cos \alpha \dot{\Omega} + m_s \Omega V_x - m_0d \sin \alpha \Omega^2 &= -\frac{1}{2r}(2M_1 - M_2 - M_3), \\ m_0d(-\sin \alpha \dot{V}_x + \cos \alpha \dot{V}_y) + I_s \dot{\Omega} + m_0d(\cos \alpha V_x + \sin \alpha V_y)\Omega &= -\frac{a}{r}(M_1 + M_2 + M_3) \end{aligned} \quad (2.4)$$

where M_1 , M_2 and M_3 are input torques applied to the wheels.

Similarly to [19], we assume that the torques produced by each of the three geared motors are linear functions of both the applied voltage and angular rotational velocity of the rotor, i.e.,

$$M_j = c_u u_j - c_v \dot{\varphi}_j, \quad j = 1, 2, 3, \quad (2.5)$$

where c_u and c_v are some constants, u_j is the voltage applied to the motor, and $c_v \dot{\varphi}_j$ is the counter electromotive torque, $j = 1, 2, 3$.

From (2.1)–(2.4) one can obtain the following dynamic equations of the robot in the world coordinate frame: $OXYZ$:

$$A(q_3)\ddot{q} + B(q_3, \dot{q}_3)\dot{q} = P(q_3)U, \quad (2.6)$$

where

$$\begin{aligned} q &= (q_1, q_2, q_3)^T, \quad q_1 = x, \quad q_2 = y, \quad q_3 = \psi, \\ U &= (U_1, U_2, U_3)^T, \quad U_i = (c_u/r)u_i, \quad i = 1, 2, 3, \\ A(q_3) &= \begin{pmatrix} m & 0 & -s(q_3) \\ 0 & m & c(q_3) \\ -s(q_3) & c(q_3) & I_s \end{pmatrix}, \\ B(q_3, \dot{q}_3) &= \begin{pmatrix} h & m_2 \dot{q}_3 & -c(q_3) \dot{q}_3 \\ -m_2 \dot{q}_3 & h & -s(q_3) \dot{q}_3 \\ 0 & 0 & 2a^2 h \end{pmatrix}, \\ P(q_3) &= \begin{pmatrix} \sin q_3 & \sin\left(q_3 + \frac{2\pi}{3}\right) & \sin\left(q_3 + \frac{4\pi}{3}\right) \\ -\cos q_3 & -\cos\left(q_3 + \frac{2\pi}{3}\right) & -\cos\left(q_3 + \frac{4\pi}{3}\right) \\ -a & -a & -a \end{pmatrix}, \end{aligned}$$

$h = 3c_v/(2r^2)$, $c(q_3) = m_0d \cos(\alpha + q_3)$, $s(q_3) = m_0d \sin(\alpha + q_3)$. Note that parameter h expresses the robot damping forces.

Let $q = q^{(0)}(t)$ be a reference trajectory which is a twice continuously differentiable function such that the following inequalities hold:

$$\begin{aligned} |q_k^{(0)}(t)| &\leq \xi_1, \quad |\dot{q}_k^{(0)}(t)| \leq \eta_1, \quad |\ddot{q}_k^{(0)}(t)| \leq \zeta_1 \quad (k = 1, 2), \\ |\dot{q}_3^{(0)}(t)| &\leq \eta_2, \quad |\ddot{q}_3^{(0)}(t)| \leq \zeta_2, \end{aligned} \quad (2.7)$$

where ξ_1 , η_k and ζ_k are some positive reals, $k = 1, 2$.

The trajectory tracking control problem to be addressed is to construct the controller $U = U(t, q, \dot{q})$ in such a way that the reference trajectory $q^{(0)}(t)$ of the robot (2.6) is globally uniformly asymptotically stable.

3. The Problem Solution

3.1. The unbounded controller design

We choose the controller as follows:

$$U = P^{-1}(q_3)(U^{(0)} + U^{(1)} + U^{(2)}), \quad (3.1)$$

where $P^{-1}P = E$

$$P^{-1}(q_3) = \frac{2}{3} \begin{pmatrix} \sin q_3 & -\cos q_3 & -\frac{1}{2a} \\ \sin\left(q_3 + \frac{2\pi}{3}\right) & -\cos\left(q_3 + \frac{2\pi}{3}\right) & -\frac{1}{2a} \\ \sin\left(q_3 + \frac{4\pi}{3}\right) & -\cos\left(q_3 + \frac{4\pi}{3}\right) & -\frac{1}{2a} \end{pmatrix},$$

$$U^{(0)} = A(q_3)\ddot{q}^{(0)}(t) + B(q_3, \dot{q}_3^{(0)}(t))\dot{q}^{(0)}(t),$$

$$U^{(1)} = (U_1^{(1)}, U_2^{(1)}, U_3^{(1)})^T,$$

$$U_j^{(1)} = -\mu_j(\dot{q}_j - \dot{q}_j^{(0)}(t)), \quad j = 1, 2, 3, \quad (3.2)$$

$$U^{(2)} = (U_1^{(2)}, U_2^{(2)}, U_3^{(2)})^T,$$

$$U_j^{(2)} = -\gamma_j(q_j - q_j^{(0)}(t)), \quad j = 1, 2, 3,$$

where the constants γ_j , $j = 1, 2, 3$ are some positive reals and μ_j , $j = 1, 2, 3$ satisfy the following inequalities:

$$h + \mu_j \geq \varepsilon, \quad j = 1, 2,$$

$$4(2a^2(h - \varepsilon) + \mu_3) \geq (m_0 d \eta_2 + m_2 \eta_1)^2 \left(\frac{1}{h - \varepsilon + \mu_1} + \frac{1}{h - \varepsilon + \mu_2} \right). \quad (3.3)$$

Here ε is some positive real.

Let us define the state trajectory tracking errors as follows:

$$e_q = q - q^{(0)}(t), \quad \dot{e}_q = \dot{q} - \dot{q}^{(0)}(t), \quad (3.4)$$

$$e_q = (e_1, e_2, e_3)^T, \quad \dot{e}_q = (\dot{e}_1, \dot{e}_2, \dot{e}_3)^T.$$

The error dynamics equations are given by

$$A(q_3^{(0)}(t) + e_3)\ddot{e}_q + B(q_3^{(0)}(t) + e_3, \dot{q}_3^{(0)}(t) + \dot{e}_3)\dot{e}_q + f(t, e_3)\dot{e}_3 = -\mu\dot{e}_q - \gamma e_q, \quad (3.5)$$

where $f = (m_2\dot{q}_2^{(0)}(t) - c(q_3^{(0)}(t) + e_3)\dot{q}_3^{(0)}(t), -s(q_3^{(0)}(t) + e_3)\dot{q}_3^{(0)}(t) - m_2\dot{q}_1^{(0)}(t), 0)^T$, $\mu = \text{diag}(\mu_1, \mu_2, \mu_3)$, $\gamma(x) = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$.

Note that the system of equations (3.5) is precompact [27] and the corresponding system of limiting equations has a similar form

$$A(q_3^{(0)*}(t) + e_3)\ddot{e}_q + B(q_3^{(0)*}(t) + e_3, \dot{q}_3^{(0)*}(t) + \dot{e}_3)\dot{e}_q + f^*(t, e_3)\dot{e}_3 = -\mu\dot{e}_q - \gamma e_q, \quad (3.6)$$

where $q_3^{(0)*}(t) = \lim_{n \rightarrow \infty} q_3^{(0)}(t_n + t)$, $\dot{q}_3^{(0)*}(t) = \lim_{n \rightarrow \infty} \dot{q}_3^{(0)}(t_n + t)$ and $f^*(t, e_3) = \lim_{n \rightarrow \infty} f(t_n + t, e_3)$.

Theorem 1. Consider the closed-loop system (3.5). Let inequalities (3.3) hold. Then the origin $\dot{x} = x = 0$ of (3.5) is globally uniformly asymptotically stable.

Proof. As a Lyapunov function candidate $V = V(t, e_q, \dot{e}_q)$, we consider the following function:

$$V = \frac{1}{2} \dot{e}_q^T A(q_3^{(0)}(t) + e_3) \dot{e}_q + \frac{1}{2} (\gamma_1 e_1^2 + \gamma_2 e_2^2 + \gamma_3 e_3^2). \quad (3.7)$$

One can easily see that V is positive definite, radially unbounded and the following holds:

$$V(t, e_q, \dot{e}_q) \rightarrow 0 \text{ uniformly on } t \in \mathbb{R} \text{ as } (e_q, \dot{e}_q) \rightarrow 0.$$

The time derivative of the function (3.7) is given by

$$\dot{V} = \frac{1}{2} \dot{e}_q^T \dot{A}(q_3^{(0)}(t) + e_3) \dot{e}_q + \dot{e}_q^T A(q_3^{(0)}(t) + e_3) \ddot{e}_q + \gamma_1 e_1 \dot{e}_1 + \gamma_2 e_2 \dot{e}_2 + \gamma_3 e_3 \dot{e}_3. \quad (3.8)$$

From (3.8), using (3.5), one can see that the time derivative of the function (3.7) becomes

$$\begin{aligned} \dot{V} = & \frac{1}{2} (\dot{q}_3^{(0)}(t) + \dot{e}_3) (\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & 0 & -c(q_3) \\ 0 & 0 & -s(q_3) \\ -c(q_3) & -s(q_3) & 0 \end{pmatrix} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} - \\ & - \dot{e}_3 (\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & m_2 & -c(q_3) \\ -m_2 & 0 & -s(q_3) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1^{(0)}(t) \\ \dot{q}_2^{(0)}(t) \\ \dot{q}_3^{(0)}(t) \end{pmatrix} - \\ & - (\dot{e}_3 + \dot{q}_3^{(0)}(t)) (\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & m_2 & -c(q_3) \\ -m_2 & 0 & -s(q_3) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} - \\ & - (\dot{e}_1, \dot{e}_2, \dot{e}_3) \text{diag}(h + \mu_1, h + \mu_2, 2a^2h + \mu_3) \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix}. \end{aligned} \quad (3.9)$$

One can easily see that the following equality holds:

$$\begin{aligned} & \frac{1}{2} (\dot{q}_3^{(0)}(t) + \dot{e}_3) (\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & 0 & -c(q_3) \\ 0 & 0 & -s(q_3) \\ -c(q_3) & -s(q_3) & 0 \end{pmatrix} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} - \\ & - (\dot{e}_3 + \dot{q}_3^{(0)}(t)) (\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & m_2 & -c(q_3) \\ -m_2 & 0 & -s(q_3) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} \equiv 0. \end{aligned} \quad (3.10)$$

From (3.9), using (3.10), one can obtain

$$\begin{aligned} \dot{V} = & -\dot{e}_3(\dot{e}_1, \dot{e}_2, \dot{e}_3) \begin{pmatrix} 0 & m_2 & -c(q_3) \\ -m_2 & 0 & -s(q_3) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1^{(0)}(t) \\ \dot{q}_2^{(0)}(t) \\ \dot{q}_3^{(0)}(t) \end{pmatrix} - \\ & -(\dot{e}_1, \dot{e}_2, \dot{e}_3) \text{diag}(h + \mu_1, h + \mu_2, 2a^2h + \mu_3) \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix}. \end{aligned} \quad (3.11)$$

Using (3.11), one can easily see that the time-derivative of V can be written as

$$\begin{aligned} \dot{V} = & -(h + \mu_1)\dot{e}_1^2 - (h + \mu_2)\dot{e}_2^2 - (2a^2h + \mu_3)\dot{e}_3^2 + \\ & + (m_0d\dot{q}_3^{(0)}(t) \cos(\alpha + e_3 + q_3^{(0)}(t)) - m_2\dot{q}_2^{(0)}(t))\dot{e}_1\dot{e}_3 + \\ & + (m_2\dot{q}_1^{(0)}(t) + m_0d\dot{q}_3^{(0)}(t) \sin(\alpha + e_3 + q_3^{(0)}(t)))\dot{e}_2\dot{e}_3. \end{aligned} \quad (3.12)$$

From (3.12), using (2.7), one can obtain the following estimation of $\dot{V}$:

$$\dot{V} \leq -(h + \mu_1)\dot{e}_1^2 - (h + \mu_2)\dot{e}_2^2 - (2a^2h + \mu_3)\dot{e}_3^2 + (m_0d\eta_2 + m_2\eta_1)|\dot{e}_3|(|\dot{e}_1| + |\dot{e}_2|). \quad (3.13)$$

From (3.13), using (3.3), one can obtain

$$\dot{V} \leq -\varepsilon(\dot{e}_1^2 + \dot{e}_2^2 + 2a^2\dot{e}_3^2) \leq 0. \quad (3.14)$$

Indeed, rewrite the estimation (3.13) as

$$\begin{aligned} \dot{V} \leq & -(h + \mu_1 - \varepsilon)\dot{e}_1^2 - (h + \mu_2 - \varepsilon)\dot{e}_2^2 - (2a^2(h - \varepsilon) + \mu_3)\dot{e}_3^2 + \\ & + (m_0d\eta_2 + m_2\eta_1)|\dot{e}_3|(|\dot{e}_1| + |\dot{e}_2|) - \varepsilon(\dot{e}_1^2 + \dot{e}_2^2 + 2a^2\dot{e}_3^2). \end{aligned} \quad (3.15)$$

Note that the quadratic form

$$-(h + \mu_1 - \varepsilon)\dot{e}_1^2 - (h + \mu_2 - \varepsilon)\dot{e}_2^2 - (2a^2(h - \varepsilon) + \mu_3)\dot{e}_3^2 + (m_0d\eta_2 + m_2\eta_1)|\dot{e}_3|(|\dot{e}_1| + |\dot{e}_2|)$$

is negative semidefinite if inequalities (3.3) hold.

Then one can obtain the estimation (3.14). From (3.14) one can easily see that $\dot{V}$ is a negative semidefinite function. Using the Persidskii theorem on uniform stability [25], one can easily find that the zero solution $\dot{e}_q = e_q = 0$ of the system (3.5) is uniformly stable.

To obtain the uniform global attractivity property of the zero solution $\dot{e}_q = e_q = 0$ of the system (3.5), we note that the set $\{(e_q, \dot{e}_q) \in \mathbb{R}^6: \dot{e}_1 = \dot{e}_2 = \dot{e}_3 = 0\}$ does not contain the solutions of the system (3.6) except for $e_1 = e_2 = e_3 = 0$.

Then, using the theorem on asymptotic stability from [2], we find that the zero solution $\dot{e}_q = e_q = 0$ of the system (3.5) is globally uniformly asymptotically stable. In other words, the controller (3.1), (3.2) solves the global trajectory tracking problem of the robot (2.6). This completes the proof. $\square$

REMARK 1. Note that the structure of the controller (3.1), (3.2) is similar to that proposed in [21] for robotic manipulators. This control structure was used in [24] for a simpler model of an omnidirectional mobile robot where the mass center of the platform coincides with its geometrical center.

3.2. The bounded controller design

Consider the following controller:

$$U = P^{-1}(q_3)(U^{(0)} + U^{(2)}), \quad (3.16)$$

where the term $U^{(2)}$ is bounded and defined as

$$U_j^{(2)} = -\gamma_j \arctan(q_j - q_j^{(0)}(t)), \quad j = 1, 2, 3, \quad (3.17)$$

where $\gamma_j > 0$, $j = 1, 2, 3$.

The error dynamics equations are

$$\begin{aligned} A(q_3^{(0)}(t) + e_3)\ddot{e}_q + B(q_3^{(0)}(t) + e_3, \dot{q}_3^{(0)}(t) + \dot{e}_3)\dot{e}_q + f(t, e_3)\dot{e}_3 = \\ = -\text{diag}(\gamma_1 \arctan e_1, \gamma_2 \arctan e_2, \gamma_3 \arctan e_3). \end{aligned} \quad (3.18)$$

Theorem 2. Consider the closed-loop system (3.18). Let the damping force of the robot be large enough that the following inequality holds:

$$h \geq \varepsilon + \frac{m_0 d \eta_2 + m_2 \eta_1}{2a}, \quad (3.19)$$

where ε is some positive real.

Then the origin $\dot{e}_q = e_q = 0$ of (3.18) is globally uniformly asymptotically stable.

Proof. As a Lyapunov function candidate, we choose the following function:

$$V = \frac{1}{2} \dot{e}_q^T A(q_3^{(0)}(t) + e_3) \dot{e}_q + \sum_{i=1}^3 \gamma_i \int_0^{e_i} \arctan e_i de_i. \quad (3.20)$$

One can easily see that V is positive definite, radially unbounded and the following holds:

$$V(t, e_q, \dot{e}_q) \rightarrow 0 \text{ uniformly on } t \in \mathbb{R} \text{ as } (e_q, \dot{e}_q) \rightarrow 0.$$

The time derivative of the function (3.20) is given by

$$\dot{V} = \frac{1}{2} \dot{e}_q^T \dot{A}(q_3^{(0)}(t) + e_3) \dot{e}_q + \dot{e}_q^T A(q_3^{(0)}(t) + e_3) \ddot{e}_q + \sum_{i=1}^3 \gamma_i \dot{e}_i \arctan e_i. \quad (3.21)$$

One can easily see that the time derivative (3.21) satisfies (3.12), where $\mu_1 = \mu_2 = \mu_3 = 0$. Then one can obtain

$$\dot{V} \leq -h(\dot{e}_1^2 + \dot{e}_2^2) - 2a^2 h \dot{e}_3^2 + (m_0 d \eta_2 + m_2 \eta_1) |\dot{e}_3| (|\dot{e}_1| + |\dot{e}_2|). \quad (3.22)$$

From (3.22), using (3.19), one can obtain the following estimation:

$$\dot{V} \leq -\varepsilon(\dot{e}_1^2 + \dot{e}_2^2 + 2a^2 \dot{e}_3^2) \leq 0. \quad (3.23)$$

To prove the estimation (3.23), rewrite (3.13) as follows:

$$\begin{aligned} \dot{V} \leq -(h - \varepsilon)(\dot{e}_1^2 + \dot{e}_2^2) - 2a^2(h - \varepsilon)\dot{e}_3^2 + \\ + (m_0 d \eta_2 + m_2 \eta_1) |\dot{e}_3| (|\dot{e}_1| + |\dot{e}_2|) - \varepsilon(\dot{e}_1^2 + \dot{e}_2^2 + 2a^2 \dot{e}_3^2). \end{aligned} \quad (3.24)$$

Note that the quadratic form

$$-(h - \varepsilon)(\dot{e}_1^2 + \dot{e}_2^2 + 2a^2 \dot{e}_3^2) + (m_0 d \eta_2 + m_2 \eta_1) |\dot{e}_3| (|\dot{e}_1| + |\dot{e}_2|)$$

is negative semidefinite if inequality (3.19) holds.

From (3.23), using the theorem on asymptotic stability from [2], we get the proof. $\square$

4. Numerical results

In this section, the performance of the proposed controllers is illustrated.

The robot parameters are given as

$$\begin{aligned} m_0 &= 20 \text{ kg}, & m_1 &= 1 \text{ kg}, \\ r &= 0.1 \text{ m}, & a &= 0.25 \text{ m}, & d &= 0.05 \text{ m}, \\ \alpha &= \pi/6 \text{ rad}, & c_\nu &= 6 \cdot 10^{-5} \text{ N} \cdot \text{m} \cdot \text{sec}, \\ \rho_0 &= 0.17 \text{ m}, & \rho_1 &= 0.05 \text{ m}, & \rho_3 &= 0.07 \text{ m}. \end{aligned} \quad (4.1)$$

Consider the ellipse tracking of the robot with a constant relative angular velocity of the platform. Choose the desired trajectory as follows:

$$q_1^{(0)}(t) = 1 + 20 \cos t \text{ m}, \quad q_2^{(0)}(t) = 2 + 30 \sin t \text{ m}, \quad q_3^{(0)}(t) = \pi/4 + 10t \text{ rad}. \quad (4.2)$$

The control gain matrices $\text{diag}(\gamma_1, \gamma_2, \gamma_3)$ and $\text{diag}(\mu_1, \mu_2, \mu_3)$ of the controller (3.1), (3.2) were chosen such that

$$\begin{aligned} \gamma_1 &= 25 \text{ N/m}, & \gamma_2 &= 25 \text{ N/m}, & \gamma_3 &= 20 \text{ N} \cdot \text{m}, \\ \mu_1 &= 35 \text{ N} \cdot \text{sec/m}, & \mu_2 &= 35 \text{ N} \cdot \text{sec/m}, & \mu_3 &= 20 \text{ N} \cdot \text{m} \cdot \text{sec}. \end{aligned} \quad (4.3)$$

In order to check the property of global tracking, we consider the simulations results using the initial conditions for the robot such that

$$\begin{aligned} q_1(0) &= 51 \text{ m}, & q_2(0) &= -68 \text{ m}, & q_3(0) &= -2.21 \text{ rad}, \\ \dot{q}_1(0) &= 25 \text{ m/sec}, & \dot{q}_2(0) &= -15 \text{ m/sec}, & \dot{q}_3(0) &= 20 \text{ rad/sec}. \end{aligned} \quad (4.4)$$

The tracking performance of the controller (3.1), (3.2) within the time interval $t = 50$ sec is shown in Figs. 2 and 3. From these results, it can be seen that the controller (3.1), (3.2) provides the convergence to the reference trajectory of the robot. The time evolution of the control inputs is shown in Fig. 4.

To verify the effectiveness of the bounded controller (3.16)–(3.17), the gain matrix is set as

$$\text{diag}(\gamma_1, \gamma_2, \gamma_3) = \text{diag}(32, 32, 1.5). \quad (4.5)$$

Let $h = 9 \text{ N} \cdot \text{sec/m}$ and $t = 120 \text{ sec}$.

The simulation results of the controller (3.16)–(3.17) are shown in Figs. 5 and 6 for the following initial conditions:

$$\begin{aligned} q_1(0) &= 101 \text{ m}, & q_2(0) &= -98 \text{ m}, & q_3(0) &= -2.21 \text{ rad}, \\ \dot{q}_1(0) &= 25 \text{ m/sec}, & \dot{q}_2(0) &= -15 \text{ m/sec}, & \dot{q}_3(0) &= 20 \text{ rad/sec}. \end{aligned} \quad (4.6)$$

The control inputs are shown in Fig. 7.

In order to compare the performances, the simulation tests were also performed with the “PD+” like controller proposed in [24] for a simpler robot model:

$$\begin{aligned} U &= P^{-1}(q_3)(U^{(0)} + U^{(2)}), \\ U^{(0)} &= A(q_3)\ddot{q}^{(0)}(t) + B(q_3, \dot{q}_3^{(0)}(t))\dot{q}^{(0)}(t), \\ U_j^{(2)} &= -\gamma_j(q_j - q_j^{(0)}(t)), \quad j = 1, 2, 3. \end{aligned} \quad (4.7)$$

Fig. 2. The robot's trajectory tracking using the controller (3.1)–(3.2).

Fig. 3. Evolution of the robot's angular position using the controller (3.1)–(3.2).

Fig. 4. Stabilizing terms of control inputs (3.1), (3.2).

Fig. 5. The robot's trajectory tracking using the controller (3.16)–(3.17).

Fig. 6. Evolution of the robot's angular position using the controller (3.16)–(3.17).

Fig. 7. Stabilizing terms of control inputs (3.16), (3.17).

We fixed the control gain parameters to $\gamma_1 = \gamma_2 = 0.5$, $\gamma_3 = 0.4$ using the trial and error approach.

Figures 8 and 9 show the simulation results for the controller (4.7), using the initial conditions (4.11). The control inputs are shown in Fig. 10. Analyzing the graphs in Figs. 8 and 9, we can see that the controller (4.7) does not ensure the trajectory tracking of the robot during the time interval $t = 120$ sec. In order to track the system (2.6), (4.7) along the reference trajectory (4.2), it is necessary to substantially increase some control gains parameters. Namely, we have to fix the control gain parameters to the following values:

$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad \gamma_3 = 0.4. \quad (4.8)$$

From these results, it can be seen that the controller (3.16), (3.17), (4.5) provides the convergence to the reference angular coordinate of the platform.

The control inputs (4.7), (4.8) and the robot's trajectory tracking are shown in Figs. 11 and 12, respectively.

Fig. 8. The robot's trajectory tracking using the controller (4.7).

Fig. 9. Evolution of the robot's angular position using the controller (4.7).

Fig. 10. Stabilizing terms of control inputs (4.7).

Fig. 11. Stabilizing terms of control inputs (4.7), (4.8).

Fig. 12. The robot's trajectory tracking using the controller (4.7)–(4.8).

Note that our controller (3.16), (3.17), (4.5) requires a significantly less control energy than (4.7), (4.8). To measure the control energy, we used the integral of the squared input (ISI) index given by

$$ISI = \int_0^T \|U(t)\|^2 dt,$$

where T is the simulation time, $U(t) = P^{-1}(q_3(t))(U^{(0)}(t, q_3(t)) + U^{(2)}(t, q(t)))$.

Table 1 shows the values of ISI. One can see from Table 1 that the lowest ISI value corresponds to the controller (3.16), (3.17), (4.5). Thus, the comparative analysis has confirmed that the proposed controller (3.16), (3.17), (4.5) provides better performance than that of [24].

Table 1. Values of ISI index of the controllers (3.16), (3.17), and (4.7), (4.8)

Index	Controller (3.16), (3.17), (4.5)	Controller (4.7), (4.8)
ISI	20682479	20744258

Compare also the performances of the controller (3.16)–(3.17) and that obtained in [6] and expressed as follows:

$$U = P^{-1}(q_3)(U^{(0)} + U^{(2)}), \quad (4.9)$$

where the term $U^{(2)}$ is defined as

$$U_j^{(2)} = -\gamma_j \arctan(q_j - q_j^{(0)}(t)), \quad j = 1, 2, \quad U_3^{(2)} = -\gamma_3 \sin \frac{q_3 - q_3^{(0)}(t)}{2}, \quad (4.10)$$

where $\gamma_j > 0$, $j = 1, 2, 3$.

For the controller (4.9)–(4.10) we fixed the gain parameters (4.5) which are equal to those of the controller (3.16)–(3.17).

In addition to the ISI index, let us use the root mean square error (RMSE) values of the closed-loop system's position and velocity which are calculated as follows:

$$RMSE(e_q) = \sqrt{\frac{1}{T} \int_0^T \|e_q(t)\|^2 dt}, \quad RMSE(\dot{e}_q) = \sqrt{\frac{1}{T} \int_0^T \|\dot{e}_q(t)\|^2 dt},$$

where the state errors e_q and $\dot{e}_q$ are given by (3.4).

Table 2 shows the values of ISI and RMSE for the initial deviations from the reference trajectory of the robot such that

$$\begin{aligned} e_1(0) &= 10 \text{ m}, & e_2(0) &= -10 \text{ m}, & e_3(0) &= -3 \text{ rad}, \\ \dot{e}_1(0) &= 5 \text{ m/sec}, & \dot{e}_2(0) &= -5 \text{ m/sec}, & \dot{e}_3(0) &= 3 \text{ rad/sec}. \end{aligned} \quad (4.11)$$

Table 2. Values of ISI and RMSE indexes of the controllers (3.16), (3.17) and (4.9), (4.10)

Index	Controller (3.16), (3.17)	Controller (4.9), (4.10)
ISI	8655024.7	8705235.5
RMSE(e_q)	28.72	46.41
RMSE($\dot{e}_q$)	6.48	6.56

As can be seen from Table 2, the lowest ISI, $\text{RMSE}(x)$ and $\text{RMSE}(\dot{x})$ values correspond to the controller (3.16)–(3.17). Thus, one can conclude that for not very large initial deviations the controller (3.16)–(3.17) is more effective than (4.9), (4.10).

5. Conclusion

This paper presents a solution to the global trajectory tracking control problem of an omnidirectional mobile robot with a displaced mass center. The proposed solution relies on the application of two control schemes employing a Lyapunov function with a semidefinite time derivative to the resulting error model. The attractivity property of the zero solution of the closed-loop system has been proved using the quasi-invariance principle for nonautonomous differential equations. The structure of the first proposed controller is a proportional-derivative term plus feedforward. The second controller is a nonlinear bounded proportional term plus feedforward. The simulation tests have shown good performances of the proposed controllers for very large initial errors. It has been shown that the proposed bounded controller is practically more useful than a conventional “PD+” when damping forces of the robot are large enough.

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