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Stable Feedback Control of a Fast Wheeled Robot

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We obtain criteria for the stability of fast straight-line motion of a wheeled robot using proportional or proportional derivative feedback control. The motion of fast robots with discrete feedback control is defined by the discrete dynamical system. The stability criteria are obtained for the discrete system for proportional and proportional-derivative feedback control.

Keywords: feedback control, stability, robotics

1. Introduction

Wheeled robots with differential driving for left and right wheels are typical of contests of fast robots. Such robots are important models for design of control at large velocities when a delay of digital feedback is noticeable. The kinematics of such robots gives us theoretical limits for a speed which is appropriate for control.

When we study only kinematics and neglect dynamics, we obtain a simplified mathematical model in which the moment of inertia and the power of motors are not taken into account. But the neglect of dynamics gives us the opportunity to consider constraints of geometry design of the robot for ideal dynamical properties. Roughly speaking, this shows us a theoretical potential for the velocity of the chosen design and control scheme.

In the most simple and therefore important case the robot can be thought of as two wheels on one geometric axis. The coordinates of the robot are the coordinates of the center of the axis (x, y) and the angle α . The angle α is the direction of the robot with respect to the axis Ox . Hence the phase space of the robot on the plane contains three components (x, y, α) , where $(x, y) \in \mathbb{R}^2$, $\alpha \in [-\pi, \pi)$.

The length of a curve in this phase space is defined for coordinates (x, y) . Therefore, one can consider this phase space as a sub-Riemannian manifold [1].

A geometric scheme of the robot and its important dimensions can be seen in Figure 1.

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Fig. 1. The coordinates of the center of the robot are (x, y) and the direction of the robot is the angle α . This coordinate triplet (x, y, α) defines the position of the robot on the plane.

The typical geometric dimensions of the kinematic model are defined as follows: D is the diameter of the wheels, which is typically 25–100 mm; $2K$ is a gauge, which is typically 100–200 mm; A is the position of the line sensor, the typical value is about 200 mm.

The paper is structured as follows. In Section 2 we derive a system of equations for the motion. We keep track of the dependence of the system parameters on the geometric dimensions of the robot and rotation speeds of the wheels. In Section 3 we write out an equation of motion for the robot under proportional feedback control for straight-line motion. We show that this motion is asymptotically stable. In Section 4 we study the motion under proportional feedback control with small delay. It is shown that this delay may bring an instability into the control. We obtain a criterion for stability of the straight-line motion under small delay. In Section 5 we consider the motion under proportional derivative feedback control and we formulate a theorem about stability of straight-line motion. In Section 6 we show that fast motion under proportional derivative feedback control with small delay may be unstable. We obtain a criterion for stability of such motion.

2. Kinematics equations

Here we derive a well-known system for the kinematics for wheeled robots. The goal of this derivation is to find the influence of dimensions on the properties of the moving robot. Besides, when one writes out the dimensionless form of kinematic equations, one can see the dependence of the parameters on the rotation velocities of the motors.

To derive the equations for the moving robot in the configuration space $(x, y, \alpha) \in \{\mathbb{R}^2 \times \mathbb{S}^1\}$, we denote the angle velocity of the left wheel by ω_L and the angle velocity of the right wheel by ω_R . The typical values of the velocities are less than 100 rad/sec.

Suppose we are given the angle velocities of the wheels and $\omega_L < \omega_R$. Then the linear speed of the left wheel is $\omega_L D/2$ and the linear speed of the right wheel is $\omega_R D/2$.

Let R be the radius of the trajectory. The left and right wheels are moving in a circle. The equation for the rotational velocity of the wheel axis is

$$\omega_L D/2 = \dot{\alpha}(R - K), \quad \omega_R D/2 = \dot{\alpha}(R + K).$$

Then we can obtain the rotational velocity of the robot and the radius of the trajectory:

$$\dot{\alpha} = \frac{D}{4K} (\omega_R - \omega_L), \quad R = \frac{\omega_R + \omega_L}{\omega_R - \omega_L} K.$$

Let (X, Y) be the coordinates of the center of rotation and let (x_0, y_0, α_0) be the current coordinates of the robot. Then we can write the following formula for X , Y and R :

$$x_0 - X = R \sin(\alpha_0), \quad y_0 - Y = -R \cos(\alpha_0).$$

Let (x_1, y_1, α_1) denote the coordinates of the robot after rotation through angle $d\alpha$. Then the change in the coordinates is

$$\begin{aligned} \alpha_1 - \alpha_0 &= d\alpha, \\ x_1 - x_0 &= R(\sin(\alpha_1) - \sin(\alpha_0)), \\ y_1 - y_0 &= -R(\cos(\alpha_1) - \cos(\alpha_0)). \end{aligned}$$

Let δ be a step of discretization. Then the recurrent system of equations for the coordinates can be rewritten as

$$\begin{aligned} \alpha_{n+1} - \alpha_n &= \frac{D}{2K} \frac{\omega_R - \omega_L}{2} \delta, \\ x_{n+1} - x_n &= \frac{\omega_R + \omega_L}{\omega_R - \omega_L} K (\sin(\alpha_{n+1}) - \sin(\alpha_n)), \\ y_{n+1} - y_n &= -\frac{\omega_R + \omega_L}{\omega_R - \omega_L} K (\cos(\alpha_{n+1}) - \cos(\alpha_n)). \end{aligned} \quad (2.1)$$

It is convenient to rewrite $x = KX$, $y = KY$ and to denote $\omega = (\omega_R + \omega_L)/2$, $\theta = (\omega_R - \omega_L)/2$, $\Delta = \delta\omega D/(2K)$. Then we obtain the system in dimensionless form:

$$\begin{aligned} \alpha_{n+1} - \alpha_n &= \frac{\theta}{\omega} \Delta, \\ X_{n+1} - X_n &= \frac{\omega}{\theta} (\sin(\alpha_{n+1}) - \sin(\alpha_n)), \\ Y_{n+1} - Y_n &= -\frac{\omega}{\theta} (\cos(\alpha_{n+1}) - \cos(\alpha_n)). \end{aligned}$$

One can transform the system of difference equations into a system of ordinary differential equations where the independent variable T is such that $T_{n+1} - T_n = \Delta$. Let us denote $\xi = \theta/\omega$. Then, as $\Delta \rightarrow 0$, one gets:

$$\alpha' = \xi, \quad X' = \cos(\alpha), \quad Y' = \cos(\alpha). \quad (2.2)$$

Here $\xi = \theta/\omega$ is the dimensionless control parameter, namely, the ratio between the velocity of rotation and the velocity of motion along the trajectory of the robot.

The system of equations (2.2) is well known. For a derivation and the use of the system for analysis of motion in a somewhat different form, see [2, 3].

3. Feedback control

Let us consider the feedback control for straight-line motion. Typically one considers a proportional-integral-derivative controller. In our case, such a controller should define the parameter θ , which is the angular velocity of the robot.

For definiteness we use Cartesian coordinates X , Y for the robot such that the axis OX is directed along the line of motion of the robot. The axis OY is perpendicular to the line of motion. In this coordinate system the curve of motion is defined by the equation $Y = 0$ [4].

There exist various control systems for such mobile robots. Below we consider proportional and proportional-derivative controllers.

We do not consider the integral control in this work. The integral part of the PID-controller is helpful for a curvilinear path when the proportional part of the PID-controller is insufficient for part of the path with a small radius of rotation. In this case the integral part of the PID-controller accumulates changes within several control steps. But the typical property of a fast robot is that it has a small frequency of control steps and the integral part does not have time to accumulate sufficient value.

A proportional feedback controller is the simplest and the most frequently used controller. Usually it operates using a deviation of the path with respect to the middle of the line sensor.

If the line sensor shows that the deviation is equal to Z , then the value of the control parameter θ is

$$\theta = \kappa_p Z.$$

Here $\kappa_p > 0$ is the coefficient of feedback control. The range of values of the coefficient is limited by the dynamical properties of the gearmotor and this range primarily depends on torque. Below we will consider the value of the coefficient $\kappa_p = O(1)$.

A formula for the value of the deviation Z from the middle of the line sensor can be obtained using the following geometrical property:

$$KZ \cos(\alpha) + KY + A \sin(\alpha) = 0. \quad (3.1)$$

Then the recurrent system for the coordinates of the robot has the form

$$\begin{aligned} \alpha_1 - \alpha_0 &= -\frac{\kappa_p}{\omega} \Delta \left(\frac{Y_0}{\cos(\alpha_0)} + \frac{A}{K} \frac{\sin(\alpha_0)}{\cos(\alpha_0)} \right), \\ X_1 - X_0 &= \frac{\omega}{\kappa_p} \frac{(\sin(\alpha_1) - \sin(\alpha_0)) \cos(\alpha_0)}{\left(Y_0 + \frac{A}{K} \sin(\alpha_0) \right)}, \\ Y_1 - Y_0 &= -\frac{\omega}{\kappa_p} \frac{(\cos(\alpha_1) - \cos(\alpha_0)) \cos(\alpha_0)}{\left(Y_0 + \frac{A}{K} \sin(\alpha_0) \right)}. \end{aligned} \quad (3.2)$$

Here one should consider ω/κ_p as a parameter of this system.

The system of ordinary differential equations for this feedback control has the form

$$\alpha' = -\frac{\kappa_p}{\omega} \left(\frac{Y}{\cos(\alpha)} + \frac{A}{K} \frac{\sin(\alpha)}{\cos(\alpha)} \right), \quad X' = \cos(\alpha), \quad Y' = \sin(\alpha). \quad (3.3)$$

One can rewrite this system as a linear differential second-order equation:

$$Y'' = -\frac{\kappa_p}{\omega} \left(Y + \frac{A}{K} Y' \right). \quad (3.4)$$

Making a change of the independent variable $\tau = T\sqrt{\kappa/\omega}$ and using a new parameter $\mu = (A/K)\sqrt{\kappa/\omega}$, we obtain

$$Y'' + \mu Y' + Y = 0.$$

The characteristic equation has the form

$$\lambda^2 + \mu\lambda + 1 = 0.$$

The real parts of the roots of this equation are $\operatorname{Re}(\lambda_{1,2}) < 0$. Therefore, the zero solution of this equation is asymptotically stable. Then we have the following theorem.

Theorem 1. *The solution of the system (3.3) $\alpha = 0$, $Y = 0$ and $X = \tau$ is asymptotically stable.*

4. Asymptotic properties of motion using feedback control

For fast linear velocities, that is, for $\mu \rightarrow 0$, the real part of the roots of the characteristic equation is small and the stability property decreases. Indeed,

$$k_1 \sim i - \frac{\mu}{2}, \quad k_2 \sim -i - \frac{\mu}{2}, \quad \mu \rightarrow 0.$$

The system with feedback control has a delay because of processing. Let us define a typical delay as δ . Then the shift with respect to the middle of the line sensor is known for the time value $t - \delta$. In terms of the variable T one gets $\Delta = (D/2K)\delta\omega$. In such cases one usually considers systems with delay instead of (2.2), see [5]. However, in our case of large values of ω , the step of discretization Δ is not small. As a result, one has to use a discrete map (3.2) instead of ordinary differential equations with delay.

The current map for X should be considered separately. The maps for α and Y are linked. Therefore, we can consider the map:

$$\alpha_{n+1} = \alpha_n + \frac{\kappa_p}{\omega} Z_n \Delta, \quad (4.1)$$

$$Y_{n+1} = Y_n - \frac{\omega}{\kappa_p} \frac{\left(\cos \left(\alpha + \frac{\kappa_p}{\omega} Z_n \Delta \right) - \cos(\alpha_n) \right)}{Z_n}, \quad (4.2)$$

where

$$Z_n = -\frac{Y + \frac{A}{K} \sin(\alpha_n)}{\cos(\alpha_n)}.$$

Let us obtain an asymptotic behavior for this map in a neighborhood of $(\alpha, y) = (0, 0)$:

$$\begin{pmatrix} \alpha_{n+1} \\ Y_{n+1} \end{pmatrix} \sim \begin{pmatrix} 1 - \frac{\kappa_p A}{\omega K} \Delta & -\frac{\kappa_p}{\omega} \Delta \\ \Delta - \frac{\kappa_p A}{\omega K} \frac{\Delta^2}{2} & 1 - \frac{\kappa_p}{\omega} \frac{\Delta^2}{2} \end{pmatrix} \begin{pmatrix} \alpha_{n+1} \\ Y_{n+1} \end{pmatrix}.$$

Let us consider quadratic form:

$$F_n = \alpha_n^2 + \frac{\kappa_p}{\omega} Y_n^2.$$

One can obtain an asymptotic approximation when $\omega \rightarrow \infty$:

$$F_{n+1} \sim F_n - \frac{2A\kappa_p}{K\omega} \Delta \alpha_n^2 + \frac{\kappa_p}{\omega} \Delta^2 \alpha_n^2.$$

Therefore if

$$K\Delta - 2A < 0,$$

then $F_{n+1} < F_n$.

Fig. 2. Stable and unstable straight-line motion under feedback control with different values of ω . The geometric dimensions of the robot are: $D = 0.05$, $K = 0.1$, $A = 0.2$. The step over time is $\delta = 0.01$. The coefficient of proportional control is $\kappa_p = 1$.

The formal asymptotic expansion shows that inequality is a necessary condition of stability for straightforward motion. Let us rewrite the condition in terms of δ . We obtain that the straightforward motion under feedback control is stable for

$$\omega < \frac{4A}{D\delta}.$$

The stability does not depend on the proportional coefficient of feedback control κ_p .

The results for the discrete map are shown in Figure 2.

The formula for F_{n+1} implies that the speed of convergence for the map to zero depends on the coefficient of the proportional feedback control κ_p . An optimal value of this coefficient can be found by optimizing the path. This is a variational problem and is not considered in this work.

5. Motion under proportional-derivative feedback control

Additional damping is a general approach to stabilize oscillations. In control systems, such an approach is applied by adding a derivative term to the feedback. In our case, such an additional term is the angular velocity:

$$\begin{aligned} \frac{d\alpha}{dT} &= \frac{\kappa_p}{\omega} Z + \frac{\kappa_d}{\omega} \frac{dZ}{dT}, \\ \frac{dX}{dT} &= \cos(\alpha), \quad \frac{dY}{dT} = \sin(\alpha). \end{aligned}$$

Here $\kappa_d > 0$ is the coefficient of derivative control.

Let us differentiate the geometric formula (3.1) to obtain an expression for $(dZ)/(dT)$. This yields:

$$\frac{dZ}{dT} \cos(\alpha) - Z \frac{d\alpha}{dT} \sin(\alpha) + \omega \sin(\alpha) + \frac{A}{K} \frac{d\alpha}{dT} \cos(\alpha) = 0.$$

Then the system of equations for α and Y has the form

$$\frac{d\alpha}{dT} = -\frac{\kappa_p}{\omega} \frac{\left(Y(T) + \frac{2A}{D} \sin(\alpha)\right)}{\cos(\alpha)} - \frac{\kappa_d}{\omega} \left(\left(Y(T) + \frac{2A}{D} \sin(\alpha)\right) \frac{d\alpha}{dT} \frac{\sin(\alpha)}{\cos(\alpha)} - \frac{\sin(\alpha)}{\cos(\alpha)} - \frac{2A}{D} \frac{d\alpha}{dT} \right),$$

$$\frac{dY}{dT} = \sin(\alpha).$$

To study the stability of the zero solution to this system, one can rewrite it as a differential second-order equation for α and obtain a linearized part of the equation in a neighborhood of zero. As a result, one obtains

$$\frac{d^2}{d\tau^2} a = - \left(\frac{d}{d\tau} a \right) k_1 \mu - \left(\frac{d}{d\tau} a \right) \mu - a,$$

where $\tau = \sqrt{\omega/\kappa_p} T$, $k_1 = \kappa_d K / (\kappa_p A)$, $\mu = \sqrt{\kappa_p/\omega} A / K$.

A characteristic equation for the second-order equation for a has the form

$$\lambda^2 = -(k_1 + 1)\mu\lambda - 1. \quad (5.1)$$

If $\kappa_1 > 0$ and $\mu > 0$, then the solution of this equation has a negative real part.

One can see that the zero is an asymptotic stable solution for a system of equations for α , Y .

Theorem 2. *The solution of a system of equations for straight-line motion under proportional derivative feedback control is asymptotically stable.*

6. Asymptotic properties of motion under proportional derivative feedback control

Let us consider solutions for a large speed, that is, for $\mu \rightarrow 0$. In this case, the real parts of the roots for the characteristic equation (5.1) have asymptotic behavior:

$$k_1 \sim i - \frac{\mu}{2}(1 + k_1), \quad k_2 \sim -i - \frac{\mu}{2}(1 + k_1), \quad \mu \rightarrow 0.$$

The real part of $\lambda_{1,2}$ is negative, but small. Hence the robot is close to instability when the speed is large.

The approximation of maps (2.1) by the system of differential equations (2.2) is not appropriate because the value of Δ is not small when ω is large. This leads to discrete maps instead of differential equations:

$$\alpha_{n+1} = \alpha_n + \frac{\kappa_p}{\omega} Z_n \Delta + \frac{\kappa_d}{\omega} (Z_n - Z_{n-1}),$$

$$X_{n+1} = X_n + \frac{\omega}{\kappa_p} \frac{\left(\sin \left(\alpha_n + \frac{\kappa_p}{\omega} Z_n \Delta + \frac{\kappa_d}{\omega} (Z_n - Z_{n-1}) \right) - \sin(\alpha_n) \right)}{Z_n + \frac{\kappa_d}{\kappa_p} \frac{(Z_n - Z_{n-1})}{\Delta}},$$

$$Y_{n+1} = Y_n - \frac{\omega}{\kappa_p} \frac{\left(\cos \left(\alpha_n + \frac{\kappa_p}{\omega} Z_n \Delta + \frac{\kappa_d}{\omega} (Z_n - Z_{n-1}) \right) - \cos(\alpha_n) \right)}{Z_n + \frac{\kappa_d}{\kappa_p} \frac{(Z_n - Z_{n-1})}{\Delta}}.$$

Fig. 3. Here one can see stable and unstable motions under proportional feedback control for different values of ω . The dimensions of the robot are: $D = 0.05$, $K = 0.1$, $A = 0.2$. The step of time discretization is $\delta = 0.01$. The coefficient of proportional feedback is $\kappa_p = 1$ and the coefficient of derivative feedback is $\kappa_d = 0.5$.

Numeric experiments show that the condition for stability of the discrete map is

$$\omega < \frac{4A}{D\delta} \left(1 + \frac{\kappa_d K}{\kappa_p A} \right).$$

Results for the discrete map are shown in Figure 3.

The stability condition for proportional derivative feedback control contains a relation of coefficients for proportional and derivative terms of the feedback. This condition does not answer the question of the optimal value of coefficients for feedback control. The search for the optimal values is an important problem, which requires additional studies.

7. Conclusions

The system of control gives kinematic stability of straight-line motion under proportional feedback when

$$\frac{4A}{D\delta} > \omega,$$

where D is the diameter of wheels; A is the distance from the line sensor, δ is the delay of the control, and ω is half the sum of the angular velocities of the wheels.

The condition for stability of straight-line motion under proportional derivative feedback looks like this:

$$\frac{4A}{D\delta} + \frac{4\kappa_d K}{\kappa_p D\delta} > \omega,$$

where $0 < \kappa_d$ is the coefficient of derivative feedback and $0 < \kappa_p$ is the coefficient of proportional feedback.

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