NONLINEAR PHYSICS AND MECHANICS

MSC 2010: 70E15, 70E20

A Sphere Held Fixed in a Poiseuille Flow near a Rough Wall

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We present here an analytical calculation of the hydrodynamic interactions between a smooth spherical particle held fixed in a Poiseuille flow and a rough wall. By the assumption of a low Reynolds number, the flow around a fixed spherical particle is described by the Stokes equations. The surface of the rigid wall has periodic corrugations, with small amplitude compared with the sphere radius. The asymptotic development coupled with the Lorentz reciprocal theorem are used to find the analytical solution of the couple, lift and drag forces exerted on the particle, generated by the second-order flow due to the wall roughness. These hydrodynamic effects are evaluated in terms of amplitude and period of roughness and also in terms of the distance between sphere and wall.

Keywords: lift force, drag force, rough wall, Stokes equations, Poiseuille flow, asymptotic development, Lorentz reciprocal theorem

Received February 17, 2021
Accepted August 17, 2021

This work was supported by the Faculty of Science, Department of Physics, Moulay Ismail University, Meknes and LGEMS Laboratory, National School of Applied Sciences, Ibn Zohr University, Agadir, Morocco.

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Introduction

The motion of a rigid spherical particle in a Poiseuille flow at a low Reynolds number is involved in a variety of applications, e.g., chemical engineering and biological processes (for example, membrane separations, flow cytometry, blood flow), the field-flow fractionation (FFF) technique in analytical chemistry [1].

There have been a series of theoretical recent works on spherical particles moving near smooth walls, on the basis of creeping-flow equations. Among them, we may quote Ho and Leal [2] who used the method of reflexions to study the inertial migration of rigid spheres, Claguea and Cornelius [3] who used the lattice Boltzmann (LB) method to study the hydrodynamic force and couple exerted on the sphere held stationary, Staben and Davis [4] who compared experimentally measured particle velocities for a range of spherical particle sizes to their previous boundary-integral simulations [5]. Bhattacharya et al. [6, 7], who applied a Cartesian representation method, used multipole expansions to study theoretically and numerically the hydrodynamic interactions of many spherical particles in suspensions confined between two parallel planar walls. On the other hand, the problem of a smooth sphere in interaction with a smooth wall in Stokes flow was treated by numerous authors. O'Neill [8] studied the translational and rotational spherical particle in a fluid at rest. The sphere held fixed in a linear shear flow has been investigated by Tozeren and Skalak [9]. Chaoui and Feuillebois [10] reconsidered these problems with a technique providing a 10^{-16} accuracy, even in the lubrication region of the order 10^{-6} of the sphere radius. In a similar way, Pasol et al. [11] treated the case of a rigid sphere held fixed in a quadratic shear flow. For hydrodynamic interactions between rigid particles and rough surfaces, a situation is much less clear both on the theoretical and on the experimental sides. Smart and Leighton [12] measured the hydrodynamic effect of the surface roughness of noncolloidal particles moving perpendicularly to a smooth wall. Smart et al. [13] considered a rough sphere rolling down a smooth inclined plane. Kunert et al. [14] measured the high-speed drainage of liquid films squeezed between a smooth sphere and a randomly rough plane by using lattice Boltzmann simulations. The flow of second order generated by the wall roughness of small amplitude has been calculated in a previous study by Assoudi et al. [15]. After that, Assoudi et al. [16] treated the motion of a spherical particle along a rough wall in a shear flow. They calculated the effects of second-order flow, forces, torque, and also the sphere velocities, generated by the wall roughness. Lamzoud et al. [17] studied the problem of a sphere settling towards a corrugated wall, in a fluid at rest.

In this paper, we focus on a smooth fixed particle entrained in a Poiseuille flow close to a rough wall. The wall roughness is modeled periodic, with small amplitudes compared with the sphere radius. We will calculate the force and torque, due to a flow generated by the wall roughness, exerted on the spherical particle by using the Lorentz reciprocal theorem [18, 19].

This paper is organized as follows. In Section 1, we will give a description of the model. The hydrodynamics forces and torque, due to the roughness, will be calculated in Section 4. Results and discussion will be presented in Section 5 and the conclusion in Section 6.

1. Description of the model

Consider a solid spherical particle of radius a , kept fixed in a Poiseuille flow $(\tilde{\mathbf{V}}_{pb}^\infty, \tilde{P}_{pb}^\infty)$ limited by two rough walls. The Cartesian coordinate system $(\tilde{X}, \tilde{Y}, \tilde{Z})$ with unit vectors $(\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z)$ is attached to the lower wall, with the plane $\tilde{Z} = 0$ located under the roughness. The sphere

$$(\tilde{\mathbf{V}}_{pb}^\infty, \tilde{P}_{pb}^\infty) \longrightarrow (\tilde{\mathbf{V}}_{pb}^{\infty(0)}, \tilde{P}_{pb}^{\infty(0)}) \quad \text{with} \quad \begin{cases} \tilde{\mathbf{V}}_{pb}^{\infty(0)} = [\tilde{k}_s \tilde{Z} + \tilde{k}_q \tilde{Z}^2] \mathbf{i}_x \\ \tilde{P}_{pb}^{\infty(0)} = 2\mu_f \tilde{k}_q \tilde{X} \end{cases} \quad (1.1)$$
$$\tilde{k}_s = -\frac{\tilde{h}}{2\mu_f} \frac{d\tilde{P}_{pb}^{\infty(0)}}{d\tilde{X}} = 4 \frac{\tilde{\mathbf{V}}_{\max}}{\tilde{h}} \quad \text{and} \quad \tilde{k}_q = \frac{1}{2\mu_f} \frac{d\tilde{P}_{pb}^{\infty(0)}}{d\tilde{X}} = -\frac{\tilde{k}_s}{\tilde{h}}, \quad (1.2)$$

In our study, we assume that the effect of the higher wall is neglected by imposing two conditions on the spherical particle. The first one is imposed on its position by taking the case where the sphere is moving close to the lower wall. The second one is imposed on its radius, such that $(a/\hbar) < 0.125$, according to Ho and Leal [2]. Let also $(\tilde{\rho}, \tilde{\phi}, \tilde{z})$ define a cylindrical polar coordinates system, with $\tilde{x} = \tilde{\rho} \cos \tilde{\phi}$ and $\tilde{y} = \tilde{\rho} \sin \tilde{\phi}$.

Fig. 1. A spherical particle held fixed in a Poiseuille flow limited by two rough walls.

The small roughness of the wall is assumed to be in the $\tilde{X}$ direction:

$$\tilde{Z}_p = a\epsilon\mathcal{R}(\tilde{X}), \quad (1.3)$$

with ε a small positive dimensionless amplitude of roughness. $\mathcal{R}(\tilde{X})$ is the shape of the roughness assumed to be periodic, it is expanded in the $(\tilde{X}, \tilde{Y}, \tilde{Z})$ frame in a Fourier series:

$$\mathcal{R}(\tilde{X}) = c_0 + \sum_{n=1}^p \left(c_n \cos(n\omega\tilde{X}) + s_n \sin(n\omega\tilde{X}) \right), \quad (1.4)$$

where $\omega = 2\pi/\tilde{L}$ is a positive constant, $\tilde{L}$ is the roughness period, and

$$\begin{aligned} c_0 &= \frac{1}{2}, \\ c_n &= \frac{2}{(2\pi n)^2 \delta(\delta - 1)} (1 - \cos(2\pi n \delta)), \\ s_n &= -\frac{2}{(2\pi n)^2 \delta(\delta - 1)} \sin(2\pi n \delta). \end{aligned}$$

Taking $p = 3$ and $\delta = 3/2$ in the Fourier expansion (1.4) is sufficient to describe the profile as shown in Fig. 1.

Let $(\tilde{\mathbf{V}}_{pb}, \tilde{P}_{pb})$ be, respectively, the velocity and pressure fields of the fluid flow perturbed by the presence of the spherical particle in the vicinity of the corrugated wall. Based on the assumption of a low Reynolds number $\left(\text{Re} = \frac{\rho_f a V_c}{\mu_f} \ll 1\right)$, where ρ_f and μ_f are, respectively, the fluid density and the dynamic viscosity and V_c is the reference velocity, the perturbed flow is generally governed by the Stokes equations:

$$\begin{cases} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{V}}_{pb} = \tilde{\nabla} \tilde{P}_{pb} \\ \tilde{\nabla} \cdot \tilde{\mathbf{V}}_{pb} = 0 \end{cases} \quad (1.5)$$

with the boundary conditions

$$\begin{cases} \tilde{\mathbf{V}}_{pb} = \mathbf{0} & \text{on the sphere surface,} \\ \tilde{\mathbf{V}}_{pb} = \mathbf{0} & \text{on the rough walls,} \\ \tilde{\mathbf{V}}_{pb} \rightarrow \tilde{\mathbf{V}}_{pb}^\infty & \text{far from the particle,} \end{cases} \quad (1.6)$$

where $\tilde{\mathbf{V}}_{pb}^\infty$ is the velocity field of the unperturbed flow (in the absence of the sphere).

2. Unperturbed flow field in the absence of the spherical particle

The unperturbed flow field $(\tilde{\mathbf{V}}_{pb}^\infty, \tilde{P}_{pb}^\infty)$, in the absence of the sphere, should be described by

$$\begin{cases} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{V}}_{pb}^\infty = \tilde{\nabla} \tilde{P}_{pb}^\infty \\ \tilde{\nabla} \cdot \tilde{\mathbf{V}}_{pb}^\infty = 0 \end{cases} \quad \text{with} \quad \begin{cases} \tilde{\mathbf{V}}_{pb}^\infty = \mathbf{0} & \text{on the rough walls,} \\ \tilde{\mathbf{V}}_{pb}^\infty \rightarrow \tilde{\mathbf{V}}_{pb}^{\infty(0)} & \text{far from the walls.} \end{cases} \quad (2.1)$$

In the case where the disturbance due to the roughness is preponderant compared with that due to the fluid inertia by assuming that $\text{Re} \ll \varepsilon \ll 1$, the velocity and pressure of the unperturbed flow field in the vicinity of the wall can be developed as follows:

$$\begin{cases} \tilde{\mathbf{V}}_{pb}^\infty = \tilde{\mathbf{V}}_{pb}^{\infty(0)} + \varepsilon \tilde{\mathbf{V}}_{pb}^{\infty(1)} + O(\varepsilon^2) \\ \tilde{P}_{pb}^\infty = \tilde{P}_{pb}^{\infty(0)} + \varepsilon \tilde{P}_{pb}^{\infty(1)} + O(\varepsilon^2) \end{cases} \quad (2.2)$$

where $(\tilde{\mathbf{V}}_{pb}^{\infty(1)}, \tilde{P}_{pb}^{\infty(1)})$ are the velocity and pressure generated by the roughness of the wall. By the adhesion condition of the velocity fields $\tilde{\mathbf{V}}_{pb}^{\infty}$ and $\tilde{\mathbf{V}}_{pb}^{\infty(0)}$ on the rough wall and the fictitious plane wall located at $\tilde{Z} = 0$, respectively, and by using the Taylor expansion, the boundary condition of the second-order velocity field $\tilde{\mathbf{V}}_{pb}^{\infty(1)}$ on a virtual plane wall $\tilde{Z} = 0$ is expressed by

$$\left[\tilde{\mathbf{V}}_{pb}^{\infty(1)} \right]_{\tilde{Z}=0} = -a\mathcal{R}(\tilde{X}) \left[\frac{\partial \tilde{\mathbf{V}}_{pb}^{\infty(0)}}{\partial \tilde{Z}} \right]_{\tilde{Z}=0}. \quad (2.3)$$

This relationship appears as a slip boundary condition for the unknown flow field on a fictitious plane wall located at $\tilde{Z} = 0$. This technique is called the flattening of the boundary condition, and it has already been used in various earlier studies [17, 20]. The substitution of $\tilde{\mathbf{V}}_{pb}^{\infty(0)}$ by its expression (1.1) in the flattened condition (2.3) leads to

$$\left[\tilde{\mathbf{V}}_{pb}^{\infty(1)} \right]_{\tilde{Z}=0} = -a\tilde{k}_s \mathcal{R}(\tilde{X}) \mathbf{i}_x. \quad (2.4)$$

This expression (2.4) is identical to that found by Assoudi et al. [16] in the case of a linear shear flow.

3. Perturbed flow field in the presence of the spherical particle

By linearity of the Stokes equations, the perturbed flow field $(\tilde{\mathbf{V}}_{pb}, \tilde{P}_{pb})$ is written as follows:

$$\begin{cases} \tilde{\mathbf{V}}_{pb} = \tilde{\mathbf{V}}_{pb}^{\infty} + \tilde{\mathbf{v}}_{pb} \\ \tilde{P}_{pb} = \tilde{P}_{pb}^{\infty} + \tilde{p}_{pb}, \end{cases} \quad (3.1)$$

where $\tilde{\mathbf{v}}_{pb}$ and $\tilde{p}_{pb}$ denote, respectively, the velocity and pressure of the perturbation flow field due to the presence of the sphere. This flow is also described by the Stokes equations

$$\begin{cases} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{v}}_{pb} = \tilde{\nabla} \tilde{p}_{pb} \\ \tilde{\nabla} \cdot \tilde{\mathbf{v}}_{pb} = 0 \end{cases} \quad \text{with} \quad \begin{cases} \tilde{\mathbf{v}}_{pb} = -\tilde{\mathbf{V}}_{pb}^{\infty} & \text{on the sphere surface,} \\ \tilde{\mathbf{v}}_{pb} = \mathbf{0} & \text{on the rough wall,} \\ \tilde{\mathbf{v}}_{pb} = \mathbf{0} & \text{far from the sphere.} \end{cases} \quad (3.2)$$

In the case of small roughness amplitude ($\varepsilon \ll 1$), this perturbation flow field can be expanded as

$$\begin{cases} \tilde{\mathbf{v}}_{pb} = \tilde{\mathbf{v}}_{pb}^{(0)} + \varepsilon \tilde{\mathbf{v}}_{pb}^{(1)} + O(\varepsilon^2) \\ \tilde{p}_{pb} = \tilde{p}_{pb}^{(0)} + \varepsilon \tilde{p}_{pb}^{(1)} + O(\varepsilon^2), \end{cases} \quad (3.3)$$

where $(\tilde{\mathbf{v}}_{pb}^{(0)}, \tilde{p}_{pb}^{(0)})$ denote the velocity and pressure of the perturbation flow field induced by a spherical particle held fixed near a virtual plane wall located at $\tilde{Z} = 0$ in the Poiseuille flow (1.1), this flow field is also governed by the Stokes equations. The analytical solution of the flow $(\tilde{\mathbf{v}}_{pb}^{(0)}, \tilde{p}_{pb}^{(0)})$ has been given by [1, 11], some details are given in Appendix A. On the other hand, $\varepsilon(\tilde{\mathbf{v}}_{pb}^{(1)}, \tilde{p}_{pb}^{(1)})$ is the perturbation flow field due to the presence of the wall roughness.

Substituting the asymptotic expansions (3.3) into Eq. (3.2), and by linearity of the problem, the perturbation flow field of the order (1) should also satisfy:

$$\left\{ \begin{array}{l} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{v}}_{pb}^{(1)} = \tilde{\nabla} \tilde{p}_{pb}^{(1)} \\ \tilde{\nabla} \cdot \tilde{\mathbf{v}}_{pb}^{(1)} = 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{ll} \tilde{\mathbf{v}}_{pb}^{(1)} = -\tilde{\mathbf{V}}_{pb}^{\infty(1)} & \text{on the sphere surface,} \\ \tilde{\mathbf{v}}_{pb}^{(1)} = -a\mathcal{R}(\tilde{x}) \left[\frac{\partial \tilde{\mathbf{v}}_{pb}^{(0)}}{\partial \tilde{z}} \right] & \text{on the plane wall at } \tilde{Z} = 0, \\ \tilde{\mathbf{v}}_{pb}^{(1)} = \mathbf{0} & \text{far from the sphere.} \end{array} \right. \quad (3.4)$$

The boundary condition on the rough wall is replaced by the flattened condition on the virtual plane wall positioned at $\tilde{Z} = 0$ in (3.4), this condition of the second-order velocity field is obtained by using the expanding Taylor series of the perturbation velocity field $\tilde{\mathbf{v}}_{pb}$ in the vicinity of $\tilde{Z} = 0$. Furthermore, the complexity of the boundary conditions in (3.4) makes the analytical resolution of the velocity $\tilde{\mathbf{v}}_{pb}^{(1)}$ of the perturbation flow field difficult. Our goal is to calculate the contribution of this flow field to the hydrodynamic forces and torques exerted on the spherical particle; for that it is sufficient to use the Lorentz reciprocal theorem [18], which allows us to determine these effects due to roughness.

4. Hydrodynamic forces and torques acting on the sphere by the roughness

Based on the assumption of a low roughness amplitude ($\varepsilon \ll 1$), and by linearity of Stokes equations, the asymptotic expansions of the hydrodynamic force and torque exerted by the flow field onto the sphere held fixed in the ambient flow field (1.1) are also obtained as follows:

$$\left\{ \begin{array}{l} \tilde{\mathbf{F}}_{pb} = \tilde{\mathbf{F}}_{pb}^{(0)} + \varepsilon \tilde{\mathbf{F}}_{pb}^{(1)} + O(\varepsilon^2) \\ \tilde{\mathbf{C}}_{pb} = \tilde{\mathbf{C}}_{pb}^{(0)} + \varepsilon \tilde{\mathbf{C}}_{pb}^{(1)} + O(\varepsilon^2), \end{array} \right. \quad (4.1)$$

where $(\tilde{\mathbf{F}}_{pb}^{(0)}, \tilde{\mathbf{C}}_{pb}^{(0)})$ are the drag force and torque of first order exerted on the spherical particle held fixed in the case of a smooth wall, respectively. And the perturbation terms $\varepsilon \tilde{\mathbf{F}}_{pb}^{(1)}$ and $\varepsilon \tilde{\mathbf{C}}_{pb}^{(1)}$ represent, respectively, the force and the torque exerted on a spherical particle generated by the wall roughness.

Again by linearity of Stokes equations, the force $\tilde{\mathbf{F}}_{pb}^{(0)}$ and torque $\tilde{\mathbf{C}}_{pb}^{(0)}$ along $\tilde{x}$ and $\tilde{y}$, respectively, may be written as sums of forces and torques for elementary flow fields:

$$\tilde{\mathbf{F}}_{pb}^{(0)} = \tilde{\mathbf{F}}_s^{(0)} + \tilde{\mathbf{F}}_q^{(0)}, \quad (4.2a)$$

$$\tilde{\mathbf{C}}_{pb}^{(0)} = \tilde{\mathbf{C}}_s^{(0)} + \tilde{\mathbf{C}}_q^{(0)}, \quad (4.2b)$$

where the subscripts $(\cdot)_s$ and $(\cdot)_q$ stand, respectively, for the flow fields due to a sphere held fixed in an ambient linear shear flow and quadratic shear flow. At order (0), the components of the forces and torques in these directions are shown in Table 1 in terms of dimensionless friction factors, see [10, 11].

By the symmetry of the model with respect to the plane $\tilde{y} = 0$, the hydrodynamic effect due to the roughness of the wall can be developed by

$$\tilde{\mathbf{F}}_{pb}^{(1)} = \tilde{\mathbf{F}}_{x,pb}^{(1)} \mathbf{i}_x + \tilde{\mathbf{F}}_{z,pb}^{(1)} \mathbf{i}_z, \quad (4.3a)$$

$$\tilde{\mathbf{C}}_{pb}^{(1)} = \tilde{\mathbf{C}}_{y,pb}^{(1)} \mathbf{i}_y. \quad (4.3b)$$

Table 1. The forces and torques of first order exerted on a sphere, in terms of friction factors.

Linear shear flow	$\tilde{F}_{x,s}^{(0)} = 6\pi a\mu_f \tilde{k}_s \tilde{l} f_{x,s}^{(0)}$	$\tilde{C}_{y,s}^{(0)} = 4\pi a^3 \mu_f \tilde{k}_s c_{y,s}^{(0)}$
Quadratic shear flow	$\tilde{F}_{x,q}^{(0)} = 6\pi a\mu_f \tilde{k}_q \tilde{l}^2 f_{x,q}^{(0)}$	$\tilde{C}_{y,q}^{(0)} = 8\pi a^3 \mu_f \tilde{k}_q \tilde{l} c_{y,q}^{(0)}$
Poiseuille flow	$\tilde{F}_{x,pb}^{(0)} = 6\pi a\mu_f \tilde{V}_{\max} f_{x,pb}^{(0)}$	$\tilde{C}_{y,pb}^{(0)} = 8\pi a^2 \mu_f \tilde{V}_{\max} c_{y,pb}^{(0)}$

To calculate these hydrodynamic effects in (4.3), we used the Lorentz reciprocal theorem [18, 19]. This theorem states a relationship between two Stokes flows embedded by the same boundary $\partial\mathcal{D}$. One of the Stokes flows here is the unknown flow $(\tilde{\mathbf{v}}_{pb}^{(1)}, \tilde{p}_{pb}^{(1)})$ and the other is a reciprocal flow, say $(\tilde{\mathbf{u}}^*, \tilde{p}^*)$, is chosen according to the problem at hand. Then the theorem is written as

$$\int_{\partial\mathcal{D}} \tilde{\mathbf{v}}_{pb}^{(1)} \cdot [\tilde{\boldsymbol{\sigma}}^* \cdot \tilde{\mathbf{dS}}] = \int_{\partial\mathcal{D}} \tilde{\mathbf{u}}^* \cdot [\tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}], \quad (4.4)$$

where $\tilde{\boldsymbol{\sigma}}_{pb}^{(1)}$ and $\tilde{\boldsymbol{\sigma}}^*$ are the stress tensors for the physical flow $(\tilde{\mathbf{v}}_{pb}^{(1)}, \tilde{p}_{pb}^{(1)})$ and the reciprocal flow $(\tilde{\mathbf{u}}^*, \tilde{p}^*)$, respectively. Let here $\partial\mathcal{D} = \mathcal{S} + \mathcal{P} + \mathcal{S}_\infty$, where $\mathcal{P}$ is the virtual plane wall surface, $\mathcal{S}$ is the sphere surface and $\mathcal{S}_\infty$ is a surface far from the sphere. Then the expression (4.4) becomes

$$\begin{aligned} \int_{\mathcal{S}} \tilde{\mathbf{v}}_{pb}^{(1)} \cdot [\tilde{\boldsymbol{\sigma}}^* \cdot \tilde{\mathbf{dS}}] + \int_{\mathcal{P}} \tilde{\mathbf{v}}_{pb}^{(1)} \cdot [\tilde{\boldsymbol{\sigma}}^* \cdot \tilde{\mathbf{dS}}] + \int_{\mathcal{S}_\infty} \tilde{\mathbf{v}}_{pb}^{(1)} \cdot [\tilde{\boldsymbol{\sigma}}^* \cdot \tilde{\mathbf{dS}}] = \\ = \int_{\mathcal{S}} \tilde{\mathbf{u}}^* \cdot [\tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}] + \int_{\mathcal{P}} \tilde{\mathbf{u}}^* \cdot [\tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}] + \int_{\mathcal{S}_\infty} \tilde{\mathbf{u}}^* \cdot [\tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}]. \end{aligned} \quad (4.5)$$

The integral on $\mathcal{S}$ on the right-hand side of (4.5) provides, depending on the chosen reciprocal flow field, either the required torque or force at $O(\varepsilon)$:

$$\begin{aligned} \tilde{\mathbf{C}}_{pb}^{(1)} &= \int_{\mathcal{S}} \tilde{\mathbf{r}} \times [\tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}] \\ \tilde{\mathbf{F}}_{pb}^{(1)} &= \int_{\mathcal{S}} \tilde{\boldsymbol{\sigma}}_{pb}^{(1)} \cdot \tilde{\mathbf{dS}}. \end{aligned}$$

The calculation of the torque $\tilde{\mathbf{C}}_{pb}^{(1)}$ exerted by the flow field on the sphere in the direction $\tilde{y}$ pushes us to choose as a reciprocal flow field that which is generated by a rotational sphere, with the constant velocity $\tilde{\boldsymbol{\Omega}}_y^* = \tilde{\Omega}_y^* \mathbf{i}_y$, in a fluid at rest. Likewise, for calculating the forces $\tilde{\mathbf{F}}_{x,pb}^{(1)}$ and $\tilde{\mathbf{F}}_{z,pb}^{(1)}$, we choose those generated by the sphere translating with the constant velocities $\tilde{\mathbf{u}}^*|_{\mathcal{S}} = \tilde{U}_x^* \mathbf{i}_x$ and $\tilde{\mathbf{u}}^*|_{\mathcal{S}} = -\tilde{U}_z^* \mathbf{i}_z$, respectively, in a fluid at rest. In all cases, the reciprocal flow field satisfies the boundary conditions

$$\tilde{\mathbf{u}}^*|_{\mathcal{P}} = \mathbf{0}, \quad \tilde{\mathbf{u}}^*|_{\mathcal{S}_\infty} = \mathbf{0}.$$

On the right-hand side of equation (4.5), the integrals on the plane wall and at infinity will disappear by using the reciprocal flow field boundary conditions. On the left-hand side, the

integral at infinity will disappear by using the boundary condition (3.4). Then Eq. (4.5) provides

$$\tilde{U}_x^* \tilde{F}_{x,pb}^{(1)} = - \int_S \tilde{\mathbf{V}}_{pb}^{\infty(1)} \cdot [\tilde{\boldsymbol{\sigma}}^{*,t} \cdot \tilde{\mathbf{dS}}] - a \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \left[\frac{\partial \tilde{\mathbf{V}}_{pb}^{(0)}}{\partial \tilde{z}} \right]_{\tilde{z}=0} \cdot [\tilde{\boldsymbol{\sigma}}^{*,t} \cdot \mathbf{i}_z] \tilde{dS}, \quad (4.6a)$$

$$\tilde{U}_z^* \tilde{F}_{z,pb}^{(1)} = - \int_S \tilde{\mathbf{V}}_{pb}^{\infty(1)} \cdot [\tilde{\boldsymbol{\sigma}}^{*,st} \cdot \tilde{\mathbf{dS}}] + a \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \left[\frac{\partial \tilde{\mathbf{V}}_{pb}^{(0)}}{\partial \tilde{z}} \right]_{\tilde{z}=0} \cdot [\tilde{\boldsymbol{\sigma}}^{*,st} \cdot \mathbf{i}_z] \tilde{dS}, \quad (4.6b)$$

$$\tilde{\Omega}_y^* \tilde{C}_{y,pb}^{(1)} = - \int_S \tilde{\mathbf{V}}_{pb}^{\infty(1)} \cdot [\tilde{\boldsymbol{\sigma}}^{*,r} \cdot \tilde{\mathbf{dS}}] - a \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \left[\frac{\partial \tilde{\mathbf{V}}_{pb}^{(0)}}{\partial \tilde{z}} \right]_{\tilde{z}=0} \cdot [\tilde{\boldsymbol{\sigma}}^{*,r} \cdot \mathbf{i}_z] \tilde{dS}. \quad (4.6c)$$

We recall that the quantities linked to the reciprocal flow are represented by a star and by a lower-case letter which indicates the nature of the flow, that is, (t) for translation, (r) for rotation and (st) for settling towards to the plane wall. Also, the quantity related to the first-order basic flow field is subscripted by a lower-case letter (pb) for Poiseuille flow.

To simplify the expression (4.6), we used the following notations:

$$\tilde{L}_{pb}^{*,\bullet} = \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \left[\frac{\partial \tilde{\mathbf{V}}_{pb}^{(0)}}{\partial \tilde{z}} \right]_{\tilde{z}=0} \cdot [\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \mathbf{i}_z] \tilde{dS}, \quad (4.7a)$$

$$\tilde{I}_{\infty}^{*,\bullet} = - \int_S \tilde{\mathbf{V}}_{pb}^{\infty(1)} \cdot [\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \tilde{\mathbf{dS}}], \quad (4.7b)$$

where the dot superscript $(\bullet)$ denotes either t , r or st . Finally, Eq. (4.6) becomes

$$\tilde{U}_x^* \tilde{F}_{x,pb}^{(1)} = -a \tilde{L}_{pb}^{*,t} + \tilde{I}_{\infty}^{*,t}, \quad (4.8a)$$

$$\tilde{U}_z^* \tilde{F}_{z,pb}^{(1)} = a \tilde{L}_{pb}^{*,st} + \tilde{I}_{\infty}^{*,st}, \quad (4.8b)$$

$$\tilde{\Omega}_y^* \tilde{C}_{y,pb}^{(1)} = -a \tilde{L}_{pb}^{*,r} + \tilde{I}_{\infty}^{*,r}. \quad (4.8c)$$

The forces $(\tilde{F}_{x,pb}^{(1)}, \tilde{F}_{z,pb}^{(1)})$ and the torque $\tilde{C}_{y,pb}^{(1)}$ are expressed in terms of $\tilde{L}_{pb}^{*,\bullet}$ and $\tilde{I}_{\infty}^{*,\bullet}$. By using the Lorentz reciprocal theorem relating the flow fields $(\tilde{\mathbf{V}}_{pb}^{\infty(1)}, \tilde{P}_{pb}^{\infty(1)})$ and $(\tilde{\mathbf{u}}^*, \tilde{p}^*)$, the integrals $\tilde{I}_{\infty}^{*,t}$, $\tilde{I}_{\infty}^{*,st}$ and $\tilde{I}_{\infty}^{*,r}$ on the right-hand side of Eq. (4.8) may be expressed, respectively, as

$$\begin{aligned} \tilde{I}_{\infty}^{*,t} &= a \tilde{k}_s \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \mathbf{i}_x \cdot [\tilde{\boldsymbol{\sigma}}^{*,t} \cdot \tilde{\mathbf{dS}}], \\ \tilde{I}_{\infty}^{*,st} &= a \tilde{k}_s \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \mathbf{i}_x \cdot [\tilde{\boldsymbol{\sigma}}^{*,st} \cdot \tilde{\mathbf{dS}}], \\ \tilde{I}_{\infty}^{*,r} &= a \tilde{k}_s \int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \mathbf{i}_x \cdot [\tilde{\boldsymbol{\sigma}}^{*,r} \cdot \tilde{\mathbf{dS}}], \end{aligned}$$

where

$$\int_{\mathcal{P}} \mathcal{R}(\tilde{x}) \mathbf{i}_x \cdot \tilde{\boldsymbol{\sigma}}^* \cdot \tilde{\mathbf{dS}} = \mu_f \int_0^\infty \int_0^{2\pi} \mathcal{R}(\tilde{\rho}, \tilde{\phi}) \left[\frac{\partial \tilde{u}_\rho^*}{\partial \tilde{z}} \cos \tilde{\phi} - \frac{\partial \tilde{u}_\phi^*}{\partial \tilde{z}} \sin \tilde{\phi} \right] \tilde{\rho} d\tilde{\rho} d\tilde{\phi}.$$

The term $\tilde{L}_{pb}^{*,\bullet}$ is an integral on the plane wall (4.7a), which contains a scalar product of the first derivative of the first-order velocity field, and the projection of the stress tensor on the $\tilde{z}$ axis. The details of this product are given in Appendix B.

The various physical quantities are now made dimensionless. The force and torque components ($\tilde{F}_{x,pb}^{(1)}$, $\tilde{F}_{z,pb}^{(1)}$, $\tilde{C}_{y,pb}^{(1)}$) are made dimensionless by using the same quantities as for the components at order (0) ($\tilde{F}_{x,pb}^{(0)}$, $\tilde{C}_{y,pb}^{(0)}$), see Table 1. The terms $\tilde{L}_{pb}^{*,t}$, $\tilde{L}_{pb}^{*,st}$ and $\tilde{L}_{pb}^{*,r}$ are made dimensionless by using the quantity $\left(\mu_f V_c \tilde{u}^* \Big|_S\right)$, where $V_c = \tilde{V}_{\max}$ is the reference velocity. Also, the terms $\tilde{I}_{\infty}^{*,\bullet}$ are made dimensionless by using the quantity $\left(a\mu_f V_c \tilde{u}^* \Big|_S\right)$. The dimensionless quantities will be denoted without tilde ($\sim$).

To simplify the notation, let $v_{\rho,\bullet}^{(0)} = \zeta_{\rho,\bullet} \cos \phi$ and $v_{\phi,\bullet}^{(0)} = \zeta_{\phi,\bullet} \sin \phi$, where the dot subscripted ($\bullet$) denotes s, q in (A.2). Then in $L_{pb}^{*,t}$ and $I_{\infty}^{*,t}$ we write $u_{\rho}^* = \zeta_{\rho,t} \cos \phi$ and $u_{\phi}^* = \zeta_{\phi,t} \sin \phi$; in $L_{pb}^{*,r}$ and $I_{\infty}^{*,r}$ we write $u_{\rho}^* = \zeta_{\rho,r} \cos \phi$ and $u_{\phi}^* = \zeta_{\phi,r} \sin \phi$, see (A.3). The dimensionless roughness effects for this problem are developed from (4.8) as follows:

$$f_{x,pb}^{(1)} = -4\lambda(l/a) \left(L_{pb}^{*,t} + I_{\infty}^{*,t} \right), \quad (4.9a)$$

$$f_{z,pb}^{(1)} = 4\lambda(l/a) \left(L_{pb}^{*,st} + I_{\infty}^{*,st} \right), \quad (4.9b)$$

$$c_{y,pb}^{(1)} = -4\lambda \left(L_{pb}^{*,r} + I_{\infty}^{*,r} \right), \quad (4.9c)$$

where

$$\begin{aligned} L_{pb}^{*,t} = & \int_{\rho=0}^{+\infty} \left(\frac{\partial \zeta_{\rho,s}}{\partial z} - \lambda \frac{\partial \zeta_{\rho,q}}{\partial z} + 1 \right) \frac{\partial \zeta_{\rho,t}}{\partial z} \mathcal{F}_1(\rho) \rho d\rho + \\ & + \int_{\rho=0}^{+\infty} \left(\frac{\partial \zeta_{\phi,s}}{\partial z} - \lambda \frac{\partial \zeta_{\phi,q}}{\partial z} - 1 \right) \frac{\partial \zeta_{\phi,t}}{\partial z} \mathcal{F}_3(\rho) \rho d\rho + \\ & + c_0 \int_{\rho=0}^{+\infty} \left(\frac{\partial \zeta_{\rho,s}}{\partial z} - \lambda \frac{\partial \zeta_{\rho,q}}{\partial z} + 1 \right) \frac{\partial \zeta_{\rho,t}}{\partial z} \rho d\rho + \\ & + c_0 \int_{\rho=0}^{+\infty} \left(\frac{\partial \zeta_{\phi,s}}{\partial z} - \lambda \frac{\partial \zeta_{\phi,q}}{\partial z} - 1 \right) \frac{\partial \zeta_{\phi,t}}{\partial z} \rho d\rho, \end{aligned} \quad (4.10a)$$

$$L_{pb}^{*,st} = \int_{\rho=0}^{+\infty} \left(\frac{\partial \zeta_{\rho,s}}{\partial z} - \lambda \frac{\partial \zeta_{\rho,q}}{\partial z} + 1 \right) \frac{\partial u_{\rho}^{st}}{\partial z} \mathcal{F}_2(\rho) \rho d\rho, \quad (4.10b)$$

$$\begin{aligned} L_{pb}^{*,r} = & \int_{\rho=0}^{+\infty} \left(\frac{1}{2} \frac{\partial \zeta_{\rho,s}}{\partial z} - \lambda \frac{\partial \zeta_{\rho,q}}{\partial z} + 1 \right) \frac{\partial \zeta_{\rho,r}}{\partial z} \mathcal{F}_1(\rho) \rho d\rho + \\ & + \int_{\rho=0}^{+\infty} \left(\frac{1}{2} \frac{\partial \zeta_{\phi,s}}{\partial z} - \lambda \frac{\partial \zeta_{\phi,q}}{\partial z} - 1 \right) \frac{\partial \zeta_{\phi,r}}{\partial z} \mathcal{F}_3(\rho) \rho d\rho + \end{aligned}$$

$$\begin{aligned}
& + c_0 \int_{\rho=0}^{+\infty} \left(\frac{1}{2} \frac{\partial \zeta_{\rho,s}}{\partial z} - \lambda \frac{\partial \zeta_{\rho,q}}{\partial z} + 1 \right) \frac{\partial \zeta_{\rho,r}}{\partial z} \rho d\rho + \\
& + c_0 \int_{\rho=0}^{+\infty} \left(\frac{1}{2} \frac{\partial \zeta_{\phi,s}}{\partial z} - \lambda \frac{\partial \zeta_{\phi,q}}{\partial z} - 1 \right) \frac{\partial \zeta_{\phi,r}}{\partial z} \rho d\rho,
\end{aligned} \tag{4.10c}$$

and

$$I_{\infty}^{*,t} = \int_{\rho=0}^{\infty} \left[\frac{\partial \zeta_{\rho,t}}{\partial z} \mathcal{F}_1(\rho) - \frac{\partial \zeta_{\phi,t}}{\partial z} \mathcal{F}_3(\rho) \right] \rho d\rho + \pi c_0 \int_{\rho=0}^{\infty} \left[\frac{\partial \zeta_{\rho,t}}{\partial z} - \frac{\partial \zeta_{\phi,t}}{\partial z} \right] \rho d\rho, \tag{4.11a}$$

$$I_{\infty}^{*,st} = \int_{\rho=0}^{\infty} \frac{\partial u_{\rho}^{st}}{\partial z} \mathcal{F}_2(\rho) \rho d\rho, \tag{4.11b}$$

$$I_{\infty}^{*,r} = \frac{1}{2} \int_{\rho=0}^{\infty} \left[\frac{\partial \zeta_{\rho,r}}{\partial z} \mathcal{F}_1(\rho) - \frac{\partial \zeta_{\phi,r}}{\partial z} \mathcal{F}_3(\rho) \right] \rho d\rho + \frac{1}{2} \pi c_0 \int_{\rho=0}^{\infty} \left[\frac{\partial \zeta_{\rho,r}}{\partial z} - \frac{\partial \zeta_{\phi,r}}{\partial z} \right] \rho d\rho, \tag{4.11c}$$

with range of variation $0 < \lambda < 0.125$. Here, $\lambda = \frac{a}{h}$ and l/a represent the ratio of the radius of the sphere to the distance separating the two parallel smooth walls and the normalized distance from the sphere center to the plane wall positioned at $\tilde{Z} = 0$, respectively, see Fig. 1. Moreover, In (4.9), $f_{x,pb}^{(1)}$, $f_{z,pb}^{(1)}$ and $c_{y,pb}^{(1)}$ denote the dimensionless friction factors of the drag force, lift force and torque coefficients due to the roughness, respectively.

The functions ζ in (4.10) and (4.11) are obtained in terms of the harmonic functions $(U_0^{\bullet}, U_1^{\bullet}, U_2^{\bullet}, Q_1^{\bullet})$, which are then expanded as series in bispherical coordinates (details are given in Appendix A). Results for the coefficients in the series giving the harmonics are provided in [10, 11]. Furthermore, the functions $\mathcal{F}_1(\rho)$, $\mathcal{F}_2(\rho)$ and $\mathcal{F}_3(\rho)$ are defined in terms of the modified Bessel functions $\mathcal{J}_n(x)$ of order n :

$$\begin{aligned}
\mathcal{F}_1(\rho) &= \pi \sum_{n=1}^{+\infty} c_n (\mathcal{J}_0(n\omega\rho) - \mathcal{J}_2(n\omega\rho)) \\
\mathcal{F}_2(\rho) &= 2\pi \sum_{n=1}^{+\infty} s_n (\mathcal{J}_1(n\omega\rho)) \\
\mathcal{F}_3(\rho) &= \pi \sum_{n=1}^{+\infty} c_n (\mathcal{J}_0(n\omega\rho) + \mathcal{J}_2(n\omega\rho)).
\end{aligned}$$

5. Results and discussion

Numerical results of second-order effects, due to the roughness, are treated in this section. The various integrals in (4.10), (4.11) are calculated by using the Gauss–Laguerre algorithm with 128 nodes, so that each integral is calculated with a precision better than $O(10^{-4})$ for $L \geq 3$. These results will be compared with those obtained by Assoudi et al. [16] in the case of a sphere held fixed, near a rough wall with a same roughness, in a linear shear flow (s). Indeed, according

to Ho and Leal [2], we will limit our study to the case $\lambda \ll 1$ in order to neglect the effect of the upper wall.

Values of the dimensionless drag force coefficient $f_{x,pb}^{(1)}$, lift force coefficient $f_{z,pb}^{(1)}$ versus the dimensionless distance l/a from the sphere center to the plane $Z = 0$ are plotted in Figs. 2a–2b, for various values of λ , in the case of the sphere located at $T = 0$ with $L = 4$ and $\delta = 3/4$. It is observed that the absolute values of these coefficients $|f_{x,pb}^{(1)}|$ and $|f_{z,pb}^{(1)}|$ decrease as expected with l/a . This means that the roughness of the wall has a great influence on the sphere when this latter is close to the wall. In Fig. 2a, when the sphere is close to the rough wall, at $l/a = 1.2$, for the Poiseuille flow, the friction coefficient $|f_{x,pb}^{(1)}|$ evolves from 0.20191 for $\lambda = 0.05$ to reach the value 0.46828 when $\lambda = 0.1$, whereas the coefficient $|f_{x,s}^{(1)A}|$, which was calculated by Assoudi et al. [16], takes the value 0.83430. This means that the intensity of the linear shear flow is greater than that of the Poiseuille flow on a sphere near the lubrication zone. In Fig. 2c the negative values of torque coefficients ($c_{y,s}^{(1)A}$, $c_{y,pb}^{(1)}$) show that the rough wall exerts a torque in the opposite direction to that exerted by a smooth wall.

Fig. 2. Dimensionless friction coefficients of order (1), due to roughness versus the dimensionless distance l/a , for different values of λ , in the case of a sphere located at $T = 0$ with $L = 4$, $\delta = 3/4$, and comparison of the results with Assoudi et al. [16]. (a) The drag force coefficients $f_{x,s}^{(1)A}$, $f_{x,pb}^{(1)}$. (b) The lift force coefficients $f_{z,s}^{(1)A}$, $f_{z,pb}^{(1)}$. (c) The torque coefficients $c_{y,s}^{(1)A}$, $c_{y,pb}^{(1)}$.

Figures 3a–3c represent the variation of the effects resulting from roughness with the dimensionless period L in the particular position of the sphere at $T = 4$ and $l/a = 1.37$, for different values of the parameter λ . The numerical results show that these dimensionless coefficients of force and torque reach their maxima at a certain period, e.g., the drag force coefficient $f_{x,pb}^{(1)}$ reaches its maximum when $L \simeq 4.5$ whatever the value of λ , the same for $f_{x,s}^{(1)A}$ as shown in Fig. 3a. The same for the lift force coefficients $f_{z,s}^{(1)A}$ and $f_{z,pb}^{(1)}$, which reach their maximum level when $L \simeq 9$ (see Fig. 3b). These results can be explained by the fact that the decrease or increase in the period means the decrease or increase in the width of the roughness cavity. In addition, it is concluded that the intensities of the efforts of order $O(\varepsilon)$ that the particle undergoes, for a linear shear flow, are more important than those of the Poiseuille flow.

Fig. 3. Dimensionless friction coefficients of order (1), due to roughness, with $\delta = 3/4$, versus the dimensionless period L , for different values of λ , in the case of sphere located at $T = 0$ with $\delta = 3/4$, and comparison with the results of Assoudi et al. [16]. (a) The drag force coefficients $f_{x,s}^{(1)A}$, $f_{x,pb}^{(1)}$. (b) The lift force coefficients $f_{z,s}^{(1)A}$, $f_{z,pb}^{(1)}$. (c) The torque coefficients $c_{x,s}^{(1)A}$, $c_{x,pb}^{(1)}$.

The evaluation of these effects in terms of the relative position of the particle T along a period is shown in Figs. 4a–4c. The plots of these coefficients show that the force and the torque vary periodically. These results may be interpreted by considering two combining effects:

- The variation of the position of the sphere, along a period, is accompanied by the variation of the closest distance separating the sphere surface and the wall surface.
- The effect due to the relative direction of the sphere relative to the nearest point of the wall surface.

Fig. 4. Dimensionless friction coefficients of order (1), due to roughness versus the relative position T/L , for different values of λ , with $L = 4$, $\delta = 3/4$ and $l/a = 1.57$, and comparison with the results of Assoudi et al. [16]. (a) The drag force coefficients $f_{x,s}^{(1)A}$, $f_{x,pb}^{(1)}$. (b) The lift force coefficients $f_{z,s}^{(1)A}$, $f_{z,pb}^{(1)}$. (c) The torque coefficients $c_{y,s}^{(1)A}$, $c_{y,pb}^{(1)}$.

6. Conclusion

This paper studied the case of a fixed spherical particle in a Poiseuille flow field limited by a rough wall. Using the asymptotic expansion based on the assumption of a low roughness amplitude, coupled with the Lorentz reciprocity theorem, we have obtained the analytical expressions of the drag and lift forces and the torque due to the roughness. The numerical results show that these effects are influenced by the roughness parameters (amplitude ε , period L) and

by the distance between the sphere center and the virtual lower plane wall, as well as the relative position of the sphere along a period. As expected, this influence of roughness is important when the particle is close to the wall, while its attenuation is observed when the distance between the sphere and the wall increases.

A. Solutions of the first-order (0) and reciprocal flow fields

In this appendix we recall some details of the solutions for the order (0) Stokes flow around a spherical particle held fixed near a lower plane wall, from [4] and the reciprocal flow fields.

1. The first-order flow field $(\tilde{\mathbf{v}}_{pb}^{(0)}, \tilde{p}_{pb}^{(0)})$

This flow field satisfies the Stokes equations

$$\left\{ \begin{array}{l} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{v}}_{pb}^{(0)} = \tilde{\nabla} \tilde{p}_{pb}^{(0)} \\ \tilde{\nabla} \cdot \tilde{\mathbf{v}}_{pb}^{(0)} = 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{ll} \tilde{\mathbf{v}}_{pb}^{(0)} = -\tilde{\mathbf{V}}_{pb}^{\infty(0)} & \text{on the sphere surface,} \\ \tilde{\mathbf{v}}_{pb}^{(0)} = \mathbf{0} & \text{on the smooth plane walls,} \\ \tilde{\mathbf{v}}_{pb}^{(0)} = \mathbf{0} & \text{far from the sphere.} \end{array} \right. \quad (A.1)$$

By linearity of the Stokes equations, this problem may be solved as the superposition of the elementary cases of

(i) a sphere held fixed in a linear shear flow $\tilde{\mathbf{V}}_s^{\infty(0)} = \tilde{k}_s \tilde{Z} \mathbf{i}_x$ near a plane wall:

$$\left\{ \begin{array}{l} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{v}}_s^{(0)} = \tilde{\nabla} \tilde{p}_s^{(0)} \\ \tilde{\nabla} \cdot \tilde{\mathbf{v}}_s^{(0)} = 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{ll} \tilde{\mathbf{v}}_s^{(0)} = -\tilde{k}_s \tilde{Z} \mathbf{i}_x & \text{on the sphere surface,} \\ \tilde{\mathbf{v}}_s^{(0)} = \mathbf{0} & \text{on the smooth plane wall,} \\ \tilde{\mathbf{v}}_s^{(0)} = \mathbf{0} & \text{far from the sphere.} \end{array} \right. \quad (A.2)$$

(ii) a sphere held fixed in a quadratic shear flow $\tilde{\mathbf{V}}_q^{\infty(0)} = \tilde{k}_q \tilde{Z}^2 \mathbf{i}_x$ near a plane wall:

$$\left\{ \begin{array}{l} \mu_f \tilde{\nabla}^2 \tilde{\mathbf{v}}_q^{(0)} = \tilde{\nabla} \tilde{p}_q^{(0)} \\ \tilde{\nabla} \cdot \tilde{\mathbf{v}}_q^{(0)} = 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{ll} \tilde{\mathbf{v}}_q^{(0)} = -\tilde{k}_q \tilde{Z}^2 \mathbf{i}_x & \text{on the sphere surface,} \\ \tilde{\mathbf{v}}_q^{(0)} = \mathbf{0} & \text{on the smooth plane wall,} \\ \tilde{\mathbf{v}}_q^{(0)} = \mathbf{0} & \text{far from the sphere.} \end{array} \right. \quad (A.3)$$

2. Solutions for the elementary first-flow problems in bispherical coordinates

Let V_c be a reference velocity, with $V_c = a \tilde{k}_s$ for a linear shear flow and $V_c = a^2 \tilde{k}_q$ for a quadratic shear flow. The reference length is a . The relevant reference pressure is $\mu_f V_c / a$. The solutions for the dimensionless pressure and velocity $(v_{\rho, \bullet}^{(0)}, v_{\phi, \bullet}^{(0)}, v_{z, \bullet}^{(0)})$ are expressed in the following form:

$$\left\{ \begin{array}{l} p_{\bullet}^{(0)} = Q_1^{\bullet} \cos \phi \\ v_{\rho, \bullet}^{(0)} = \frac{1}{2} \left(\frac{\rho}{c} Q_1^{\bullet} + (U_2^{\bullet} + U_0^{\bullet}) \right) \cos \phi \\ v_{\phi, \bullet}^{(0)} = \frac{1}{2} (U_2^{\bullet} - U_0^{\bullet}) \sin \phi \\ v_{z, \bullet}^{(0)} = \frac{1}{2} \left(\frac{z}{c} Q_1^{\bullet} + 2W_1^{\bullet} \right) \cos \phi, \end{array} \right. \quad (A.4)$$

where the constant $c = \sqrt{\left(\frac{l}{a}\right)^2 - 1}$, the symbol $(\bullet)$ represents any of the letters (s) and (q) , which correspond to the elementary flow fields: linear and quadratic shear flows, and the following harmonic functions are expanded as series:

$$W_1^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} [A_n^\bullet \sinh \gamma_n \xi] P_n'(\mu), \quad (A.5a)$$

$$Q_1^\bullet = c^{M-1} (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} [B_n^\bullet \cosh \gamma_n \xi + C_n^\bullet \sinh \gamma_n \xi] P_n'(\mu), \quad (A.5b)$$

$$U_0^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sum_{n=0}^{\infty} [D_n^\bullet \cosh \gamma_n \xi + E_n^\bullet \sinh \gamma_n \xi] P_n(\mu), \quad (A.5c)$$

$$U_2^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sin^2 \eta \sum_{n=0}^{\infty} [F_n^\bullet \cosh \gamma_n \xi + G_n^\bullet \sinh \gamma_n \xi] P_n''(\mu). \quad (A.5d)$$

Here $(M = 1)$ stands for linear shear flow, $M = 2$ for quadratic shear flow, $\mu = \cos \eta$, $\gamma_n = n + \frac{1}{2}$, $P_n(\mu)$ is the Legendre polynomial of order n and $(\)' = \frac{d}{d\mu}$. The coefficients $B_n^\bullet$, $C_n^\bullet$, $D_n^\bullet$, $G_n^\bullet$, $E_n^\bullet$, $F_n^\bullet$ may then be expressed in terms of the $A_n^\bullet$ and the numerical results for these coefficients are provided in [10, 11].

3. Solutions for the reciprocal flow fields $(\tilde{u}^{*,\bullet}, \tilde{p}^{*,\bullet})$

Let V_c be a reference velocity, with $V_c = \tilde{U}_x^*$ for translation, $V_c = \tilde{\Omega}_y^*$ for rotation. The solutions for the dimensionless pressure and velocity $(u_\rho^{*,\bullet}, u_\phi^{*,\bullet}, u_z^{*,\bullet})$ are searched for in the following form:

$$\begin{cases} p^{*,\bullet} = Q_1^\bullet \cos \phi \\ u_\rho^{*,\bullet} = \frac{1}{2} \left(\frac{\rho}{c} Q_1^\bullet + (U_2^\bullet + U_0^\bullet) \right) \cos \phi \\ u_\phi^{*,\bullet} = \frac{1}{2} (U_2^\bullet - U_0^\bullet) \sin \phi \\ u_z^{*,\bullet} = \frac{1}{2} \left(\frac{z}{c} Q_1^\bullet + 2W_1^\bullet \right) \cos \phi. \end{cases} \quad (A.6)$$

The symbol $(\bullet)$ represents any of the letters (t) and (r) , which correspond to the reciprocal flow fields: translation and rotation, and the following harmonic functions are expanded as series:

$$W_1^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} [A_n^\bullet \sinh \gamma_n \xi] P_n'(\mu), \quad (A.7a)$$

$$Q_1^\bullet = c^{M-1} (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} [B_n^\bullet \cosh \gamma_n \xi + C_n^\bullet \sinh \gamma_n \xi] P_n'(\mu), \quad (A.7b)$$

$$U_0^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sum_{n=0}^{\infty} [D_n^\bullet \cosh \gamma_n \xi + E_n^\bullet \sinh \gamma_n \xi] P_n(\mu), \quad (A.7c)$$

$$U_2^\bullet = c^M (\cosh \xi - \mu)^{1/2} \sin^2 \eta \sum_{n=0}^{\infty} [F_n^\bullet \cosh \gamma_n \xi + G_n^\bullet \sinh \gamma_n \xi] P_n''(\mu). \quad (A.7d)$$

Here ($M = 0$) stands for translation, ($M = 1$) for rotation. The coefficients $B_n^\bullet$, $C_n^\bullet$, $D_n^\bullet$, $G_n^\bullet$, $E_n^\bullet$, $F_n^\bullet$ may then be expressed in terms of the $A_n^\bullet$ and the numerical results for these coefficients are provided in [10, 11].

The solution for the reciprocal flow field for a sphere settling normally towards a smooth plane wall was derived in the same way by [21] with the method of bispherical coordinates. The dimensionless reciprocal flow field ($u^{*,\bullet}$, $p^{*,\bullet}$) is expressed in cylindrical coordinates:

$$\begin{cases} p^{st} = \frac{Q_0^{st}}{c} \\ u_\rho^{st} = \rho \frac{Q_0^{st}}{2c} + U_1^{st} \\ u_z^{st} = z \frac{Q_0^{st}}{2c} + U_0^{st}, \end{cases} \quad (A.8)$$

where (U_0^{st} , U_1^{st} , Q_0^{st}) are harmonic functions. These functions are then expanded as series in bispherical coordinates as in (A.7).

B. Analytical calculation of the term $\left[\frac{\partial \tilde{\mathbf{v}}_{pb}^{(0)}}{\partial \tilde{z}} \cdot (\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \mathbf{i}_z) \right]_{\tilde{z}=0}$

In all three integrals (4.7), the stress tensor $\tilde{\boldsymbol{\sigma}}^{*,\bullet}$ depends on the reciprocal flow chosen to calculate $\tilde{F}_{pb,x}^{(1)}$, $\tilde{F}_{pb,z}^{(1)}$ and $\tilde{C}_{pb}^{(1)}$ individually. The projection of the stress tensor on $\mathbf{i}_z$ is expressed in cylindrical coordinates:

$$\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \mathbf{i}_z = \mu_f \left(\frac{\partial \tilde{u}_z^*}{\partial \tilde{\rho}} + \frac{\partial \tilde{u}_\rho^*}{\partial \tilde{z}} \right) \mathbf{i}_\rho + \mu_f \left(\frac{\partial \tilde{u}_\phi^*}{\partial \tilde{z}} + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{u}_z^*}{\partial \tilde{\phi}} \right) \mathbf{i}_\phi + \left(-\tilde{p}^* + 2\mu_f \frac{\partial \tilde{u}_z^*}{\partial \tilde{z}} \right) \mathbf{i}_z. \quad (B.1)$$

From the adhesion condition on the plane wall at $\tilde{z} = 0$, the three components of the velocity field of the reciprocal flow ($\tilde{u}_\rho^*$, $\tilde{u}_\phi^*$, $\tilde{u}_z^*$) are independent of $(\tilde{\rho}, \tilde{\phi})$, thus:

$$\left[\frac{\partial \tilde{u}_\rho^*}{\partial \tilde{\rho}} \right]_{\tilde{z}=0} = \left[\frac{\partial \tilde{u}_\phi^*}{\partial \tilde{\phi}} \right]_{\tilde{z}=0} = \left[\frac{\partial \tilde{u}_z^*}{\partial \tilde{\rho}} \right]_{\tilde{z}=0} = \left[\frac{\partial \tilde{u}_z^*}{\partial \tilde{\phi}} \right]_{\tilde{z}=0} = 0.$$

Then, from the continuity equation,

$$\left[\frac{\partial \tilde{u}_z^*}{\partial \tilde{z}} \right]_{\tilde{z}=0} = 0.$$

The equation (B.1) on $\tilde{z} = 0$ becomes

$$[\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \mathbf{i}_z]_{\tilde{z}=0} = \mu_f \frac{\partial \tilde{u}_\rho^*}{\partial \tilde{z}} \mathbf{i}_\rho + \mu_f \frac{\partial \tilde{u}_\phi^*}{\partial \tilde{z}} \mathbf{i}_\phi - \tilde{p}^* \mathbf{i}_z. \quad (B.2)$$

The derivative with respect to $\tilde{z}$ which appears in the relation (4.7) has thus the expression in the cylindrical coordinate system ($\mathbf{i}_\rho$, $\mathbf{i}_\phi$, $\mathbf{i}_z$):

$$\frac{\partial \tilde{\mathbf{v}}_{pb}^{(0)}}{\partial \tilde{z}} = \frac{\partial \tilde{v}_{\rho,pb}^{(0)}}{\partial \tilde{z}} \mathbf{i}_\rho + \frac{\partial \tilde{v}_{\phi,pb}^{(0)}}{\partial \tilde{z}} \mathbf{i}_\phi + \frac{\partial \tilde{v}_{z,pb}^{(0)}}{\partial \tilde{z}} \mathbf{i}_z. \quad (B.3)$$

Similarly, the adhesion condition for $\tilde{\mathbf{v}}_{pb}^{(0)}$ on $\tilde{z} = 0$ with the continuity equation gives

$$\left[\frac{\partial \tilde{v}_{z,pb}^{(0)}}{\partial \tilde{z}} \right]_{\tilde{z}=0} = 0. \quad (B.4)$$

Thus, from (B.2) and (B.4),

$$\left[\frac{\partial \tilde{\mathbf{v}}_{pb}^{(0)}}{\partial \tilde{z}} \cdot (\tilde{\boldsymbol{\sigma}}^{*,\bullet} \cdot \mathbf{i}_z) \right]_{\tilde{z}=0} = \mu_f \left[\frac{\partial \tilde{v}_{\rho,pb}^{(0)}}{\partial \tilde{z}} \frac{\partial \tilde{u}_{\rho}^*}{\partial \tilde{z}} + \frac{\partial \tilde{v}_{\phi,pb}^{(0)}}{\partial \tilde{z}} \frac{\partial \tilde{u}_{\phi}^*}{\partial \tilde{z}} \right]_{\tilde{z}=0}. \quad (B.5)$$

References

- [1] Pasol, L., Martin, M., Ekiel-Jezewska, M. L., Wajnryb, E., Bławdziewicz, J., and Feuillebois, F., Motion of Sphere Parallel to Plane Walls in a Poiseuille Flow. Application to Field-Flow Fractionation and Hydrodynamic Chromatography, *Chem. Eng. Sci.*, 2011, vol. 66, no. 18, pp. 4078–4089.
- [2] Ho, B. P. and Leal, L. G., Inertial Migration of Rigid Spheres in Two-Dimensional Unidirectional Flows, *J. Fluid Mech.*, 1974, vol. 65, no. 2, pp. 365–400.
- [3] Clague, D. S. and Cornelius, P. J., The Hydrodynamic Force and Torque on a Bounded Sphere in Poiseuille Flow, *Int. J. Numer. Methods Fluids*, 2001, vol. 35, no. 1, pp. 55–70.
- [4] Staben, M. E., Zinchenko, A. Z., and Davis, R. H., Motion of a Particle between Two Parallel Plane Walls in Low-Reynolds-Number Poiseuille Flow, *Phys. Fluids*, 2003, vol. 15, no. 6, pp. 1711–1733.
- [5] Staben, M. E. and Davis, R. H., Particle Transport in Poiseuille Flow in Narrow Channels, *Int. J. Multiph. Flow*, 2005, vol. 31, no. 5, pp. 529–547.
- [6] Bhattacharya, S., Bławdziewicz, J., and Wajnryb, E., Many-Particle Hydrodynamic Interactions in Parallel-Wall Geometry: Cartesian-Representation Method, *Phys. A*, 2005, vol. 356, nos. 2–4, pp. 294–340.
- [7] Bhattacharya, S., Bławdziewicz, J., and Wajnryb, E., Hydrodynamic Interactions of Spherical Particles in Poiseuille Flow between Two Parallel Walls, *Phys. Fluids*, 2006, vol. 18, no. 5, 053301, 20 p.
- [8] O'Neill, M. E., A Slow Motion of Viscous Liquid Caused by a Slowly Moving Solid Sphere, *Mathematika*, 1964, vol. 11, no. 1, pp. 67–74.
- [9] Tözeren, A. and Skalak, R., Stress in a Suspension near Rigid Boundaries, *J. Fluid Mech.*, 1977, vol. 82, no. 2, pp. 289–307.
- [10] Chaoui, M. and Feuillebois, F., Creeping Flow around a Sphere in a Shear Flow Close to a Wall, *Q. J. Mech. Appl. Math.*, 2003, vol. 56, no. 3, pp. 381–410.
- [11] Pasol, L., Sellier, A., and Feuillebois, F., A Sphere in a Second Degree Polynomial Creeping Flow Parallel to a Wall, *Q. J. Mech. Appl. Math.*, 2006, vol. 59, no. 4, pp. 587–614.
- [12] Smart, J. R. and Leighton, Jr., D. T., Measurement of the Hydrodynamic Surface Roughness of Non-colloidal Spheres, *Phys. Fluids A*, 1989, vol. 1, no. 1, pp. 52–60.
- [13] Smart, J. R., Beimfohr, S., and Leighton, Jr., D. T., Measurement of the Translational and Rotational Velocities of a Noncolloidal Sphere Rolling Down a Smooth Inclined Plane at Low Reynolds Number, *Phys. Fluids A*, 1993, vol. 5, no. 1, pp. 13–24.
- [14] Kunert, C., Harting, J., and Vinogradova, O. I., Random-Roughness Hydrodynamic Boundary Conditions, *Phys. Rev. Lett.*, 2010, vol. 105, no. 1, 016001, 4 p.
- [15] Assoudi, R., Lamzoud, K., and Chaoui, M., Influence of the Wall Roughness on a Linear Shear Flow, *FME Trans.*, 2019, vol. 46, no. 2, pp. 272–277.
- [16] Assoudi, R., Chaoui, M., Feuillebois, F., and Allouche, H., Motion of a Spherical Particle along a Rough Wall in a Shear Flow, *Z. Angew. Math. Phys.*, 2018, vol. 69, no. 5, pp. 1–30.

- [17] Lamzoud, K., Assoudi, R., Bouisfi, F., and Chaoui, M., A Spherical Particle Settling towards a Corrugated Wall, *Russian J. Nonlinear Dyn.*, 2019, vol. 15, no. 2, pp. 125–134.
- [18] Lorentz, H. A., A General Theorem Concerning the Motion of a Viscous Fluid and a Few Consequences Derived from It, *Zittingsverlag Akad. Wet. Amsterdam*, 1896, vol. 5, pp. 168–175.
- [19] *The Centenary of a Paper on Slow Viscous Flow by the Physicist H. A. Lorentz*, Dordrecht: Springer, 1996.
- [20] Lecoq, N., Boundary Conditions for Creeping Flow along Periodic or Random Rough Surfaces, Experimental and Theoretical Results, *J. Phys. Conf. Ser.*, 2012, vol. 392, no. 1, 012010, 19 p.
- [21] Pasol, L., Chaoui, M., Yahiaoui, S., and Feuillebois, F., Analytical Solution for a Spherical Particle near a Wall in Axisymmetrical Polynomial Creeping Flows, *Phys. Fluids*, 2005, vol. 17, no. 7, 073602, 13 p.