NONLINEAR PHYSICS AND MECHANICS

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Modeling of Nonlinear Waves in Two Coaxial Physically Nonlinear Shells with a Viscous Incompressible Fluid Between Them, Taking into Account the Inertia of its Motion

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This article investigates longitudinal deformation waves in physically nonlinear coaxial elastic shells containing a viscous incompressible fluid between them. The rigid nonlinearity of the shells is considered. The presence of a viscous incompressible fluid between the shells, as well as the influence of the inertia of the fluid motion on the amplitude and velocity of the wave, are taken into account.

A numerical study of the model constructed in the course of this work is carried out by using a difference scheme for the equation similar to the Crank–Nicolson scheme for the heat equation.

In the case of identical initial conditions in both shells, the deformation waves in them do not change either the amplitude or the velocity. In the case of setting different initial conditions in the coaxial shells, the amplitude of the solitary wave in the first shell decreases from the value specified at the initial instant of time, and in the second, the amplitude grows from zero until they equalize, that is, energy is transferred.

The movement occurs in a negative direction. This means that the velocity of deformation wave is subsonic.

Keywords: nonlinear waves, elastic cylindrical shells, viscous incompressible fluid, Crank–Nicolson difference scheme

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1. Introduction

The study of the wave process in elastic shells is widely used in the new structures in modern engineering industry. The propagation of deformation waves in nonlinear shells was considered in [10–13]. In [14–16] the interaction of the shell with a viscous incompressible fluid was considered without taking into account wave phenomena.

In the case of a related problem the equations of fluid dynamics and the ones of the elastic body dynamics are solved simultaneously, taking into account the boundary conditions. This approach is used for studying the oscillations of elastic bodies [17, 18], and also in this article when studying the deformation waves of nonlinear elastic shells containing a viscous fluid of constant density, taking into account the inertia of its motion. It is impossible to study the models of deformation waves using the methods of qualitative analysis in the case of filling the space between shells with a viscous incompressible fluid [2, 3, 19–23]. This leads to the necessity of applying numerical methods [4, 5].

This paper addresses the problem of obtaining mathematical models of the wave process in infinitely long physically nonlinear coaxial cylindrical elastic shells [16] with rigid nonlinearity by means of the perturbation method, the problem parameter being small. They differ from the known ones in that they take into account the presence of an incompressible viscous fluid between the shells. The research is carried out on the basis of the related hydroelasticity problems described by the shell dynamics and incompressible viscous fluid equations with the corresponding boundary conditions in the form of a system of generalized MKdV equations. The effects of the incompressible viscous fluid between the shells on the behavior of the deformation wave in the coaxial shells are revealed. The presence of a deformation wave in the outer shell leads to the appearance of a deformation wave in the inner shell, which was not present at the initial instant, and there is a “transfer of energy” (through the liquid layer) from the outer shell to the inner one, accompanied by a decrease in the wave amplitude in the outer shell. As a result, a decrease in the speed of its distribution takes place. In this case an increase in amplitude occurs in the inner shell. In the course of time the amplitudes become equal due to their changes. The fluid movement inertia changes the wave velocity.

A numerical study of the model constructed in the course of this work is carried out by using a difference scheme similar to the Crank–Nicolson scheme in the case of the heat equation [29–32].

2. Defining and resolving relations of the physically nonlinear theory of shells

A. A. Ilyushin’s deformation plasticity theory [6, 24] in the case of rigid nonlinearity [7, 25] connects the components of the stress tensor σ_x , σ_Θ with the components of the strain tensor ε_x , ε_Θ and the square of the strain intensity ε_u , in the form

$$\begin{aligned}\sigma_x^{(i)} &= \frac{E}{1 - \mu_0^2} \left(\varepsilon_x^{(i)} + \mu_0 \varepsilon_\Theta^{(i)} \right) \left(1 + \frac{m}{E} \varepsilon_u^{(i)2} \right), \\ \sigma_\Theta^{(i)} &= \frac{E}{1 - \mu_0^2} \left(\varepsilon_\Theta^{(i)} + \mu_0 \varepsilon_x^{(i)} \right) \left(1 + \frac{m}{E} \varepsilon_u^{(i)2} \right),\end{aligned}\tag{2.1}$$

$$\varepsilon_u^{(i)2} = \frac{4}{3} \left(\mu_1 \left(\varepsilon_x^{(i)2} + \varepsilon_\Theta^{(i)2} \right) - \mu_2 \varepsilon_x^{(i)} \varepsilon_\Theta^{(i)} \right),$$

$$\mu_1 = \frac{1}{3} \left[1 + \frac{\mu_0 - 1}{(1 - \mu_0)^2} \right], \quad \mu_2 = \frac{1}{3} \left[1 - \frac{2\mu_0 - 1}{(1 - \mu_0)^2} \right],$$

where E is Young's modulus; m is the constant of the material, determined from the experiments on tension or compression; μ_0 is Poisson's ratio of the shell material. Let us consider axisymmetric coaxial cylindrical shells.

We use the following notation: δ is the width of the slit occupied by the fluid; R_1 is the radius of the inner surface of the outer shell; R_2 is the radius of the outer surface of the inner shell ($R_1 = R_2 + \delta$); $R^{(i)}$ are the radii of the median surfaces; $h_0^{(i)}$ are the shell thicknesses; $\left(R_1 = R^{(1)} - \frac{h_0^{(1)}}{2} \right)$; $\left(R_2 = R^{(2)} + \frac{h_0^{(2)}}{2} \right)$; $U^{(i)}$ are the longitudinal elastic shells' displacements; $W^{(i)}$ are the deflections directed to the center of curvature ($i = 1$ for the outer shell, $i = 2$ for the inner one).

Let us write down the relation between the components of the strains and elastic displacements in the form of [8]

$$\varepsilon_x^{(i)} = \frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - z \frac{\partial^2 W^{(i)}}{\partial x^2}, \quad \varepsilon_\Theta^{(i)} = -\frac{W^{(i)}}{R^{(i)}}, \quad (2.2)$$

where x is the longitudinal coordinate along the median surface and z is the normal coordinate in the shell $\left(-\frac{h_0^{(i)}}{2} \leq z \leq \frac{h_0^{(i)}}{2} \right)$. We write down the square of the strain intensity in the form

$$\varepsilon_u^{(i)2} = \frac{4}{3} \left\langle \mu_1 \left\{ \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - z \frac{\partial^2 W^{(i)}}{\partial x^2} \right]^2 + \frac{W^{(i)2}}{R^{(i)2}} \right\} + \right.$$

$$\left. + \mu_2 \frac{W^{(i)}}{R^{(i)}} \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - z \frac{\partial^2 W^{(i)}}{\partial x^2} \right] \right\rangle. \quad (2.3)$$

Fig. 1. Elastic infinitely long coaxial cylindrical shells.

We determine the forces in the middle surface of the shell and the moment using the following formulas

$$N_x^{(i)} = \int_{-\frac{h_0^{(i)}}{2}}^{\frac{h_0^{(i)}}{2}} \sigma_x^{(i)} dz, \quad N_\Theta^{(i)} = \int_{-\frac{h_0^{(i)}}{2}}^{\frac{h_0^{(i)}}{2}} \sigma_\Theta^{(i)} dz, \quad M_x^{(i)} = \int_{-\frac{h_0^{(i)}}{2}}^{\frac{h_0^{(i)}}{2}} \sigma_x^{(i)} z dz. \quad (2.4)$$

The asymptotic analysis performed in [18] showed that the expressions in square brackets of formulas (2.3) can be considered on the median surface ($z = 0$) for longitudinal waves:

$$\varepsilon_u^{(i)2} = \frac{4}{3} \left\{ \mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \frac{W^{(i)2}}{R^{(i)2}} \right] + \mu_2 \frac{W^{(i)}}{R^{(i)}} \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right] \right\}. \quad (2.5)$$

By substituting (2.1), (2.2), (2.5) into (2.4), we find

$$\begin{aligned} N_x^{(i)} = & \frac{Eh_0^{(i)}}{1 - \mu_0^2} \left\langle \frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} + \frac{4m}{3E} \left\{ \left[\frac{\partial U^{(i)}}{\partial x} + \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} \right] \left[\mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \right. \right. \right. \\ & \left. \left. \left. + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right] \right\} \right\rangle, \end{aligned} \quad (2.6)$$

$$\begin{aligned} N_\Theta^{(i)} = & \frac{Eh_0^{(i)}}{1 - \mu_0^2} \left\langle \mu_0 \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right] - \frac{W^{(i)}}{R^{(i)}} + \right. \\ & \left. + \frac{4m}{3E} \left\{ \left[\mu_0 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) - \frac{W^{(i)}}{R^{(i)}} \right] \left[\mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right] \right\} \right\rangle, \\ M_x^{(i)} = & -\frac{Eh_0^{(i)3}}{12(1 - \mu_0^2)} \frac{\partial^2 W^{(i)}}{\partial x^2} \left\langle 1 + \frac{4m}{3E} \left\{ \mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right\} \right\rangle. \end{aligned} \quad (2.7)$$

The asymptotic analysis [18] showed that in (2.7) the first term is much larger than the remaining terms and they can be discarded because M_x itself is much less than the force N_x . Therefore, we obtain from (2.7)

$$M_x^{(i)} = -\frac{Eh_0^{(i)3}}{12(1 - \mu_0^2)} \frac{\partial^2 W^{(i)}}{\partial x^2}. \quad (2.8)$$

The dynamics equations of the shells (Fig. 1) are written down as

$$\begin{aligned} \frac{\partial N_x^{(i)}}{\partial x} &= \rho_0 h_0^{(i)} \frac{\partial^2 U^{(i)}}{\partial t^2} - \left[q_x^{(i)} - W^{(i)} \frac{\partial q_x^{(i)}}{\partial r} + U^{(i)} \frac{\partial q_x^{(i)}}{\partial x} \right]_R, \\ \frac{\partial^2 M_x^{(i)}}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial W^{(i)}}{\partial x} N_x^{(i)} \right) + \frac{1}{R^{(i)}} N_\Theta^{(i)} &= \\ &= \rho_0 h_0^{(i)} \frac{\partial^2 W^{(i)}}{\partial t^2} - (-1)^{i-1} \left[q_n - W^{(i)} \frac{\partial q_n}{\partial r} + U^{(i)} \frac{\partial q_n}{\partial x} \right]_R, \end{aligned} \quad (2.9)$$

where t is the time; $\rho_0^{(i)}$ is the density of the shell material; $q_x^{(i)}$, q_n are the stresses from the side of the liquid inside the annular section; r , x are the cylindrical coordinates; $i = 1$ for the outer shell, $i = 2$ for the inner one. By substituting (2.6) into (2.9), we obtain the permutation equations

$$\begin{aligned} & \frac{E h_0^{(i)}}{1 - \mu_0^2} \frac{\partial}{\partial x} \left\langle \frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} + \frac{4}{3} \frac{m}{E} \left\{ \left[\frac{\partial U^{(i)}}{\partial x} + \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} \right] \left[\mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \right. \right. \right. \right. \\ & \left. \left. \left. + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right] \right\} \right\rangle = \\ & = \rho_0 h_0^{(i)} \frac{\partial^2 U^{(i)}}{\partial t^2} - \left[q_x^{(i)} - W^{(i)} \frac{\partial q_x^{(i)}}{\partial r} + U^{(i)} \frac{\partial q_x^{(i)}}{\partial x} \right]_R - \frac{E h_0^{(i)3}}{12 (1 - \mu_0^2)} \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 W^{(i)}}{\partial x^2} \right) + \\ & + \frac{E h_0^{(i)}}{1 - \mu_0^2} \frac{\partial}{\partial x} \left\langle \frac{\partial W^{(i)}}{\partial x} \left\{ \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} \right] + \right. \right. \\ & \left. \left. + \frac{4}{3} \frac{m}{E} \left\{ \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 - \mu_0 \frac{W^{(i)}}{R^{(i)}} \right] \left[\mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \right. \right. \right. \right. \\ & \left. \left. \left. + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right] \right\} \right\} \right\rangle + \frac{E h_0^{(i)}}{1 - \mu_0^2} \frac{1}{R^{(i)}} \left\langle \mu_0 \left[\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right] - \right. \\ & \left. - \frac{W^{(i)}}{R^{(i)}} + \frac{4}{3} \frac{m}{E} \left\{ \left[\mu_0 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) - \frac{W^{(i)}}{R^{(i)}} \right] \times \right. \right. \\ & \left. \left. \times \left[\mu_1 \left[\left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^2 + \left(\frac{W^{(i)}}{R^{(i)}} \right)^2 \right] + \mu_2 \left(\frac{\partial U^{(i)}}{\partial x} + \frac{1}{2} \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right) \frac{W^{(i)}}{R^{(i)}} \right] \right\} \right\rangle = \\ & = \rho_0 h_0^{(i)} \frac{\partial^2 W^{(i)}}{\partial t^2} - (-1)^{i-1} \left[q_n - W^{(i)} \frac{\partial q_n}{\partial r} + U^{(i)} \frac{\partial q_n}{\partial x} \right]_R. \end{aligned} \quad (2.10)$$

3. The asymptotic method for studying the equations of shells with a liquid

We used the two-scale perturbation method for solving system (2.10). The estimates made in dimensionless variables characterize the problems under consideration. For wave problems the shell is considered infinite. For longitudinal waves, dimensionless variables and dimensionless parameters are introduced in the shell. We take the wavelength as the characteristic length l , and u_m , w_m as the characteristic values of elastic displacements

$$\begin{aligned} W^{(i)} &= w_m u_3^{(i)}, \quad U^{(i)} = u_m u_1^{(i)}, \quad x^* = \frac{x}{l}, \quad t^* = \frac{c_0}{l} t, \quad r^* = \frac{r}{R}^{(i)}, \\ w_m &= h_0, \quad u_m = \frac{h_0 l}{R^{(i)}}. \end{aligned} \quad (3.1)$$

$c_0 = \sqrt{\frac{E}{\rho(1-\mu_0)}}$ is the propagation velocity of longitudinal elastic waves in the shell. Set

$$\begin{aligned} \frac{h_0^{(i)}}{R^{(i)}} &= \varepsilon \ll 1, \quad \frac{R^{(i)2}}{l^2} = O(\varepsilon), \quad \frac{w_m}{h_0^{(i)}} = O(1), \\ \frac{u_m}{l} \frac{R^{(i)}}{h_0^{(i)}} &= O(1), \quad \frac{m\varepsilon}{E} = O(1), \quad \frac{h_0^{(i)2}}{l^2} = \frac{h_0^{(i)2}}{R^{(i)2}} \cdot \frac{R^{(i)2}}{l^2} = \varepsilon^3, \end{aligned} \quad (3.2)$$

where ε is the small parameter of the problem. We introduce independent variables in the form

$$\xi = x^* - \sqrt{1 - \mu_0^2} t^*, \quad \tau = \varepsilon t^*, \quad (3.3)$$

where τ is the fast time. In these variables (3.3) we obtain the equations as in [9, 26, 33, 34] by keeping terms of order ε and ε^2 in Eqs. (2.10) and discarding terms with higher degrees

$$\begin{aligned} &\frac{\partial}{\partial \xi} \left\langle \frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} - \mu_0 \frac{w_m}{R^{(i)}} u_3^{(i)} + \frac{4m}{3E} \left(\frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} - \mu_0 \frac{w_m}{R^{(i)}} u_3^{(i)} \right) \times \right. \\ &\times \left[\mu_1 \left[\left(\frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} \right)^2 + \left(\frac{w_m}{R^{(i)}} \right)^2 u_3^{(i)2} \right] + \mu_2 \frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} \frac{w_m}{R^{(i)}} u_3^{(i)} \right] \Bigg\rangle = \\ &= \frac{u_m}{l} \left[(1 - \mu_0^2) \frac{\partial^2 u_1^{(i)}}{\partial \xi^2} - 2\varepsilon \sqrt{1 - \mu_0^2} \frac{\partial^2 u_1^{(i)}}{\partial \xi \partial \tau} \right] - \frac{l}{\rho_0 h_0^{(i)} c_0^2} q_x^{(i)}, \\ &\mu_0 \frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} - \frac{w_m}{R^{(i)}} u_3^{(i)} + \frac{4m}{3E} \left(\mu_0 \frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} - \frac{w_m}{R^{(i)}} u_3^{(i)} \right) \times \\ &\times \left[\mu_1 \left[\left(\frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} \right)^2 + \left(\frac{w_m}{R^{(i)}} \right)^2 u_3^{(i)2} \right] + \mu_2 \frac{u_m}{l} \frac{\partial u_1^{(i)}}{\partial \xi} \frac{w_m}{R^{(i)}} u_3^{(i)} \right] = \\ &= \frac{R^{(i)2}}{l^2} \frac{w_m}{R^{(i)}} \left[(1 - \mu_0^2) \frac{\partial^2 u_3^{(i)}}{\partial \xi^2} - 2\varepsilon \sqrt{1 - \mu_0^2} \frac{\partial^2 u_3^{(i)}}{\partial \xi \partial \tau} \right] - \frac{R^{(i)}}{\rho_0 h_0^{(i)} c_0^2} (-1)^{i-1} q_n. \end{aligned} \quad (3.4)$$

We represent the dependent variables as an asymptotic expansion

$$u_1^{(i)} = u_{10}^{(i)} + \varepsilon u_{11}^{(i)} + \dots, \quad u_3^{(i)} = u_{30}^{(i)} + \varepsilon u_{31}^{(i)} + \dots \quad (3.5)$$

We obtain the system of equations by substituting (3.5) into (3.4) and keeping terms of order ε

$$\begin{aligned} \frac{\partial}{\partial \xi} \left\langle \frac{\partial u_{10}^{(i)}}{\partial \xi} - \mu_0 \frac{w_m l}{u_m R^{(i)}} u_{30}^{(i)} \right\rangle &= (1 - \mu_0^2) \frac{\partial^2 u_{10}^{(i)}}{\partial \xi^2}, \\ \mu_0 \frac{\partial u_{10}^{(i)}}{\partial \xi} - \frac{w_m l}{u_m R^{(i)}} u_{30}^{(i)} &= 0. \end{aligned} \quad (3.6)$$

We get from this system

$$\frac{w_m l}{u_m R^{(i)}} u_{30}^{(i)} = \mu_0 \frac{\partial u_{10}^{(i)}}{\partial \xi}. \quad (3.7)$$

Thus, $u_{10}^{(i)}$ is an arbitrary function, since the shell has infinite length, the wave velocity is equal to $\sqrt{\frac{E}{\rho_0}}$, the wave velocity in the rod. We obtain the system of equations in the approximation ε^2

$$\begin{aligned} \frac{\partial^2 u_{10}^{(i)}}{\partial \xi \partial \tau} + \frac{m}{E\varepsilon} \left(\frac{u_m}{l} \right)^2 2\sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{\partial u_{10}^{(i)}}{\partial \xi} \right)^2 \frac{\partial^2 u_{10}^{(i)}}{\partial \xi^2} + \\ + \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{\mu_0^2 \sqrt{1 - \mu_0^2}}{2} \frac{\partial^4 u_{10}^{(i)}}{\partial \xi^4} = - \frac{1}{2\sqrt{1 - \mu_0^2}} \frac{l^2}{\varepsilon u_m \rho_0 h_0^2 c_0^2} \left[q_x^{(i)} - \mu_0 \frac{R}{l} (-1)^{i-1} \frac{\partial q_n}{\partial \xi} \right]. \end{aligned} \quad (3.8)$$

The resulting equations are generalized Modified Korteweg–de Vries (MKdV) equations for $\frac{\partial u_{10}^{(i)}}{\partial \xi}$.

In the absence of fluid the right-hand side of the equation is zero and the system of Eq. (3.8) disintegrates into two independent modified Korteweg–de Vries (MKdV) equations. It is necessary to determine the right-hand side by solving the equations of hydrodynamics.

4. The study of stresses acting on the shell from the side of the fluid inside

The equation of motion of an incompressible viscous fluid and the continuity equation in a cylindrical coordinate system (r, Θ, x) in the case of an axisymmetric flow [3] are written as

$$\begin{aligned} \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_x \frac{\partial V_r}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= \nu \left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial x^2} - \frac{V_r}{r^2} \right), \\ \frac{\partial V_x}{\partial t} + V_r \frac{\partial V_x}{\partial r} + V_x \frac{\partial V_x}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= \nu \left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r} \frac{\partial V_x}{\partial r} + \frac{\partial^2 V_x}{\partial x^2} \right), \\ \frac{\partial V_r}{\partial t} + \frac{V_r}{r} + \frac{\partial V_x}{\partial x} &= 0. \end{aligned} \quad (4.1)$$

The conditions for the fluid to stick to the boundary of the shells and the fluid are satisfied at $r = R_i - W^{(i)}$ (Fig. 1)

$$V_x = \frac{\partial U^{(i)}}{\partial t}, \quad V_r = -\frac{\partial W^{(i)}}{\partial t}, \quad (4.2)$$

where t is the time; r, x are the cylindrical coordinates; V_r, V_x are the projections on the axis of the cylindrical coordinate system of the velocity vector; p is pressure in the fluid; ρ is the fluid density; ν is the kinematic viscosity coefficient. The stresses from the side of the fluid layer are determined by the formulas

$$\begin{aligned} q_n &= \left[P_{rr} \cos \left(\widehat{-\bar{n}^{(i)}, \bar{n}_r} \right) + P_{rx} \cos \left(\widehat{-\bar{n}^{(i)}, \bar{i}} \right) \right] \Big|_{r=R_i-W^{(i)}}, \\ q_x &= - \left[P_{rx} \cos \left(\widehat{-\bar{n}^{(i)}, \bar{n}_r} \right) + P_{xx} \cos \left(\widehat{-\bar{n}^{(i)}, \bar{i}} \right) \right] \Big|_{r=R_i-W^{(i)}}, \\ P_{rr} &= -p + 2\rho\nu \frac{\partial V_r}{\partial r}, \quad P_{rx} = \rho\nu \left(\frac{\partial V_x}{\partial r} + \frac{\partial V_r}{\partial x} \right), \quad P_{xx} = -p + 2\rho\nu \frac{\partial V_x}{\partial x}. \end{aligned} \quad (4.3)$$

According to Euler's approach, we have

$$\begin{aligned} \cos \left(\widehat{-\bar{n}^{(i)}, \bar{n}_r} \right) &= \frac{R_i - W^{(i)}}{|\bar{N}|}, \quad \cos \left(\widehat{-\bar{n}^{(i)}, \bar{i}} \right) = -\frac{R_i - W^{(i)}}{|\bar{N}|} \frac{\partial W^{(i)}}{\partial x}, \\ \cos \left(\widehat{-\bar{n}^{(i)}, \bar{n}_r} \right) &= \frac{1}{\left(1 + \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^{\frac{1}{2}}}, \quad \cos \left(\widehat{-\bar{n}^{(i)}, \bar{i}} \right) = \frac{\frac{\partial W^{(i)}}{\partial x}}{\left(1 + \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^{\frac{1}{2}}}, \\ |\bar{N}| &= (R_i - W^{(i)}) \left(1 + \left(\frac{\partial W^{(i)}}{\partial x} \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.4)$$

Here $\bar{n}$ is the normal to the middle surface of the i th shell, $\bar{n}_r, \bar{n}_\Theta, \bar{i}$ are the unit vectors of the basis (r, Θ, x) of the cylindrical coordinate system, the center of which is located on the geometric axis. If we carry down the stress on the unperturbed surface of the shell, we can assume that $-\bar{n} = \bar{n}_r$ and $\cos \left(\widehat{-\bar{n}^{(i)}, \bar{n}_r} \right) = 1, \cos \left(\widehat{-\bar{n}^{(i)}, \bar{i}} \right) = 0$.

5. Ring cross section

We introduce dimensionless variables and parameters

$$\begin{aligned} V_r &= w_m \frac{c_0}{l} v_r, \quad V_x = w_m \frac{c_0}{\delta} v_x, \quad r = R_2 + \delta r^*, \quad t^* = \frac{c_0}{l} t, \quad x^* = \frac{x}{l}, \\ p &= \frac{\rho \nu c_0 l R_i w_m}{\delta^3} P + p_0, \quad \psi = \frac{\delta}{R_2} = o(1), \\ \lambda &= \frac{w_m}{\delta} = \frac{w_m R_2}{R_2 \delta} = o\left(\frac{\varepsilon}{\psi}\right), \quad \frac{w_m}{R_2} = \frac{w_m}{\delta} \frac{\delta}{R_2} = \lambda \psi, \\ \frac{w_m}{l} &= \frac{w_m}{\delta} \frac{\delta}{R_i} \frac{R_i}{l} = \lambda \psi \varepsilon^{\frac{1}{2}}, \quad \frac{\delta}{l} = \frac{\delta}{R_i} \frac{R_i}{l} = \psi \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (5.1)$$

We obtain hydrodynamics equations in the introduced dimensionless variables

$$\begin{aligned} & \left(\frac{\delta}{l}\right)^2 \left\{ \frac{\delta c_0}{\nu} \frac{\delta}{l} \left[\frac{\partial v_r}{\partial t^*} + \lambda \left(v_r \frac{\partial v_r}{\partial r^*} + v_x \frac{\partial v_r}{\partial x^*} \right) \right] \right\} + \frac{\partial P}{\partial r^*} = \\ & = \left(\frac{\delta}{l}\right)^2 \left[\frac{\partial^2 v_r}{\partial r^{*2}} + \frac{\delta}{R_2(1+\psi r^*)} \frac{\partial v_r}{\partial r^*} - \frac{\delta^2 v_r}{R_2^2(1+\psi r^*)^2} + \frac{\delta^2}{l^2} \frac{\partial^2 v_r}{\partial x^{*2}} \right], \\ & \frac{\delta c_0}{\nu} \frac{\delta}{l} \left[\frac{\partial v_r}{\partial t^*} + \lambda \left(v_r \frac{\partial v_r}{\partial r^*} + v_x \frac{\partial v_r}{\partial x^*} \right) \right] + \frac{\partial P}{\partial r^*} = \frac{\partial^2 v_r}{\partial r^{*2}} + \frac{\delta}{R_2(1+\psi r^*)} \frac{\partial v_x}{\partial r^*} + \frac{\delta^2}{l^2} \frac{\partial^2 v_x}{\partial x^{*2}}, \\ & \frac{\partial v_r}{\partial r^*} + \frac{\delta}{R_2(1+\psi r^*)} + \frac{\partial v_x}{\partial x^*} = 0 \end{aligned} \quad (5.2)$$

and boundary conditions

$$v_x = \frac{\delta}{l} \frac{u_m}{w_m} \frac{\partial u_1^{(i)}}{\partial t^*}, \quad v_r = -\frac{\partial u_3^{(i)}}{\partial t^*} \quad (5.3)$$

at $r^* = 1 - \lambda u_3^{(1)}$ and $r^* = -\lambda u_3^{(2)}$.

Let us consider the asymptotic expansions of the velocity components and pressure in small parameters of the problem, $\frac{\delta}{l} \ll 1$ and $\frac{\delta}{R_2} \ll 1$. For the first terms of these expansions, that is, $\frac{\delta}{l} = 0$ and $\frac{\delta}{R_2} = 0$ (the hydrodynamic lubrication theory with the corresponding Reynolds number $\frac{\delta}{l} \frac{\delta c_0}{\nu} < 1$), we obtain hydrodynamics equations in the form

$$\begin{aligned} \frac{\partial P}{\partial r^*} &= 0, \quad \widetilde{\text{Re}} \left[\frac{\partial v_x}{\partial t^*} + \lambda \left(v_x \frac{\partial v_x}{\partial x^*} + v_r \frac{\partial v_x}{\partial r^*} \right) \right] + \frac{\partial P}{\partial x^*} = \frac{\partial^2 v_x}{\partial r^{*2}}, \\ \frac{\partial v_r}{\partial r^*} + \frac{\partial v_x}{\partial x^*} &= 0, \quad \widetilde{\text{Re}} = \frac{\delta}{l} \frac{\delta c_0}{\nu} \end{aligned} \quad (5.4)$$

and boundary conditions: $v_r = -\frac{\partial u_3^{(i)}}{\partial t^*}$, $v_x = 0$ for $r^* = 1 - \lambda u_3^{(1)}$ and $r^* = -\lambda u_3^{(2)}$. By expanding pressure and velocity components in powers of a small parameter λ

$$P = P^0 + \lambda P^1 + \dots, \quad v_x = v_x^0 + \lambda v_x^1 + \dots, \quad v_r = v_r^0 + \lambda v_r^1 + \dots, \quad (5.5)$$

we obtain equations for the first terms of the expansion

$$\frac{\partial P^0}{\partial r^*} = 0, \quad \widetilde{\text{Re}} \frac{\partial v_x^0}{\partial t^*} + \frac{\partial P^0}{\partial x^*} = \frac{\partial^2 v_x^0}{\partial r^{*2}}, \quad \frac{\partial v_r^0}{\partial r^*} + \frac{\partial v_x^0}{\partial x^*} = 0 \quad (5.6)$$

and boundary conditions

$$\begin{aligned} v_r^0 &= -\frac{\partial u_3^{(1)}}{\partial t^*}, \quad v_x^0 = 0, \quad \text{where } r^* = 1, \\ v_r^0 &= -\frac{\partial u_3^{(2)}}{\partial t^*}, \quad v_x^0 = 0, \quad \text{where } r^* = 0. \end{aligned} \quad (5.7)$$

Up to ψ , λ we obtain

$$\begin{aligned} q_x^{(i)} &\approx -\rho \nu \frac{w_m c_0}{\delta^2} \frac{\partial v_{x^*}^0}{\partial r^*} \bigg|_{r_2^* = -\lambda u_3^{(2)}}, \\ &\quad r_1^* = 1 - \lambda u_3^{(1)} \\ q_n &\approx -\frac{\rho \nu c_0 l w_m}{\delta^3} P^0. \end{aligned} \quad (5.8)$$

We use the iteration method as in [18]. At the first step of the iteration we set $\widetilde{\text{Re}} = 0$ (creeping fluid flows, hydrodynamic theory of lubrication) [23]. Taking the boundary conditions into account, we obtain from the equations of fluid motion

$$P^0 = 12 \int \left[\int \left(\frac{\partial u_3^{(2)}}{\partial t^*} - \frac{\partial u_3^{(1)}}{\partial t^*} \right) dx^* \right] dx^*, \quad \frac{\partial v_x^0}{\partial t^*} = (r^{*2} - r^*) 6 \int \left(\frac{\partial^2 u_3^{(2)}}{\partial t^{*2}} - \frac{\partial^2 u_3^{(1)}}{\partial t^{*2}} \right) dx^*. \quad (5.9)$$

Substituting the found values $\frac{\partial v_x^0}{\partial t^*}$ into the equations of fluid dynamics (5.6), at the second step of the iteration we find

$$\begin{aligned} P^0 &= \iint \left[12 \left(\frac{\partial u_3^{(2)}}{\partial t^*} - \frac{\partial u_3^{(1)}}{\partial t^*} \right) + \frac{6}{5} \widetilde{\text{Re}} \left(\frac{\partial^2 u_3^{(2)}}{\partial t^{*2}} - \frac{\partial^2 u_3^{(1)}}{\partial t^{*2}} \right) \right] dx^* dx^*, \\ \frac{\partial v_x^0}{\partial r^*} &= (2r^* - 1) \int \left[6 \left(\frac{\partial^2 u_3^{(2)}}{\partial t^{*2}} - \frac{\partial^2 u_3^{(1)}}{\partial t^{*2}} \right) + \frac{1}{10} \widetilde{\text{Re}} \left(\frac{\partial^2 u_3^{(2)}}{\partial t^{*2}} - \frac{\partial^2 u_3^{(1)}}{\partial t^{*2}} \right) \right] dx^*. \end{aligned} \quad (5.10)$$

Taking into account the introduced variables $\xi = x^* - t^* \sqrt{1 - \mu_0^2}$ and $\tau = \varepsilon t^*$ at $\varepsilon \ll 1$, we find

$$\begin{aligned} P^0 &= \int \left[12 \sqrt{1 - \mu_0^2} (u_{30}^{(1)} - u_{30}^{(2)}) - \frac{6}{5} \widetilde{\text{Re}} (1 - \mu_0^2) \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right) \right] dx^*, \\ \frac{\partial P^0}{\partial \xi} &= \left[12 \sqrt{1 - \mu_0^2} (u_{30}^{(1)} - u_{30}^{(2)}) - \frac{6}{5} \widetilde{\text{Re}} (1 - \mu_0^2) \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right) \right], \\ \frac{\partial v_x^0}{\partial r^*} &= (2r^* - 1) \left[12 \sqrt{1 - \mu_0^2} (u_{30}^{(1)} - u_{30}^{(2)}) - \frac{\widetilde{\text{Re}}}{10} (1 - \mu_0^2) \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right) \right], \\ \frac{\partial v_x^0}{\partial r^*} \Big|_{r^*=1} &= 6 \sqrt{1 - \mu_0^2} (u_{30}^{(1)} - u_{30}^{(2)}) - \frac{\widetilde{\text{Re}}}{10} (1 - \mu_0^2) \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right), \\ \frac{\partial v_x^0}{\partial r^*} \Big|_{r^*=0} &= - \frac{\partial v_x^0}{\partial r^*} \Big|_{r^*=1}. \end{aligned} \quad (5.11)$$

The equations of system (3.8) include the expressions

$$q_x^{(i)} - \mu_0 \frac{R^{(i)}}{l} \frac{\partial q_n}{\partial \xi} (-1)^{i-1}. \quad (5.12)$$

Therefore, we obtain from (5.12) the following expressions:

$$\begin{aligned} q_x^{(1)} - \mu_0 \frac{R^{(1)}}{l} \frac{\partial q_n}{\partial \xi} &= \mu_0 \frac{R^{(1)}}{l} \frac{\rho \nu c_0 l w_m}{\delta^3} 12 \sqrt{1 - \mu_0^2} \left[(u_{30}^{(1)} - u_{30}^{(2)}) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R^{(1)}} \right) - \right. \\ &\quad \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R^{(1)}} \right) \right], \\ q_x^{(2)} + \mu_0 \frac{R^{(2)}}{l} \frac{\partial q_n}{\partial \xi} &= \mu_0 \frac{R^{(2)}}{l} \frac{\rho \nu c_0 l w_m}{\delta^3} 12 \sqrt{1 - \mu_0^2} \left[(u_{30}^{(2)} - u_{30}^{(1)}) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R^{(1)}} \right) - \right. \\ &\quad \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial u_{30}^{(2)}}{\partial \xi} - \frac{\partial u_{30}^{(1)}}{\partial \xi} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R^{(1)}} \right) \right]. \end{aligned} \quad (5.13)$$

Let us find the expression on the right-hand sides of the system of equations (3.8). Considering that $w_m l u_{30}^{(i)} = \mu_0 u_m R^{(i)} u_{30\xi}^{(i)}$ and that $R^{(1)} = R^{(2)} = R$, $h^{(1)} = h^{(2)} = h_0$ due to the smallness of ψ , λ for the first equation of the system (3.8) ($i = 1$), we obtain

$$\begin{aligned} & -6\mu_0^2 \frac{\rho l}{\rho_0 h_0} \frac{\nu}{R c_0 \varepsilon} \left(\frac{R}{\delta}\right)^3 \left[\left(\frac{\partial^2 u_{10}^{(2)}}{\partial \xi} - \frac{\partial^2 u_{10}^{(1)}}{\partial \xi} \right) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R} \right) - \right. \\ & \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial^2 u_{30}^{(2)}}{\partial \xi^2} - \frac{\partial^2 u_{30}^{(1)}}{\partial \xi^2} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R} \right) \right]. \end{aligned} \quad (5.14)$$

For the second equation of this system ($i = 2$) we obtain

$$\begin{aligned} & -6\mu_0^2 \frac{\rho l}{\rho_0 h_0} \frac{\nu}{R c_0 \varepsilon} \left(\frac{R}{\delta}\right)^3 \left[\left(\frac{\partial^2 u_{10}^{(1)}}{\partial \xi} - \frac{\partial^2 u_{10}^{(2)}}{\partial \xi} \right) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R} \right) - \right. \\ & \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial^2 u_{30}^{(1)}}{\partial \xi^2} - \frac{\partial^2 u_{30}^{(2)}}{\partial \xi^2} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R} \right) \right]. \end{aligned} \quad (5.15)$$

6. The equations of coaxial shells dynamics

Taking into account the obtained right-hand sides (5.14), (5.15) of the system of equations (3.8), this system of equations takes the form

$$\begin{aligned} & \frac{\partial^2 u_{10}^{(1)}}{\partial \xi \partial \tau} + \frac{m}{E \varepsilon} \left(\frac{u_m}{l}\right)^2 2\sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{\partial u_{10}^{(1)}}{\partial \xi}\right)^2 \frac{\partial^2 u_{10}^{(1)}}{\partial \xi^2} + \\ & + \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{\mu_0^2 \sqrt{1 - \mu_0^2}}{2} \frac{\partial^4 u_{10}^{(1)}}{\partial \xi^4} = - \frac{1}{2\sqrt{1 - \mu_0^2}} \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} \mu_0 \frac{R}{l} \frac{\rho \nu c_0 l w_m}{\delta^3} \times \\ & \times 12\sqrt{1 - \mu_0^2} \left[\left(u_{30}^{(1)} - u_{30}^{(2)} \right) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R} \right) - \right. \\ & \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial u_{30}^{(1)}}{\partial \xi} - \frac{\partial u_{30}^{(2)}}{\partial \xi} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R} \right) \right], \\ & \frac{\partial^2 u_{10}^{(2)}}{\partial \xi \partial \tau} + \frac{m}{E \varepsilon} \left(\frac{u_m}{l}\right)^2 2\sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{\partial u_{10}^{(2)}}{\partial \xi}\right)^2 \frac{\partial^2 u_{10}^{(2)}}{\partial \xi^2} + \\ & + \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{\mu_0^2 \sqrt{1 - \mu_0^2}}{2} \frac{\partial^4 u_{10}^{(2)}}{\partial \xi^4} = - \frac{1}{2\sqrt{1 - \mu_0^2}} \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} \mu_0 \frac{R}{l} \frac{\rho \nu c_0 l w_m}{\delta^3} \times \\ & \times 12\sqrt{1 - \mu_0^2} \left[\left(u_{30}^{(2)} - u_{30}^{(1)} \right) \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R} \right) - \right. \\ & \left. - \frac{1}{10} \widetilde{\text{Re}} \sqrt{1 - \mu_0^2} \left(\frac{\partial u_{30}^{(2)}}{\partial \xi} - \frac{\partial u_{30}^{(1)}}{\partial \xi} \right) \left(1 - \frac{1}{12} \frac{\delta}{\mu_0 R} \right) \right]. \end{aligned} \quad (6.1)$$

It is possible to introduce the notation $u_{10\xi}^{(1)} = c_3\phi^{(1)}$, $u_{10\xi}^{(2)} = c_3\phi^{(2)}$, $\eta = c_1\xi$, $t = c_2\tau$, where

$$c_2 = 6\mu_0^2 \frac{\rho l}{\rho_0 h_0 \varepsilon} \left(\frac{R}{\delta} \right)^2 \left(1 - \frac{1}{2} \frac{\delta}{\mu_0 R} \right) \frac{\nu}{\delta c_0}, \quad c_1 = \left[c_2 \varepsilon \left(\frac{l}{R} \right)^2 \frac{2}{\mu^2 \sqrt{1 - \mu_0^2}} \right]^{\frac{1}{3}},$$

$$c_3 = \left[6 \frac{c_2}{c_1} \frac{E \varepsilon}{m} \left(\frac{l}{u_m} \right)^2 \frac{1}{2 \sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2)} \right]^{\frac{1}{2}}. \quad (6.2)$$

Taking

$$\sigma_1 = 6\mu_0^2 \frac{\rho l}{\rho_0 h_0} \left(\frac{R}{\delta} \right)^2 \frac{\delta}{l} \frac{1}{\varepsilon} \frac{\sqrt{1 - \mu_0^2}}{10} \left(1 - \frac{\delta}{12\mu_0 R} \right), \quad (6.3)$$

we obtain the system of equations

$$\begin{aligned} \phi_t^{(1)} + 6\phi^{(1)}\phi_\eta^{(1)} + \phi_{\eta\eta\eta}^{(1)} + \phi^{(1)} - \phi^{(2)} - \sigma_1 \left(\phi_\eta^{(1)} - \phi_\eta^{(2)} \right) &= 0, \\ \phi_t^{(2)} + 6\phi^{(2)}\phi_\eta^{(2)} + \phi_{\eta\eta\eta}^{(2)} + \phi^{(2)} - \phi^{(1)} - \sigma_1 \left(\phi_\eta^{(2)} - \phi_\eta^{(1)} \right) &= 0. \end{aligned} \quad (6.4)$$

The system of equations has the following exact solution:

$$\phi^{(1)} = \phi^{(2)} = \pm k \operatorname{ch}^{-1} (k\eta + 4k^3 t). \quad (6.5)$$

Due to the presence of fluid between the shells, a numerical solution of the system of equations under the initial condition is required:

$$\phi^{(1)} = k \operatorname{ch}^{-1} (k\eta), \quad \phi^{(2)} = 0. \quad (6.6)$$

7. Computational experiment

We consider a difference scheme for the equations which is similar to the Crank–Nicolson scheme for the heat equation [27–32] for numerical modeling

$$\begin{aligned} & \frac{u_j^{(1)n+1} - u_j^{(1)n}}{\tau} + 2 \frac{\left(u_{j+1}^{(1)3n+1} - u_{j-1}^{(1)3n+1} \right) + \left(u_{j+1}^{(1)3n} - u_{j-1}^{(1)3n} \right)}{4h} + \\ & + \frac{\left(u_{j+2}^{(1)n+1} - 2u_{j+1}^{(1)n+1} + 2u_{j-1}^{(1)n+1} - u_{j-2}^{(1)n+1} \right)}{4h^3} + \frac{\left(u_{j+2}^{(1)n} - 2u_{j+1}^{(1)n} + 2u_{j-1}^{(1)n} - u_{j-2}^{(1)n} \right)}{4h^3} + \\ & + \frac{u_j^{(1)n+1} - u_j^{(1)n}}{2} - \frac{u_j^{(2)n+1} - u_j^{(2)n}}{2} - \sigma_1 \left(\frac{\left(u_{j+1}^{(1)n+1} - u_{j-1}^{(1)n+1} \right) + \left(u_{j+1}^{(1)n} - u_{j-1}^{(1)n} \right)}{4h} - \right. \\ & \left. \frac{\left(u_{j+1}^{(2)n+1} - u_{j-1}^{(2)n+1} \right) + \left(u_{j+1}^{(2)n} - u_{j-1}^{(2)n} \right)}{4h} \right) = 0, \end{aligned} \quad (7.1)$$

$$\begin{aligned}
 & \frac{u_j^{(2)n+1} - u_j^{(2)n}}{\tau} + 2 \frac{\left(u_{j+1}^{(2)3n+1} - u_{j-1}^{(2)3n+1} \right) + \left(u_{j+1}^{(2)3n} - u_{j-1}^{(2)3n} \right)}{4h} + \\
 & + \frac{\left(u_{j+2}^{(2)n+1} - 2u_{j+1}^{(2)n+1} + 2u_{j-1}^{(2)n+1} - u_{j-2}^{(2)n+1} \right)}{4h^3} + \frac{\left(u_{j+2}^{(2)n} - 2u_{j+1}^{(2)n} + 2u_{j-1}^{(2)n} - u_{j-2}^{(2)n} \right)}{4h^3} + \\
 & + \frac{u_j^{(2)n+1} - u_j^{(2)n}}{2} - \frac{u_j^{(1)n+1} - u_j^{(1)n}}{2} - \sigma_1 \left(\frac{\left(u_{j+1}^{(2)n+1} - u_{j-1}^{(2)n+1} \right) + \left(u_{j+1}^{(2)n} - u_{j-1}^{(2)n} \right)}{4h} - \right. \\
 & \left. \frac{\left(u_{j+1}^{(1)n+1} - u_{j-1}^{(1)n+1} \right) + \left(u_{j+1}^{(1)n} - u_{j-1}^{(1)n} \right)}{4h} \right) = 0.
 \end{aligned}$$

The step in the dimensionless coordinate is $h = 0.2$, and the step in dimensionless time τ is chosen to be $0.5h$. The obtained approximation is coordinated with the system of equations (6.4). As the differential scheme coordinated with it is not overt, it converges to the solution of the above-mentioned system of equations (6.4) with initial condition (6.6). This differential scheme possesses stability and the second order of accuracy in the coordinate and in time.

Fig. 2. Influence of the absence of the inertia of the fluid motion ($\sigma_1 = 0$).

Fig. 3. Influence of the inertia of the fluid motion ($\sigma_1 = 1$).

8. Results and conclusions

The presence of a deformation wave in the outer shell led to the appearance of a deformation wave in the inner shell, which was not present at the initial instant. The wave velocity decreases. According to Fig. 2, the effect of viscous fluid stress on the shells leads to a decrease in the deformation wave amplitude in the first shell, and in the second one the deformation wave amplitude grows until the average value of the amplitudes in both shells is achieved, the energy transfer taking place through the fluid layer. As a result, energy volumes in both shells become equal. The deformation wave velocity is subsonic. According to Fig. 3, the wave velocity decreases (the graph shifts to the left), due to the influence of the inertia of the fluid motion ($\sigma_1 = 1$).

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