MATHEMATICAL PROBLEMS OF NONLINEARITY

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Precessional Motion of a Rigid Body Acted upon by Three Irreducible Fields

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We consider a quit general problem of motion of an asymmetric rigid body about a fixed point, acted upon by an irreducible skew combination of gravitational, electric and magnetic fields. Two of those three fields are uniform and the third has a more complicated structure. The existence of precessional motions about a nonvertical axis is established. Conditions on the parameters of the system are obtained. An alternative physical interpretation is given in the framework of the problem of motion of a rigid body immersed in an incompressible perfect fluid, acted upon by torques due to two uniform fields.

Keywords: rigid body, precessional motion, three irreducible fields

1. Introduction

The motion of a rigid body, fixed at one of its points is said to be a regular precession if the body rotates uniformly about an axis, called the figure axis, fixed in it, while that axis precesses with the same, or another, uniform angular velocity about an axis fixed in space, the precession axis. Routh [12] showed that regular precessions of an asymmetric rigid body, having for the axis of precession the vertical and for the figure axis one of the three principal axes of inertia, are dynamically impossible.

The possibility of the regular precession of an asymmetric heavy rigid body about an axis inclined to the vertical was pointed out by Grioli [3, 4] on a dynamical basis. Later, Gulyaev [5] obtained Grioli's solution by direct integration of the Euler–Poisson equations of motion. The question of the existence of other Grioli-type solutions of the classical problem of the motion of a rigid body has been investigated by many authors, see, e.g., [7, 8]. Grioli's result was generalized by Kharlamova [9] to the case of a gyrostat acted upon by a single gravity field.

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The existence of a Grioli-type precession in the problem of the motion of a rigid body in a liquid is shown in [13, 14] when the figure axis is perpendicular to the precession axis. The case when the figure and precession axes are not perpendicular to each other is considered in [10].

The equations of motion of the problem of motion of a rigid body by inertia in a liquid were given in a new form [16] isomorphic to those of a body moving about a fixed point and subject to the effect of a conservative combination of coaxial potential and gyroscopic forces. Physical interpretation of those forces involves a uniform magnetic field, which induces a scalar magnetic potential of magnetized parts of the body, a vector potential representing Lorentz forces on fixed electric charge distribution on the surface of the body and two other fields, gravitational and electric, whose potentials are symmetric around the direction of the magnetic field and involve only first and second harmonics. The analogy of the two problems has made it possible to construct integrable cases and particular solutions in the second problem from all known ones of the first. The second problem turned out to be extremely flexible and has gone a long way in development and generalization in the same framework of axial combinations of two or three fields. For details the reader may be referred to [17] and [18].

Although enormous effort was invested in the problem of motion of a rigid body about a fixed point in a uniform field and in an axial combination of nonuniform fields, the problem involving skew fields acting on the body has escaped attention most of the time.

Conditions for the existence of a regular precession in the problem of the motion of a rigid body (gyrostat) acted upon by an irreducible combination of three fields are obtained in [20]. This generalizes results obtained in [19] for the case of a single rigid body acted upon by two irreducible fields.

The present article considers a rigid body moving in an ideal incompressible fluid while acted upon by an irreducible combination of two uniform fields. We show that the body can perform a Grioli-type precession and obtain some previous results as a special case.

2. Equations of Motion

To describe the attitude dynamics of a rigid body, two coordinate systems $OXYZ$ and $Oxyz$ are introduced: the system $OXYZ$ is fixed in space with unit vectors α , β and γ along its axes, while the system $Oxyz$ is rigidly attached to the body with unit vectors i , j and k which, in general, are not directed along the body principal axes at O . So, the inertia tensor is given by

$$\mathbf{I} = \begin{pmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{pmatrix}. \quad (2.1)$$

We consider a dynamical problem defined by the Euler equation

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + \mathbf{K}\boldsymbol{\gamma} + \boldsymbol{\kappa}) = \boldsymbol{\alpha} \times \frac{\partial V}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial V}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}}, \quad (2.2)$$

where $\boldsymbol{\omega}$ denotes the angular velocity of the body, $\mathbf{K} = (K_{ij})$ is a 3×3 symmetric matrix, $\boldsymbol{\kappa}$ is the gyrostatic moment and the dot denotes differentiation with respect to time. Equations (2.2) can be adapted by a proper choice of the system's parameters and the potential V to describe many problems in rigid body dynamics. Let us consider the case when

$$V = \mathbf{a} \cdot \boldsymbol{\alpha} + \mathbf{b} \cdot \boldsymbol{\beta} + \left(\frac{1}{2} \mathbf{J}\boldsymbol{\gamma} + \mathbf{s} \right) \cdot \boldsymbol{\gamma}, \quad (2.3)$$

where $\mathbf{J} = (J_{ij})$ is a 3×3 symmetric matrix. Then the problem under consideration is defined by the equations

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + \mathbf{K}\boldsymbol{\gamma} + \boldsymbol{\kappa}) = \boldsymbol{\alpha} \times \mathbf{a} + \boldsymbol{\beta} \times \mathbf{b} + \boldsymbol{\gamma} \times (\mathbf{J}\boldsymbol{\gamma} + \mathbf{s}), \quad (2.4a)$$

$$\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \quad \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \quad (2.4b)$$

where the dynamical equation (2.4a) is augmented by the Poisson kinematical equations (2.4b). A physical interpretation of the problem (2.4) when $\mathbf{a} = \mathbf{b} = \mathbf{0}$ is given in [16]. It is shown that the problem describes the inertial motion of a rigid body in an ideal incompressible fluid. In the generic case, the potential is interpreted as representing the action of three crossed fields on a body about a fixed point; one of them is nonlinear (axisymmetric about $\boldsymbol{\gamma}$) and the other two are uniform in the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ directions, with their centers at $\mathbf{a}$ and $\mathbf{b}$, respectively.

3. Precessional Motion

Without loss of generality, we let $\boldsymbol{\gamma}$ and $\mathbf{k}$ be the precession and figure axes, respectively. If we denote the constant angle between the two axes by θ_0 , then the regular precession is defined by

$$\boldsymbol{\gamma} \cdot \mathbf{k} = a_0, \quad \boldsymbol{\omega} = \Omega(\boldsymbol{\gamma} + \mathbf{k}), \quad (3.1)$$

where $a_0 = \cos \theta_0$. The precession (3.1) can be defined in terms of Euler's angles defining the angular orientation of the system $Oxyz$ relative to $OXYZ$: ψ denotes the angle of precession around the Z -axis, θ is the angle of nutation between the z - and Z -axes, and ϕ is the angle of rotation of the body around the z -axis. We first recall that (see [11]):

$$\begin{aligned} \boldsymbol{\omega} &= (\dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi) \mathbf{i} + (\dot{\psi} \sin \theta \cos \phi - \dot{\theta} \sin \phi) \mathbf{j} + (\dot{\psi} \cos \theta + \dot{\phi}) \mathbf{k}, \\ \boldsymbol{\alpha} &= (\cos \psi \cos \phi - \cos \theta \sin \psi \sin \phi) \mathbf{i} - (\cos \psi \sin \phi + \cos \theta \sin \psi \cos \phi) \mathbf{j} + \sin \theta \sin \psi \mathbf{k}, \\ \boldsymbol{\beta} &= (\sin \psi \cos \phi + \cos \theta \cos \psi \sin \phi) \mathbf{i} - (\sin \psi \sin \phi - \cos \theta \cos \psi \cos \phi) \mathbf{j} - \sin \theta \cos \psi \mathbf{k}, \\ \boldsymbol{\gamma} &= \sin \theta \sin \phi \mathbf{i} + \sin \theta \cos \phi \mathbf{j} + \cos \theta \mathbf{k}. \end{aligned} \quad (3.2)$$

The motion (3.1) can be defined by

$$\theta = \theta_0, \quad \psi = \phi \equiv u = \Omega(t - t_0). \quad (3.3)$$

Substituting (3.3) into (3.2), we get

$$\begin{aligned} \boldsymbol{\omega} &= (a'_0 \Omega \sin u, a'_0 \Omega \cos u, \epsilon_1 \Omega), \\ \boldsymbol{\alpha} &= (\epsilon_1 \cos^2 u - a_0, -\epsilon_1 \sin u \cos u, a'_0 \sin u), \\ \boldsymbol{\beta} &= (\epsilon_1 \sin u \cos u, -\epsilon_1 \sin^2 u + a_0, -a'_0 \cos u), \\ \boldsymbol{\gamma} &= (a'_0 \sin u, a'_0 \cos u, a_0), \end{aligned} \quad (3.4)$$

where

$$a'_0 = \sqrt{1 - a_0^2} = \sin \theta_0, \quad \epsilon_1 = 1 + a_0 = 2 \cos^2(\theta_0/2). \quad (3.5)$$

For $a_0 = \pm 1$, the motion degenerates to a permanent rotation about $\boldsymbol{\gamma} \equiv \pm \mathbf{k}$. Since we are interested here in precessional motions, we will assume in what follows that $a_0 \neq \pm 1$.

4. The solution

One can easily see that (3.4) is a solution of the kinematical Poisson equations (2.4b). So, we turn our attention now to the dynamical problem. Substituting (3.4) into the dynamical equation (2.4a), we get the set of three scalar equations

$$\begin{aligned}
 & [b_0^2 (D\Omega^2 - J_{23}) + \epsilon_1 b_3] \cos^2 u + [b_0^2 (E\Omega^2 + J_{13}) - \epsilon_1 a_3] \cos u \sin u \\
 & - b_0 \{ [A + \epsilon_1 (C - B)] \Omega^2 + \kappa_3 \Omega + a_0 (J_{22} - J_{33}) - b_2 - s_3 \} \cos u \\
 & - b_0 [(\epsilon_1 + 1) F\Omega^2 + a_0 J_{12} + a_2] \sin u - \epsilon_1^2 D\Omega^2 + \epsilon_1 \kappa_2 \Omega - \epsilon_1 b_3 - a_0 (a_0 J_{23} - b_3 + s_2) = 0, \\
 & - [b_0^2 (E\Omega^2 + J_{13}) - \epsilon_1 a_3] \cos^2 u + [b_0^2 (D\Omega^2 + J_{23}) + \epsilon_1 b_3] \cos u \sin u \\
 & + b_0 \{ [\epsilon_1 (A - C) - B] \Omega^2 - \kappa_3 \Omega + a_0 (J_{33} - J_{11}) - a_1 + s_3 \} \sin u \\
 & + b_0 [-(\epsilon_1 + 1) F\Omega^2 - a_0 J_{12} + b_1] \cos u + (E\Omega^2 + J_{13}) b_0^2 - E\Omega^2 \epsilon_1^2 + \Omega \epsilon_1 \kappa_1 \\
 & - a_0 (a_0 J_{13} + a_3 + s_1) = 0, \\
 & [2b_0^2 (F\Omega^2 + J_{12}) - \epsilon_1 (a_2 - b_1)] \cos^2 u + \{ b_0^2 [(B - A)\Omega^2 + J_{11} - J_{22}] - \epsilon_1 (a_1 + b_2) \} \cos u \sin u \\
 & + b_0 ((\epsilon_1 - 1) E\Omega^2 - \kappa_1 \Omega + a_0 J_{13} + s_1) \cos u + b_0 [(\epsilon_1 - 1) D\Omega^2 + \kappa_2 \Omega - J_{23} a_0 - s_2] \sin u \\
 & - b_0^2 (F\Omega^2 + J_{12}) - \epsilon_1 b_1 + a_0 (a_2 + b_1) = 0,
 \end{aligned} \tag{4.1}$$

whose solution is a set of conditions on the parameters sufficient for the system to admit the desired solution (3.4).

Since recently, Computer Algebra Systems (CAS) have been playing an increasingly important role in mechanics that goes beyond the numerical approximation methods of computational mechanics. In rigid body dynamics, CAS help, for example, in constructing new integrable cases [18], particular solutions, e.g., polynomial solutions [6, 21] and, generally speaking, in verifying solutions obtained via analytical methods.

Using a Computer Algebra System capable of doing symbolic computations, one can prove that (3.4) is a solution of Eqs. (4.1), and hence of the problem (2.4) if

$$a_2 = -b_1 = -\epsilon_2 \Omega (2F\Omega - K_{12}), \quad a_3 = \epsilon_2 (E\Omega^2 - \Omega K_{23} + J_{13}) \tag{4.2a}$$

$$b_2 = [A - \epsilon_1 (B - C)] \Omega^2 - (\epsilon_1 K_{22} - a_0 K_{33} - k_3) \Omega + a_0 (J_{22} - J_{33}) - s_3 \tag{4.2b}$$

$$b_3 = -\epsilon_2 (D\Omega^2 - \Omega K_{23} + J_{23}) \tag{4.2c}$$

$$s_1 = 2a_0 E\Omega^2 + \epsilon_2 K_{13} \Omega - J_{13}, \quad s_2 = 2a_0 D\Omega^2 + \epsilon_2 K_{23} \Omega - J_{23} \tag{4.2d}$$

$$\kappa_1 = 3a_0 E\Omega + (1 - 2a_0) K_{13} - \epsilon_2 \Omega^{-1} J_{13}, \tag{4.2e}$$

$$\kappa_2 = 3a_0 D\Omega + (1 - 2a_0) K_{23} - \epsilon_2 \Omega^{-1} J_{23}, \tag{4.2f}$$

$$J_{11} = 3(A - B)\Omega^2 + 2(K_{11} - K_{22})\Omega + J_{22}, \quad J_{12} = -\Omega (3F\Omega - 2K_{12}), \tag{4.2g}$$

in which $\epsilon_2 = 1 - a_0$, and the angular velocity Ω is given by

$$\Omega = \frac{1}{2} \frac{\lambda_2 \pm \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{\lambda_1}, \tag{4.3}$$

where

$$\begin{aligned}
 \lambda_1 &= -a_0 (2A - 3B + C) - A + B + C, \\
 \lambda_2 &= \epsilon_2 K_{11} + a_0 (2K_{22} - K_{33}) - \kappa_3, \\
 \lambda_3 &= a_0 (J_{22} - J_{33}) + a_1 - s_3.
 \end{aligned} \tag{4.4}$$

The potential (2.3) is given now by

$$\begin{aligned}
 V = & a_1 \alpha_1 - \epsilon_2 \Omega (2F\Omega - K_{12}) (\alpha_2 - \beta_1) + \epsilon_2 (E\Omega^2 - \Omega K_{23} + J_{13}) \alpha_3 + \\
 & + \{ [A - \epsilon_1 (B - C)] \Omega^2 - (\epsilon_1 K_{22} - a_0 K_{33} - \kappa_3) \Omega + a_0 (J_{22} - J_{33}) - s_3 \} \beta_2 - \\
 & - \epsilon_2 (D\Omega^2 - \Omega K_{23} + J_{23}) \alpha_3 + \frac{1}{2} [3(A - B)\Omega^2 + 2(K_{11} - K_{22})\Omega + J_{22}] \gamma_1^2 - \\
 & - \Omega (3F\Omega - 2K_{12}) \gamma_1 \gamma_2 + J_{13} \gamma_1 \gamma_3 + \frac{1}{2} J_{22} \gamma_2^2 + J_{23} \gamma_2 \gamma_3 + \frac{1}{2} J_{33} \gamma_3^2 + \\
 & + (2a_0 E\Omega^2 + \epsilon_2 K_{13} \Omega - J_{13}) \gamma_1 + (2a_0 D\Omega^2 + \epsilon_2 K_{23} \Omega - J_{23}) \gamma_2 + s_3 \gamma_3.
 \end{aligned} \tag{4.5}$$

The solution (4.2) has the free parameters: $a_0 \in [-1, 1]$, a_1 , b_1 , κ_3 , c_1 and all the values of K_{ij} and J_{ij} except J_{11} and J_{22} , in addition to the inertia moments of the body. It is remarkable that all nonfree parameters of the system are given explicitly in terms of the free ones.

5. Special cases

5.1. A gyrostat under three uniform fields

Let us consider the case of a gyrostat, with a gyrostatic moment κ acted upon by irreducible three uniform fields derived from the potential

$$V = \mathbf{a} \cdot \boldsymbol{\alpha} + \mathbf{b} \cdot \boldsymbol{\beta} + \mathbf{s} \cdot \boldsymbol{\gamma}. \tag{5.1}$$

The motion is governed by the equations

$$\begin{aligned}
 \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\kappa}) &= \boldsymbol{\alpha} \times \mathbf{a} + \boldsymbol{\beta} \times \mathbf{b} + \boldsymbol{\gamma} \times \mathbf{s}, \\
 \dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} &= \mathbf{0}, \quad \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}.
 \end{aligned} \tag{5.2}$$

The problem is the special version of (2.4) when $\mathbf{K} = \mathbf{J} = \mathbf{0}$. Assuming in (4.2) zero values of all K_{ij} and J_{ij} , we get from (4.2g) the conditions $B = A$, $F = 0$ on the inertia moments. This means that the z -axis is perpendicular to one of the two circular cross-sections of the ellipsoid of inertia of the body at the fixed point and F vanishes for any other pair of axes in that plane. Thus, we can rotate the $x - y$ axes in their plane to satisfy the additional condition $D = 0$. We end with the inertia tensor

$$\mathbf{I} = \begin{pmatrix} A & 0 & -E \\ 0 & A & 0 \\ -E & 0 & C \end{pmatrix}, \tag{5.3}$$

and the following conditions for the existence of the precessional motion

$$a_2 = b_1 = b_3 = s_2 = \kappa_2 = 0, \quad a_3 = \epsilon_2 E\Omega^2, \tag{5.4a}$$

$$b_2 = (a_0 A - C) \Omega^2 + \kappa_3 \Omega - s_3, \quad s_1 = 2a_0 E\Omega^2, \quad \kappa_1 = 3a_0 E\Omega^2. \tag{5.4b}$$

The values of λ_i are

$$\lambda_1 = \epsilon_1 C - a_0 A, \quad \lambda_2 = -\kappa_3, \quad \lambda_3 = a_1 - s_3. \tag{5.5}$$

Substituting (5.5) into (4.3), we obtain the angular velocity in the form

$$\Omega = \frac{1}{2} \frac{\kappa_3 \pm \sqrt{\kappa_3^2 + 4(a_1 - s_3)(a_0 A - \epsilon_1 C)}}{\epsilon_1 C - a_0 A}. \tag{5.6}$$

Using conditions (5.4), the potential (2.3) can be brought to the form

$$\begin{aligned} V &= a_1\alpha_1 + a_3\alpha_3 + b_2\beta_2 + s_1\gamma_1 + s_3\gamma_3 = \\ &= \mathbf{i} \cdot (a_1\mathbf{a} + s_1\boldsymbol{\gamma}) + b_2(\mathbf{j} \cdot \boldsymbol{\beta}) + \mathbf{k} \cdot (a_3\boldsymbol{\alpha} + s_3\boldsymbol{\gamma}). \end{aligned} \quad (5.7)$$

The third term in (5.7) can be interpreted as the potential of a uniform gravity field

$$\mathbf{g} = a_3\boldsymbol{\alpha} + s_3\boldsymbol{\gamma}, \quad (5.8)$$

pointing vertically upwards. Then the axis of precession $\boldsymbol{\gamma}$ is inclined to the vertical at an angle

$$\delta = \tan^{-1} \frac{a_3}{s_3} = \tan^{-1} \frac{\epsilon_2 E \left[\kappa_3 \pm \sqrt{\kappa_3^2 + 4(a_1 - s_3)(a_0 A - \epsilon_1 C)} \right]^2}{4s_3(\epsilon_1 C - a_0 A)^2}, \quad (5.9)$$

which depends on the angle θ_0 between the axis of precession $\boldsymbol{\gamma}$ and the axis $\mathbf{k}$ of proper rotation.

In the special case of a rigid body ($\kappa_1 = \kappa_3 = 0$), we see from (5.4b) that the precessional motion is possible only when $a_0 = 0$ ($\theta_0 = \pi/2$, $a'_0 = \epsilon_1 = \epsilon_2 = 1$). Then we get results established recently by Yehia [20].

5.2. The two-field Kowalewski top

The version of the problem when

$$\mathbf{a} = 0, \quad A = 2C, \quad E = 0 \quad (5.10)$$

describes a Kowalevski gyrostat in two perpendicular fields. In this case, (5.4b) gives

$$b_2 = C(2a_0 - 1)\Omega^2 + \kappa_3\Omega - s_3, \quad s_1 = \kappa_1 = 0, \quad (5.11)$$

that is, the gyrostatic momentum is directed along the axis of dynamical symmetry. The velocity Ω of precession can be obtained from (5.6) as

$$\Omega = \frac{\kappa_3 \pm \sqrt{\kappa_3^2 + 4s_3\epsilon_2 C}}{2\epsilon_2 C}. \quad (5.12)$$

The potential (5.7) is reduced to

$$V = b_2(\mathbf{j} \cdot \boldsymbol{\beta}) + s_3(\mathbf{k} \cdot \boldsymbol{\gamma}) = b_2\beta_2 + s_3a_0. \quad (5.13)$$

The first term in V can be interpreted as the potential of a uniform gravity field in the direction of $\boldsymbol{\beta}$ and then the vector of the body's center of mass is $b_2\mathbf{j}$. The second term in V is the potential of a field in the direction of $\boldsymbol{\gamma}$ whose center of application is $s_3\mathbf{k}$.

6. Concluding Remarks

We add two uniform fields to what is called the generalized problem of the motion of a body [1, 15]. The main result of the present article is that the rigid body in the generic case, which describes a rigid body moving in an ideal incompressible fluid while acted upon by an irreducible combination of two uniform fields, can perform the Grioli-type precessional motion (3.4) under conditions (4.2), (4.3) and (4.4). As far as we know, only the special case

when the body moves by inertia in the fluid, that is, $\mathbf{a} = \mathbf{b} = \mathbf{0}$, is considered in the literature, see, for example, [2, 10, 13, 14].

As a special case, we get a solution of the problem of precessional motions of a gyrostat acted upon by irreducible three uniform fields. In the latter case, it is known that the gyrostat can perform a Grioli-type precessional motion when the axis of proper rotation is perpendicular to the axis of precession [20]. We construct here a new solution of the problem assuming an arbitrary value of the angle between the two axes, from which the results of [20] are obtained under the perpendicularity condition. The results are more specialized to prove the existence of the considered class of motion for a Kowalevski gyrostat in two perpendicular uniform fields.

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