NONLINEAR PHYSICS AND MECHANICS

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Another Special Case of Vibrations of a Rolling Tire

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We investigate a special case of vibrations of a loaded tire rolling at constant speed. A previously proposed analytical model of a radial tire is considered. The surface of the tire is a flexible tread combined with elastic sidewalls. In the undeformed state, the tread is a circular cylinder. The tread is reinforced with inextensible cords. The tread is the part of the tire that makes actual contact with the ground plane. In the undeformed state, the sidewalls are represented by parts of two tori and consist of incompressible rubber described by the Mooney–Rivlin model. The previously obtained partial differential equation which describes the tire radial in-plane vibrations about steady-state regime of rolling is investigated. Analyzing the discriminant of the quartic polynomial, which is the function of the frequency of the tenth degree and the function of the angular velocity of the sixth degree, the rare case of a root of multiplicity three is discovered. The angular velocity of rotation, the tire speed and the natural frequency, corresponding to this case, are determined analytically. The mode shape of vibration in the neighborhood of the singular point is determined analytically.

Keywords: radial tire, analytical model, rolling, modal analysis, vibrations, multiple roots

1. Introduction

Natural frequency (NF) is the frequency at which a mechanical system tends to oscillate in the absence of external or damping force. When a system vibrates at a frequency of applied external force and this frequency is equal to the NF, then the system vibration amplitude highly increases. This circumstance could lead to damage. That is why it is very important to know the NF of the structure. For each NF the corresponding mode shape (MS) exists. A mode is a standing wave which oscillates but whose peak amplitude does not move in space.

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The vibrations of tires were studied by many authors. Precession of the standing waves in a thin elastic ring rotating with constant angular velocity was considered in [1]. The investigation of the small oscillations about the steady motion of uniform endless chain in the form of a circle revolving in its own plane was also presented in [2] as an example (pp. 248–250). The research on the analysis of tire vibrations started in the 60s focusing on the differences between the then standard bias ply tires and the new radial tires [3–5]. The in-plane flexible ring model, introduced in [4], is much older than the simpler rigid ring models. This model uses a circular beam supported on an elastic foundation to represent the motion of the tread-band of a pneumatic tire in the plane of the wheel. A model to study the vibration transmission properties of tires at higher frequencies when the tread pattern produces airborne noise was proposed in [5]. The ring model (where variations across the width direction are neglected) was studied in [6] and applied to tire noise analysis by many researchers. The inertial properties of elastic waves during rotation were discussed in [7] for inextensible ring rotating with variable angular velocity. The precession of the standing waves of an axisymmetric shell rotating with constant angular velocity was considered in [8]. The vibrations of flexible extensible rotating ring were considered in [9] taking into account the geometrical non-linearity. Flexural vibration of a thin rotating ring was investigated experimentally and from some theoretical points of view in [10]. The free non-linear vibrations of a rotating thin ring with a constant speed were studied in [11] when the ring has both the in-plane and out-of-plane motions. The study [12] presents steady response functions concerning the radial, circumferential and lateral vibrations using the three-dimensional flexible ring-based model. A new physically explicit theory of tire standing waves was developed in [13]. The paper [14] presents experimental modal analysis of a non-rotating tire. These experimental results were compared with the modes of a theoretical tire ring model. The paper [15] presents the measurement with a laser Doppler vibrometer and analysis of rolling tire vibrations due to road impact excitations. Based on the results of experimental and numerical analyses, the effect of rotation on the tire dynamic behaviour was investigated in [16]. The effects of rotation on the NF of a loaded tire were studied in [17] using finite element model. An overview on sophisticated modeling approaches for the finite element computation of rolling tires was outlined in [18, 19].

Current problems of investigating the dynamics require a fast calculating models. Therefore, the problem of constructing models, which simulate complex dynamic processes and do not require significant computational resources is very important. A model of a reinforced tire was proposed in [20]. In the case of a wheel rolling without slipping in contact area, an unknown in advance, a complete system of equations of motion was obtained. The steady-state regime of rolling at constant speed was investigated. The tire model proposed was used in studying the vibrations of an unloaded and loaded non-rotating tire [21]. In [22] we study the vibrations of an unloaded and loaded tire rolling at constant speed. Supposing that all the roots of the characteristic equation are different, the NF were determined numerically and MS were determined analytically for a loaded rotating tire. The quantities of obtained NF for the unloaded and loaded non-rotating cases were compared with the results of two experiments.

The idea to study in detail the roots of a characteristic equation arose. It was observed that for any value of the angular velocity of a wheel there are several frequencies at which two roots can coincide. For these frequencies the frequency function which defines infinite spectrum of NF tends to zero. This means that they must probably be added to the spectrum of NF. If there are multiple roots, then the MS is represented in a different form, and the problem must be solved differently. In the process of research an even rarer special case of two pairs of multiple roots was discovered [23]. Later another rare special case of a root of multiplicity three was also discovered. This special case is considered in this paper.

2. Tire model

Assume that the wheel with a reinforced tire consists of a disc joined to the sidewalls and of a tread. The wheel disc is a rigid body. In the undeformed state, the sidewalls are represented by parts of two tori. The elastic sidewalls of incompressible rubber are described by the Mooney–Rivlin model [24]. The tread is the part of the tire that makes actual contact with the ground plane. The tread is reinforced with inextensible cords. In the undeformed state, the tread is a circular cylinder of radius r . The tread deformations are considered taking into account the exact nonlinear conditions of inextensibility of reinforcing cords. Due to nonlinear geometric constraints in the deformed state, the tread retains its cylindrical shape, which is not circular for a typical configuration. The median line of the tread is inextensible. Assume that the whole mass of the tire is distributed uniformly along the plane median line of the tread with linear density ρ .

Let (X_1, X_3) denote the coordinates of the mass center of the disc C in the inertial frame OX_1X_3 (Fig. 1). Introduce a moving frame Cy_1y_3 with its origin C and axes fixed to the disc, θ is a rotation angle. Using the Lagrangian specification we determine the position of median line points by angle φ with respect to a Cy_1y_3 . After two rotations by angle $\theta + \varphi$ we obtain a new moving frame Cx_1x_3 and $ru(\varphi, t)$, $rv(\varphi, t)$ are, respectively, the radial and tangential components of the displacement vector of median line points in the Cx_1x_3 . The contact area of the tire and the plane can be represented by a rectangle of constant width, equal to the tread width, and of variable length. The external longitudinal force, the vertical load and the wheel torque are applied to the wheel disc. Suppose that the wheel rolls without slipping and without jumping. This means that the velocity of points of the tread in the contact area vanishes. The reactions of these constraints in the contact area are the functions of two variables φ and t . Using the Hamilton–Ostrogradsky variational principle one can obtain [20, 22] a complete system of

Fig. 1. The model of a wheel with a reinforced tire.

fourteen equations in fourteen unknowns. The steady-state regime of rolling of a loaded tire at constant speed, without slipping in the contact area, was considered in [20].

3. Tire vibrations

The problem of vibrations of a tire about steady-state regime of rolling was investigated in [22]. Suppose that the wheel rotates with constant angular velocity Ω . Putting $\alpha = \varphi + \Omega t - \pi/2$ we pass from the Lagrangian specification to the Eulerian specification. The contact area length $r\Delta\alpha = r(\alpha_2 - \alpha_1)$ is defined by two constants α_1, α_2 (Fig. 1).

REMARK 1. In determining the frequency of tire vibrations, the length of the contact area is taken as constant, since within the model chosen its variation determines the second order of smallness correction to the frequency.

We represent the functions $ru(\varphi, t)$, $rv(\varphi, t)$ (Fig. 1) determining the shape of the deformed median line of the tread in the form

$$ru(\varphi, t) = rU(\alpha) + rU_{\text{vib}}(\alpha, t), \quad rv(\varphi, t) = rV(\alpha) + rV_{\text{vib}}(\alpha, t),$$

where rU , rV describe the steady-state motion and rU_{vib} , rV_{vib} describe the vibrations of the tire about the steady-state motion. The terms rU_{vib} , rV_{vib} are the radial and tangential components of the displacement vector of median line points. The function V_{vib} satisfies the partial differential equation [22]

$$\rho r^3 \ddot{V}_{\text{vib}}'' - \rho r^3 \ddot{V}_{\text{vib}} + 2\rho r^3 \Omega \dot{V}_{\text{vib}}''' + 2\rho r^3 \Omega \dot{V}_{\text{vib}}' + a_0 V_{\text{vib}}^{(\text{IV})} + a_1 V_{\text{vib}}'' + a_2 V_{\text{vib}} = 0 \quad (3.1)$$

and the boundary conditions

$$V_{\text{vib}}(\alpha_1 + 2\pi) = 0, \quad V_{\text{vib}}(\alpha_2) = 0, \quad V_{\text{vib}}'(\alpha_1 + 2\pi) = 0, \quad V_{\text{vib}}'(\alpha_2) = 0. \quad (3.2)$$

Here the coefficients a_0 , a_1 , a_2 are constant. They are determined analytically and depend on the geometric parameters of the tire and on the internal tire pressure. Using the method of separation of variables (the Fourier method), we will represent

$$V_{\text{vib}}(\alpha, t) = e^{i\omega t} X(\alpha),$$

where ω is an angular frequency, $\nu = \omega/(2\pi)$ is a NF in Hertz, $X(\alpha)$ is a MS. Substituting this expression into Eqs. (3.1) and (3.2), we obtain the ordinary differential equation

$$a_0 X^{(\text{IV})} + 2\rho r^3 \Omega \omega i X''' + (a_1 - \rho r^3 \omega^2) X'' + 2\rho r^3 \Omega \omega i X' + (a_2 + \rho r^3 \omega^2) X = 0 \quad (3.3)$$

and the boundary conditions

$$X(\alpha_1 + 2\pi) = 0, \quad X(\alpha_2) = 0, \quad X'(\alpha_1 + 2\pi) = 0, \quad X'(\alpha_2) = 0. \quad (3.4)$$

The general solution of (3.3) can be written as

$$X(\alpha) = Ge^{p\alpha}.$$

So, the characteristic equation reads

$$a_0 p^4 + 2\rho r^3 \Omega \omega i p^3 + (a_1 - \rho r^3 \omega^2) p^2 + 2\rho r^3 \Omega \omega i p + (a_2 + \rho r^3 \omega^2) = 0. \quad (3.5)$$

Fig. 2. The discriminant $D(\omega)$ of the quartic polynomial for (a) $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, (b) $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$.

Solving this equation one can obtain [22] four roots $p_1(\omega, \Omega)$, $p_2(\omega, \Omega)$, $p_3(\omega, \Omega)$, $p_4(\omega, \Omega)$. In the general case $p_i \neq p_j$ (this case was investigated in [22]) the solution $X(\alpha)$ can be represented in the form

$$X(\alpha) = G_1 e^{p_1 \alpha} + G_2 e^{p_2 \alpha} + G_3 e^{p_3 \alpha} + G_4 e^{p_4 \alpha}. \quad (3.6)$$

The coefficients $G_i(p_1, p_2, p_3, p_4) = G_i(\omega, \Omega)$ are determined [22] from the boundary conditions (3.4). The homogeneous system (3.4) has a nonzero solution if its determinant $f(p_1, p_2, p_3, p_4) = f(\omega, \Omega)$ vanishes. So, if the angular velocity $\Omega = \Omega_0$ is fixed, then we obtain the frequency equation $f(\omega, \Omega_0) = f(\omega) = 0$ which defines infinite spectrum of NF. The function $f(\omega) = \text{Re}(f(\omega)) + i \text{Im}(f(\omega))$ is a complex-valued function, but it assumes real or purely imaginary values for real-valued arguments.

The quartic polynomial (3.5) has the discriminant

$$D(\omega, \Omega) = a_0^6 (p_1 - p_2)^2 (p_1 - p_3)^2 (p_1 - p_4)^2 (p_2 - p_3)^2 (p_2 - p_4)^2 (p_3 - p_4)^2$$

which is the function of the frequency of the tenth degree and the function of the angular velocity of the sixth degree [23]. This discriminant vanishes if and only if at least two roots are equal. The plot of $D(\omega)$ for angular velocity $\Omega = 5.084$ and $45.0282 \text{ rad}\cdot\text{s}^{-1}$ is shown in Fig. 2. So one multiple root (root of multiplicity two, for example, p_1, p_1, p_3, p_4) is located between 98.7 and 98.8 Hz for $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$ (Fig. 2a) and between 96 and 97 Hz for $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$ (Fig. 2b). These cases of a roots of multiplicity two are not considered in this paper. Yet, there are a more interesting situations (roots of multiplicity three) around 99.2 Hz (Fig. 2a) and 101.3 Hz (Fig. 2b). The functions $p_j(\omega) = \text{Re}(p_j(\omega)) + i \text{Im}(p_j(\omega))$ are represented in Figs. 3a and 3b for $\Omega = 4 \text{ rad}\cdot\text{s}^{-1}$, in Figs. 3c and 3d for $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, in Figs. 3e and 3f for $\Omega = 6 \text{ rad}\cdot\text{s}^{-1}$ and in Figs. 4a and 4b for $\Omega = 40 \text{ rad}\cdot\text{s}^{-1}$, in Figs. 4c and 4d for $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$, in Figs. 4e and 4f for $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$. So, these figures show the roots evolution as a function of angular velocity. Most likely, Figs. 3c and 3d and Figs. 4c and 4d are close to a root of multiplicity three.

REMARK 2. Carrying out calculations we used the input data (Input I) as in [22]: the tire size (205/55 R16), the mass of tire and the quantity of internal pressure, $a_0 = -8045.37 \text{ N}\cdot\text{m}$, $a_1 = 55661.5 \text{ N}\cdot\text{m}$, $a_2 = -58109.6 \text{ N}\cdot\text{m}$.

Fig. 3. The functions $p_1(\omega)$, $p_2(\omega)$, $p_3(\omega)$, $p_4(\omega)$, (a) $\text{Rep}_j(\omega)$, $\Omega = 4 \text{ rad}\cdot\text{s}^{-1}$, (b) $\text{Imp}_j(\omega)$, $\Omega = 4 \text{ rad}\cdot\text{s}^{-1}$, (c) $\text{Rep}_j(\omega)$, $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, (d) $\text{Imp}_j(\omega)$, $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, (e) $\text{Rep}_j(\omega)$, $\Omega = 6 \text{ rad}\cdot\text{s}^{-1}$, (f) $\text{Imp}_j(\omega)$, $\Omega = 6 \text{ rad}\cdot\text{s}^{-1}$.

Fig. 4. The functions $p_1(\omega)$, $p_2(\omega)$, $p_3(\omega)$, $p_4(\omega)$, (a) $\text{Rep}_j(\omega)$, $\Omega = 40 \text{ rad}\cdot\text{s}^{-1}$, (b) $\text{Imp}_j(\omega)$, $\Omega = 40 \text{ rad}\cdot\text{s}^{-1}$, (c) $\text{Rep}_j(\omega)$, $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$, (d) $\text{Imp}_j(\omega)$, $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$, (e) $\text{Rep}_j(\omega)$, $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$, (f) $\text{Imp}_j(\omega)$, $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$.

3.1. A special case of vibrations of a rolling tire

Let us consider the special case of a root of multiplicity three, when $p_1 = p_2 = p_3$. In this case, Eq. (3.5) reads

$$a_0 p^4 + 2\rho r^3 \Omega \omega i p^3 + (a_1 - \rho r^3 \omega^2) p^2 + 2\rho r^3 \Omega \omega i p + (a_2 + \rho r^3 \omega^2) = a_0 (p - p_1)^3 (p - p_4).$$

It is necessary to determine the angular velocity of rotation, the NF and MS corresponding to this case. Multiplying the factors on the right-hand side and identifying the coefficients of each power of p , one might obtain:

$$\begin{aligned} 3p_1 + p_4 &= -\frac{2\rho r^3 \Omega \omega i}{a_0}, & p_1^2(p_1 + 3p_4) &= -\frac{2\rho r^3 \Omega \omega i}{a_0}, \\ 3p_1(p_1 + p_4) &= \frac{a_1 - \rho r^3 \omega^2}{a_0}, & p_1^3 p_4 &= \frac{a_2 + \rho r^3 \omega^2}{a_0}. \end{aligned} \quad (3.7)$$

Solving this system, we calculate p_1 , p_4 , Ω analytically

$$p_1^2 = \frac{a_1 - 6a_2 - 7\rho r^3 \omega^2}{a_1 - 6a_0 - \rho r^3 \omega^2}, \quad p_4 = \frac{a_2 + \rho r^3 \omega^2}{a_0} \frac{1}{p_1^3}, \quad \Omega = \frac{(a_1 - \rho r^3 \omega^2)^2 - 36a_0(a_2 + \rho r^3 \omega^2)}{6\rho r^3 \omega(a_1 - 6a_0 - \rho r^3 \omega^2)} \frac{i}{p_1},$$

where ω satisfies

$$g(\omega) = \frac{g_6 \omega^6 + g_4 \omega^4 + g_2 \omega^2 + g_0}{a_0^3} = 0,$$

$$g_6 = -10(\rho r^3)^3, \quad g_0 = a_1^3 - 9a_0 a_1^2 - 9a_1^2 a_2 - 108a_0 a_2^2 - 108a_0^2 a_2 + 108a_0 a_1 a_2,$$

$$g_4 = (-225a_0 + 21a_1 - 9a_2)(\rho r^3)^2, \quad g_2 = (-108a_0^2 - 12a_1^2 + 126a_0 a_1 - 324a_0 a_2 + 18a_1 a_2)\rho r^3.$$

The plot of $g(\omega)$ is shown in Fig. 5. The equation $g(\omega) = 0$ has three positive roots: 1) $\nu = 99.219...$ Hz (corresponding angular velocity $\Omega = 5.084...$ rad·s⁻¹ and, respectively, the tire speed $\dot{X}_1 = r\Omega = 5.8...$ km·h⁻¹); 2) $\nu = 101.288...$ Hz (corresponding angular velocity $\Omega = 45.0282...$ rad·s⁻¹ and, respectively, the tire speed $\dot{X}_1 = r\Omega = 51.37...$ km·h⁻¹); 3) $\nu = 197.8...$ Hz (corresponding angular velocity is a purely imaginary, this case will not be considered).

Fig. 5. The sixth degree polynomial $g(\omega)$. The right figure zooms the left figure around 100 Hz.

The plots of determinant of the homogeneous system (3.4) $\text{Re}(f(\omega))$, $\text{Im}(f(\omega))$ for $\Delta\alpha = 0.3$ rad are shown in Fig. 6a for angular velocity $\Omega = 4 \text{ rad}\cdot\text{s}^{-1}$, in Fig. 6c for $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, in Fig. 6e for $\Omega = 6 \text{ rad}\cdot\text{s}^{-1}$ and in Fig. 6b for $\Omega = 40 \text{ rad}\cdot\text{s}^{-1}$, in Fig. 6d for $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$, in Fig. 6f for $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$.

Fig. 6. The function $f(\omega)$, — $\text{Re}(f(\omega))$, - - - $\text{Im}(f(\omega))$, (a) $\Omega = 4 \text{ rad}\cdot\text{s}^{-1}$, (b) $\Omega = 40 \text{ rad}\cdot\text{s}^{-1}$, (c) $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, (d) $\Omega = 45.0282 \text{ rad}\cdot\text{s}^{-1}$, (e) $\Omega = 6 \text{ rad}\cdot\text{s}^{-1}$, (f) $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$.

in Fig. 6f for $\Omega = 50 \text{ rad}\cdot\text{s}^{-1}$. So, these figures show the evolution of $f(\omega)$ as a function of angular velocity. One can see (Figs. 6c and 6d) a singular point of the curve, corresponding to the special case of a root of multiplicity three.

In this special case, one cannot use the formula (3.6). Thus, one must search the MS in the form

$$X(\alpha) = (G_1 + G_2\alpha + G_3\alpha^2)e^{p_1\alpha} + G_4e^{p_4\alpha}, \quad (3.8)$$

where the coefficients G_1, G_2, G_3, G_4 are determined from the boundary conditions (3.4)

$$\begin{aligned} G_1e^{p_1(\alpha_1+2\pi)} + G_2(\alpha_1+2\pi)e^{p_1(\alpha_1+2\pi)} + G_3(\alpha_1+2\pi)^2e^{p_1(\alpha_1+2\pi)} + G_4e^{p_4(\alpha_1+2\pi)} &= 0, \\ G_1e^{p_1\alpha_2} + G_2\alpha_2e^{p_1\alpha_2} + G_3\alpha_2^2e^{p_1\alpha_2} + G_4e^{p_4\alpha_2} &= 0, \\ \left[G_1p_1 + G_2(1+p_1(\alpha_1+2\pi)) + G_3(\alpha_1+2\pi)(2+p_1(\alpha_1+2\pi)) \right] e^{p_1(\alpha_1+2\pi)} + G_4p_4e^{p_4(\alpha_1+2\pi)} &= 0, \\ G_1p_1e^{p_1\alpha_2} + G_2(1+p_1\alpha_2)e^{p_1\alpha_2} + G_3\alpha_2(2+p_1\alpha_2)e^{p_1\alpha_2} + G_4p_4e^{p_4\alpha_2} &= 0. \end{aligned} \quad (3.9)$$

The solution of this system reads

$$\begin{aligned} G_1 &= e^{-p_1(\alpha_1+2\pi)} \left[- \left((\alpha_1+2\pi)\alpha_2 - \alpha_2(\Delta\alpha-2\pi)(1-(p_4-p_1)(\alpha_1+2\pi)) \right) e^{p_1(\Delta\alpha-2\pi)} \right. \\ &\quad \left. + (\alpha_1+2\pi)^2e^{p_4(\Delta\alpha-2\pi)} \right] G_5^*, \\ G_2 &= e^{-p_1(\alpha_1+2\pi)} \left[\left(2(\alpha_1+2\pi) + (p_4-p_1)(\alpha_2^2 - (\alpha_1+2\pi)^2) \right) e^{p_1(\Delta\alpha-2\pi)} \right. \\ &\quad \left. - 2(\alpha_1+2\pi)e^{p_4(\Delta\alpha-2\pi)} \right] G_5^*, \\ G_3 &= e^{-p_1(\alpha_1+2\pi)} \left[- \left(1 + (p_4-p_1)(\Delta\alpha-2\pi) \right) e^{p_1(\Delta\alpha-2\pi)} + e^{p_4(\Delta\alpha-2\pi)} \right] G_5^*, \\ G_4 &= e^{-p_4(\alpha_1+2\pi)} \left[-(\Delta\alpha-2\pi)^2e^{p_1(\Delta\alpha-2\pi)} \right] G_5^*, \end{aligned} \quad (3.10)$$

where G_5^* is an arbitrary constant. The homogeneous system (3.9) has a nonzero solution (3.10) if its determinant vanishes

$$\begin{aligned} f &= e^{(3p_1+p_4)(\alpha_1+2\pi)} \left[(\Delta\alpha-2\pi)(p_1-p_4) \left(e^{(p_1+p_4)(\Delta\alpha-2\pi)} + e^{2p_1(\Delta\alpha-2\pi)} \right) \right. \\ &\quad \left. + 2 \left(e^{(p_1+p_4)(\Delta\alpha-2\pi)} - e^{2p_1(\Delta\alpha-2\pi)} \right) \right] (\Delta\alpha-2\pi) \\ &= e^{2p_1(\alpha_1+2\pi)} e^{(p_1+p_4)\alpha_2} (\Delta\alpha-2\pi) f_1 \left((p_1-p_4)(\Delta\alpha-2\pi) \right) = 0, \quad f_1(x) = x + 2 + (x-2)e^x. \end{aligned} \quad (3.11)$$

Substituting (3.10) into (3.9) for verification, one can obtain three identities (the first three equations) and the equation $f \cdot e^{-(3p_1+p_4)(\alpha_1+2\pi)} = 0$ (the fourth equation). If f vanishes, then the fourth equation of (3.9) is fulfilled identically. But the multiplier $f_1(x)$ appearing in (3.11) does not vanish except $x = 0$, that is, for $p_1 = p_4$. So, this is a case of a root of multiplicity four ($p_1 = p_2 = p_3 = p_4$) [23]¹. Thus, the homogeneous system (3.9) has only a zero solution $G_1 = G_2 = G_3 = G_4 = 0$ and the corresponding MS $X(\alpha)$ vanishes.

Let us consider the neighborhood of the singular point. If $p_1 \rightarrow p_2 \rightarrow p_3$ it is necessary to use the formula (3.6). The frequency function tends to zero $f(\omega) \rightarrow 0$. Thus, (3.6) practically satisfies the boundary conditions (3.4): the first three equations of (3.4) are the

¹A root of multiplicity four is impossible for the tire geometry chosen. But theoretically, it is possible to choose such a geometry of the tire ($a_2 = -a_1$), that this case of a root of multiplicity four can be implemented.

Fig. 7. The MS in the neighborhood of the singular point for (a) $\Omega = 5.084 \text{ rad}\cdot\text{s}^{-1}$, $\nu = 99.219 \text{ Hz}$, (b) $\Omega = 45.028 \text{ rad}\cdot\text{s}^{-1}$, $\nu = 101.288 \text{ Hz}$.

three identities and the left side of the fourth equation $X'(\alpha_2)$ tends to zero. This means that the tangential component of the displacement vector of median line points meets the conditions $rV_{\text{vib}}(\alpha_1 + 2\pi) = 0$, $rV_{\text{vib}}(\alpha_2) = 0$. But the radial component fulfills the conditions $rU_{\text{vib}}(\alpha_1 + 2\pi) = 0$, $rU_{\text{vib}}(\alpha_2) \rightarrow 0$ (Fig. 7). As G_i tend to zero, the amplitude of vibrations also tends to zero. The MS in the neighborhood of the singular point are represented in Fig. 7. The shape of the deformed median line of the tread in the steady-state regime of rolling of a loaded tire is represented in the figure by the dotted line. The vibrations about the steady-state motion correspond to the solid line.

4. Conclusions

Multiple roots correspond to internal resonances (structural resonance), which are often irremovable, because the product is already manufactured. As a rule, the wear and damage could occur in the neighborhood of internal resonance. That is why it is necessary to investigate this situation in detail. A special case of a root of multiplicity three is considered in this paper. In contrast to [22], if one has a situation of multiple roots, then the solution is represented in a different form, and the problem must be solved differently. If the geometry of the tire and the internal tire pressure are fixed, then the angular velocity of rotation ($5.084\dots$ and $45.0282\dots \text{ rad}\cdot\text{s}^{-1}$) the tire speed ($5.8\dots$ and $51.37\dots \text{ km}\cdot\text{h}^{-1}$) and the frequency ($99.219\dots$ and $101.288\dots \text{ Hz}$), corresponding to this case, are determined analytically. As for mode shape, here one has an interesting situation. If $p_1 \rightarrow p_2 \rightarrow p_3$ it is necessary to use the formulae obtained in [22]. The frequency function from the frequency equation which defines infinite spectrum of natural frequencies, tends to zero. Thus, the solution practically satisfies the boundary conditions. The amplitude of vibrations tends to zero. If $p_1 = p_2 = p_3$, one has a special case and cannot use the formulae obtained in [22]. It is necessary to use the formulae obtained in this paper. For the case of a root of multiplicity three the frequency function is a constant function and has a different structure. This constant function is not equal to zero at the singular point. So the amplitude of vibrations is equal to zero and hence the corresponding mode shape also vanishes.

Conflict of Interest

The author declares that he has no conflict of interest.

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