MATHEMATICAL PROBLEMS OF NONLINEARITY

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Armbruster – Guckenheimer – Kim Hamiltonian System in 1:1 Resonance

M. Alvarez-Ramírez, A. García, J. Vidarte

This article deals with the autonomous two-degree-of-freedom Hamiltonian system with Armbruster – Guckenheimer – Kim galactic potential in 1:1 resonance depending on two parameters. We detect periodic solutions and KAM 2-tori arising from linearly stable periodic solutions not found in earlier papers. These are established by using reduction, normalization, averaging and KAM techniques.

Keywords: galactic potential, Hamiltonian system, normalization and reduction, KAM tori, reduced space, periodic orbits

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Martha Alvarez-Ramírez
mar@xanum.uam.mx, malvarez66@gmail.com
Antonio García
agar@xanum.uam.mx
Departamento de Matemáticas, UAM–Iztapalapa
09340 Iztapalapa, Ciudad de México, México

Jhon Vidarte
jhon.vidarte@ucsc.cl
Departamento de Matemática y Física Aplicadas
Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile

1. Setting of the problem

In this paper we study the Hamiltonian with Armbruster–Guckenheimer–Kim (shortly, AGK-Hamiltonian) potential:

$$\mathcal{H}(x, y) = \frac{1}{2}(y_1^2 + y_2^2) - \frac{\mu}{2}(x_1^2 + x_2^2) - \frac{a}{4}(x_1^2 + x_2^2)^2 - \frac{b}{2}x_1^2x_2^2, \quad (1.1)$$

near the fixed point $(0, 0, 0, 0)$, where $\mu < 0$, $a \neq 0$ and b are real parameters.

In 1989, Simonelli and Gollub [21] carried out an experimental work with surface waves in square and rectangular containers subject to vertical oscillations. Their methods allowed stable and unstable fixed points (sinks, sources or saddles) and bifurcation sequences. Motivated by these experiments, Armbruster et al. [3] derived a dynamical model starting with a normal form given by four first-order differential equations depending on several parameters. Their system provided a general description of a codimension-two bifurcation problem. To simplify the problem, they performed a rescaling of the variables and parameters. When the scaling parameter was set to zero, the differential equations became the AGK-Hamiltonian system, where an important feature is its D_4 -symmetry. They found a large parameter region of chaotic behavior and argued that the existence of chaos in a given region suggested that the Hamiltonian is nonintegrable. In [2], Andrle proposed a similar potential AGK-Hamiltonian as a dynamical model for a stellar system with an axis and a plane of symmetry.

Many studies have been focused on the study of the dynamics of AGK-Hamiltonian for several values of the parameters. Without being exhaustive, some interesting works are the following. Llibre and Roberto [12] used classical averaging theory of first order to examine the $\mathcal{C}^1$ nonintegrability in the sense of Liouville and Arnold, and also studied the existence of periodic orbits. In [7], Elmandou considered some dynamical aspects for the AGK potential in a rotating reference frame and examined the nonintegrability by means of Painlevé analysis. Additionally, he employed the Lyapunov method to seek periodic solutions near the equilibrium positions. Acosta-Humánez et al. [1] applied Morales–Ramis theory to prove that the AGK-Hamiltonian is rationally integrable if and only if $b = 0$, $a = -b$ or $b = 2a$. In fact, these are the only Liouville integrable cases. El-Sabaa et al. [8] made a topological study of the integral manifolds for the integrable cases $b = 2a$ and $b = -a$, finding the structure of the bifurcation of the level sets. As expected, if bifurcations of Liouville tori take place, the level set of the first integrals becomes degenerate, and the existence of families of periodic solutions is possible, and they were given in terms of Jacobi’s elliptic functions. In addition, phase portraits associated to integrable systems are shown. However, the complete description of the dynamics of both integrable cases is not given. The gaps in the aforementioned work were completed by Llibre and Valls in [13], who obtained and characterized the full global flow.

The goal of this paper is to provide results on the existence of periodic solutions, their stability and KAM 2-tori for $\mu < 0$, $a, b \in \mathbb{R}$ and positive energy in a neighborhood of the origin. This is achieved by performing an explosion of the origin and a combination of different techniques such as normal forms, symplectic reduction and averaging theories. The key idea of our analysis is that the origin is an elliptic fixed point of the Hamiltonian system associated to (1.1), and that can be seen as a quartic perturbation of two coupled oscillators in 1:1 resonance. We are able to detect periodic solutions and KAM 2-tori which were not reported in previous works. This is done in a set of coordinates that fits better in the problem and allows one to avoid certain technicalities of other approaches related to the 1:1 resonance. We have used *Mathematica* software to perform all calculations and to make the figures that are essential in the understanding of the results.

We would like to point out that Deprit [5] introduced the so-called Lissajous variables which deal with perturbations of coupled oscillators in 1-1 resonance. Later on, Deprit and Elipe [6] used the Lissajous transformation in the study of perturbed elliptic oscillators shown by an example where the perturbing term is a quartic polynomial. They proved that the reduced phase space is a two-dimensional sphere, where each point represents an orbit. Indeed, the potential in the Hamiltonian (1.1) is a quartic polynomial, however, we have chosen to use a similar, but different, approach, which can be preferable when applying theorems given in the Appendix. Therefore, we will draw the same conclusions as those in [6].

The outline of this paper is as follows. In Section 2 we perform a symplectic scaling of the AGK Hamiltonian system for $\mu < 0$ by introducing the small positive parameter ε . Then we do several symplectic reductions and normalizations in order to put the Hamiltonian $\mathcal{H}_0 + \varepsilon^2 \mathcal{H}_2$ and the Hopf map in the convenient coordinates (L, Q, ℓ, P) . Now the case $\varepsilon = 0$ is studied and it ends with the averaged Hamiltonian. In Section 3 we consider this in terms of the coordinates (Q, P) . We obtain conditions on the parameters a and b for the existence of the nondegenerate critical points, as well as their linear stability. Section 4 is devoted to the use of KAM theory to establish the persistence of KAM 2-tori in the full system, and the elliptic periodic solutions, more precisely, we use the Han, Li and Yi theorem (Theorem 7 in Appendix) that is valid for highly degenerate Hamiltonian systems. Here there is also the application of the Reeb theorem (Theorem 5 in the Appendix) to find that each nondegenerate critical point gives rise to a family of periodic solutions of the full system. In order to make this paper self-contained, we decided to include an appendix. As far as the main theorems are concerned, we present results on averaging theory and KAM theory and stability for Hamiltonian systems.

2. Normalization and reduction

In this section we go through processes of reduction and averaging in order to enlighten the properties of the AGK-Hamiltonian near the origin.

In the AGK-Hamiltonian (1.1), we scale the variables by

$$\begin{aligned} x_1 &\rightarrow \varepsilon(-\mu)^{-1/4}x_1, & x_2 &\rightarrow \varepsilon(-\mu)^{-1/4}x_2, \\ y_1 &\rightarrow \varepsilon(-\mu)^{1/4}y_1, & y_2 &\rightarrow \varepsilon(-\mu)^{1/4}y_2, \end{aligned} \quad (2.1)$$

where ε is a positive small parameter; this is symplectic with multiplier ε^2 . Applying this change of coordinates to the Hamiltonian (1.1), it assumes the form

$$\mathcal{H}_\varepsilon(x, y) = \mathcal{H}_0(x, y) + \varepsilon^2 \mathcal{H}_2(x), \quad (2.2)$$

after rescaling time by $dt \rightarrow (-\mu)^{-1/2}dt$ and resetting $(-\mu)^{-3/2}\varepsilon^2 \rightarrow \varepsilon^2$; where

$$\mathcal{H}_0 = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) \quad (2.3)$$

and

$$\mathcal{H}_2 = -\left(\frac{a}{4}(x_1^2 + x_2^2)^2 + \frac{b}{2}x_1^2x_2^2\right). \quad (2.4)$$

Thus, we have a Hamiltonian perturbation of two harmonic oscillators in 1:1 resonance.

For a moment, we are going to focus on the unperturbed case $\varepsilon = 0$. The equations of motion derived from $\mathcal{H}_0$ are

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -x_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_2 = -y_1.$$

We recall that every solution of this system is periodic with period 2π .

In order to describe the orbit space of (2.3), we fix the value $\mathcal{H}_0 = h \in \mathbb{R}^+$, such that it describes the 3-sphere of radius $\sqrt{2h}$ given by

$$M_h = \{(x_1, x_2, y_1, y_2) : x_1^2 + x_2^2 + y_1^2 + y_2^2 = 2h\}.$$

Thus, we localize around the origin by performing three different changes of symplectic variables. First, we use action-angle variables $(I, \phi) = (I_1, I_2, \phi_1, \phi_2)$

$$\begin{aligned} x_1 &= \sqrt{2I_1} \cos \phi_1, & x_2 &= \sqrt{2I_2} \cos \phi_2, \\ y_1 &= \sqrt{2I_1} \sin \phi_1, & y_2 &= \sqrt{2I_2} \sin \phi_2, \end{aligned} \quad (2.5)$$

where $0 \leq I_1, I_2 \leq h$ and $\mathcal{H}_0 = I_1 + I_2$. We see that $I_1 = \frac{1}{2}(x_1^2 + y_1^2)$ and $I_2 = \frac{1}{2}(x_2^2 + y_2^2)$ are first integrals and not simultaneously equal to zero. As the next step, we introduce the linear symplectic change of variables (L, J, ℓ, θ) through the transformation given by

$$L = I_1 + I_2, \quad J = I_2, \quad \ell = \phi_1, \quad \theta = \phi_2 - \phi_1,$$

where $0 \leq \ell, \theta < 2\pi$ and $0 \leq J \leq h$. In this symplectic transformation, the unperturbed Hamiltonian (2.3) is converted into $\mathcal{H}_0 = L$.

Now we introduce the rectangular symplectic coordinates by (J, θ) through the transformation

$$Q = \sqrt{2J} \cos \theta, \quad P = \sqrt{2J} \sin \theta, \quad (2.6)$$

where $0 \leq Q^2 + P^2 = 2J \leq 2L = 2h$. The Hamiltonian function of the harmonic oscillator now reads $\mathcal{H}_0 = L$.

In addition, if $x_1^2 + y_1^2 \neq 0$, then $0 \leq Q^2 + P^2 < 2L = 2h$. Finally, we summarize all the previous successive changes of coordinates as the transformation $T_1: \Omega \rightarrow M_h$ defined by

$$\begin{aligned} x_1 &= \sqrt{2L - Q^2 - P^2} \cos \ell, & y_1 &= \sqrt{2L - Q^2 - P^2} \sin \ell, \\ x_2 &= Q \cos \ell - P \sin \ell, & y_2 &= P \cos \ell + Q \sin \ell, \end{aligned} \quad (2.7)$$

where $\Omega = \{(L, Q, \ell, P) : Q^2 + P^2 < 2L, \ell \in \mathbb{S}^1\}$. We stress the fact that the image of T_1 excludes the set $x_1^2 + y_1^2 = 0$. In this case we have that $x_2^2 + y_2^2 \neq 0$ and can perform a similar coordinate change, namely, $T_2: \Omega \rightarrow M_h$ defined by

$$\begin{aligned} x_1 &= Q \cos \ell - P \sin \ell, & y_1 &= P \cos \ell + Q \sin \ell, \\ x_2 &= \sqrt{2L - Q^2 - P^2} \cos \ell, & y_2 &= \sqrt{2L - Q^2 - P^2} \sin \ell. \end{aligned} \quad (2.8)$$

We have that $T_2 = R \circ T_1$, where $R(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$. Hence, the set

$$A = \{T_1: \Omega \rightarrow M_h, \quad T_2: \Omega \rightarrow M_h\}$$

becomes an atlas of symplectic charts, and therefore M_h is a symplectic manifold.

The inverse of the transformations T_1 and T_2 are expressed, respectively, as

$$\begin{aligned} L &= \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2), & \ell &= \arctan\left(\frac{y_1}{x_1}\right), \\ Q &= \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}}, & P &= \frac{x_1y_2 - x_2y_1}{\sqrt{x_1^2 + y_1^2}}, \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} L &= \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2), & \ell &= \arctan\left(\frac{y_2}{x_2}\right), \\ Q &= \frac{x_1x_2 + y_1y_2}{\sqrt{x_2^2 + y_2^2}}, & P &= -\frac{x_1y_2 - x_2y_1}{\sqrt{x_2^2 + y_2^2}}. \end{aligned} \quad (2.10)$$

The Hamiltonian in both charts has the form $\mathcal{H}_0 = L$. So, the equations of motion derived from it are

$$\dot{L} = 0, \quad \dot{Q} = 0, \quad \dot{\ell} = -1, \quad \dot{P} = 0. \quad (2.11)$$

Hence, $\ell(t) = -t + \ell_0$ and L , Q and P are constant. Since ℓ is an angle variable, these solutions are 2π -periodic.

For our purpose we need to reduce the phase space of the two-dimensional harmonic oscillator by using the symmetry given by the S^1 -action generated by the flow of the Hamiltonian (2.3). Keeping $\mathcal{H}_0 = h$ fixed, we address this issue for the Hopf fibration given by the map

$$\Pi: M_h \equiv \mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad (2.12)$$

where the great circles of $\mathbb{S}^3$ correspond one-to-one to points on the two-dimensional sphere $\mathbb{S}^2$. Thus, the orbit mapping becomes a symplectic reduction of the surface $M_h \equiv \mathbb{S}^3$ to the reduced phase space $\mathbb{S}^2$. It is given explicitly as follows:

$$\begin{aligned} \Pi: \mathbb{S}^3 &\rightarrow \mathbb{S}^2, \\ (x_1, x_2, y_1, y_2) &\rightarrow (w_1, w_2, w_3), \end{aligned} \quad (2.13)$$

where

$$w_1 = 2(x_1y_2 - x_2y_1), \quad w_2 = 2(x_1x_2 + y_1y_2), \quad w_3 = x_1^2 + y_1^2 - x_2^2 - y_2^2, \quad (2.14)$$

and satisfies the relation

$$(x_1^2 + x_2^2 + y_1^2 + y_2^2)^2 = w_1^2 + w_2^2 + w_3^2 = 4h^2. \quad (2.15)$$

For this, we refer the reader to [16].

Hence, relations (2.9) and (2.10) imply that the coordinates Q and P in terms of (2.14) are transformed to

$$Q = \frac{w_2}{\sqrt{2(w_3 + 2h)}}, \quad P = \frac{w_1}{\sqrt{2(w_3 + 2h)}}, \quad (2.16)$$

or

$$Q = \frac{w_2}{\sqrt{2(2h - w_3)}}, \quad P = -\frac{w_1}{\sqrt{2(2h - w_3)}}. \quad (2.17)$$

Let $U_1 = \mathbb{S}^2 \setminus \{(0, 0, -2h)\}$ and $U_2 = \mathbb{S}^2 \setminus \{(0, 0, 2h)\}$. It follows that

$$\begin{aligned} \psi_j: \quad U_j &\rightarrow \mathbb{R}^2 & j &= 1, 2 \\ (w_1, w_2, w_3) &\rightarrow (Q, P) \end{aligned}$$

are local symplectic charts for the reduced space $\mathbb{S}^2$, respectively.

We shall specify an atlas $\tilde{A}$ on the 2-sphere with two charts defined as the composition of the inclusion $(L, Q, P) \rightarrow (L, Q, 0, P)$

$$\tilde{T}_1: \tilde{\Omega} \rightarrow \mathbb{S}^2, \quad \tilde{T}_2: \tilde{\Omega} \rightarrow \mathbb{S}^2,$$

where $\tilde{\Omega} = \{(L, Q, P): Q^2 + P^2 < 2h \text{ and } L = h\}$. Then locally the Hopf map is

$$\begin{aligned} \Pi: \Omega &\rightarrow \Omega_1, \\ (L, Q, \ell, P) &\rightarrow (L, Q, P). \end{aligned} \quad (2.18)$$

Indeed, periodic orbits of the system (2.11) are sent to orbits in $\mathbb{S}^2$, namely, fixed points.

It is worth noting that the changes of coordinates (2.7) and (2.8) correspond to a particular case of the 1:1 resonance in local symplectic maps for resonant Hamiltonian systems with n degrees of freedom (for more details, see [5] and [17]).

Now we come back to the analysis of the Hamiltonian $\mathcal{H}_\varepsilon$, which is a smooth Hamiltonian perturbation of two harmonic oscillators in 1:1 resonance by a quartic potential. Let us observe that the parameter ε measures the smallness of the quartic perturbation. The sets $M_{h,\varepsilon} = \{(x_1, x_2, y_1, y_2): \mathcal{H}_\varepsilon = h\}$, where $h > 0$ and $\varepsilon \geq 0$ is small enough, are invariant manifolds.

It follows that there is a homotopy $R: M_h \times [0, \varepsilon_0] \rightarrow \mathbb{R}^4$ such that for each $0 \leq \varepsilon \leq \varepsilon_0$ the map $R_\varepsilon = R(\cdot, \varepsilon)$ is a diffeomorphism between the 3-sphere M_h and $M_{h,\varepsilon}$. Moreover, the set $A_\varepsilon = \{\phi_1: \Omega \rightarrow M_h, \phi_2: \Omega \rightarrow M_h\}$, where $\phi_k = R_\varepsilon \circ T_k$, is an atlas of $M_{h,\varepsilon}$, see [9, 11].

In each chart of A_ε , the Hamiltonian $\mathcal{H}_\varepsilon$ reads as

$$\mathcal{H}_\varepsilon = L + \varepsilon^2 \mathcal{H}_2(L, Q, \ell, P). \quad (2.19)$$

We can eliminate the angle ℓ from the Hamiltonian to a certain order with the aid of the Lie transformation method. We end up to order four in the small parameter with the following Hamiltonian:

$$\mathcal{H}_\varepsilon = L + \varepsilon^2 \overline{\mathcal{H}}_2(L, Q, P) + O(\varepsilon^4), \quad (2.20)$$

where

$$\overline{\mathcal{H}}_2 = -\frac{1}{16} [6aL^2 + (2L - Q^2 - P^2)(3bQ^2 + (b - 2a)P^2)], \quad (2.21)$$

see [14, 16]. We observe that $\dot{L} = O(\varepsilon^4)$ and L is not longer a constant, but its variation is very small. Therefore, the map (2.12), which in the charts of A_ε is (2.18), does not send orbits into orbits.

3. The averaged Hamiltonian

Let $\tilde{H}_\varepsilon = L + \varepsilon^2 \overline{\mathcal{H}}_2$ be the Hamiltonian equal to $\mathcal{H}_\varepsilon$ up to terms of order four in ε . We notice that $\overline{\mathcal{H}}_2$ is independent of ℓ , and thus its conjugate variable L is constant.

In order to obtain the Hamiltonian $\tilde{H}_\varepsilon$ on the reduced space $\mathbb{S}^2$, we replace L by $h > 0$ in (2.21), eliminate the factor $1/16$ by scaling the time, and remove the constant term $-6aL^2$, arriving at

$$\hat{\mathcal{H}} = -(2h - Q^2 - P^2)[3bQ^2 + (b - 2a)P^2]. \quad (3.1)$$

So the equations of motion become

$$\begin{aligned} \dot{Q} &= 4P[(2b - a)Q^2 + (2a - b)(h - P^2)], \\ \dot{P} &= 4Q[3b(h - Q^2) + (a - 2b)P^2]. \end{aligned} \quad (3.2)$$

Proposition 1. *Let $\delta = 2a/b$. The system (3.2) has the following critical points on $\mathbb{S}^2$ (see Table 1):*

1. *For $\delta \in \mathbb{R} \setminus \{-2, 1\}$, there are six isolated critical points, $\mathcal{O}_j$, $j = 1, \dots, 6$.*
2. *For $\delta = -2$, there are two isolated critical points: $\mathcal{O}_1$ and $\mathcal{O}_2$. In addition, there is a continuous set of critical points given by $Q^2 + P^2 = h$.*
3. *For $\delta = 1$, there are two isolated critical points: $\mathcal{O}_3$ and $\mathcal{O}_4$. Additionally, the line $Q = 0$ is a continuous set of critical points.*
4. *For $b = 0$ there are two isolated critical points: $\mathcal{O}_5$ and $\mathcal{O}_6$. Moreover, the line $P = 0$ is a continuous set of critical points.*

Proof. We recall that the system (3.2) is the flow in local coordinates. A direct computation shows that for $\delta \in \mathbb{R} \setminus \{-2, 1\}$ the system (3.2) has nine critical points, but only five of them satisfy the condition $Q^2 + P^2 < 2h$. In addition, the point $(0, 0)$ corresponds to different critical points in the flow on $\mathbb{S}^2$ and the other four appear in both charts, therefore there are six critical points.

The rest of the statement is obtained in a similar way. $\square$

Table 1. Critical points of $\hat{\mathcal{H}}$ in the variables (Q, P) and in $\mathbb{S}^2$

Critical point	$\tilde{T}_1$	$\tilde{T}_2$	$\mathbb{S}^2$
$\mathcal{O}_1$	$(0, 0)$	—	$(0, 0, 2h)$
$\mathcal{O}_2$	—	$(0, 0)$	$(0, 0, -2h)$
$\mathcal{O}_3$	$(\sqrt{h}, 0)$	$(\sqrt{h}, 0)$	$(0, 2h, 0)$
$\mathcal{O}_4$	$(-\sqrt{h}, 0)$	$(-\sqrt{h}, 0)$	$(0, -2h, 0)$
$\mathcal{O}_5$	$(0, \sqrt{h})$	$(0, -\sqrt{h})$	$(2h, 0, 0)$
$\mathcal{O}_6$	$(0, -\sqrt{h})$	$(0, \sqrt{h})$	$(-2h, 0, 0)$

The number of isolated critical points of the system (3.2) in the plane of parameters a and b is shown in Fig. 1.

REMARK 1. Each critical point $\mathcal{O}_j$ on $\mathbb{S}^2$ is associated with a periodic orbit on $\mathbb{S}^3$ parameterized by ℓ . In fact, from (2.12), this curve is $\Pi^{-1}(\mathcal{O}_j)$. A direct computation shows the projection of $\Pi^{-1}(\mathcal{O}_j)$ on the configuration plane $x_1 x_2$, see Table 2.

Table 2. Critical points and their projection onto the configuration plane $x_1 x_2$

Critical point	projection of $\Pi^{-1}(\mathcal{O}_j)$ on the configuration plane $x_1 x_2$
$\mathcal{O}_1$	segment in the x_1 axis
$\mathcal{O}_2$	segment in the x_2 axis
$\mathcal{O}_3$	segment of line outside the coordinate axes
$\mathcal{O}_4$	segment of line outside the coordinate axes
$\mathcal{O}_5$	circles with center at the origin and radius $\sqrt{h}$
$\mathcal{O}_6$	circles with center at the origin and radius $\sqrt{h}$

Fig. 1. Number of isolated critical points of the system (3.2) in the plane of parameters a and b . The green and orange lines correspond to $b = 0$ and $2a - b = 0$ ($\delta = 1$), respectively. The blue line is $a + b = 0$ ($\delta = -2$).

Now, for convenience, we define the sets of critical points

$$\mathcal{R} = \{\mathcal{O}_1, \mathcal{O}_2\}, \quad \mathcal{M} = \{\mathcal{O}_3, \mathcal{O}_4\}, \quad \mathcal{C} = \{\mathcal{O}_5, \mathcal{O}_6\}.$$

The next result comprises the information on the stability of the critical points.

Proposition 2. *Under the assumptions of Proposition 1, the linear stability of the critical points is as follows:*

1. *The critical points in $\mathcal{R}$ are stable for $\delta < 1$ and unstable for $\delta > 1$.*
2. *The critical points in $\mathcal{M}$ are stable for $\delta > -2$ and unstable for $\delta < -2$.*
3. *The critical points in $\mathcal{C}$ are stable for $\delta < -2$ or $\delta > 1$ and unstable for $-2 < \delta < 1$.*
4. *For $b = 0$ and $a \neq 0$ the critical points in $\mathcal{C}$ are stable.*

Proof. Let $A = \mathbb{J}D^2\widehat{\mathcal{H}}$, where $\mathbb{J}$ denotes the standard skew-symmetric matrix of order 2. Its characteristic polynomial is $p(\lambda) = \lambda^2 + \det A$. The theorem follows from Table 3. $\square$

The change in the number of critical points of the reduced Hamiltonian (3.1) is related to the emergence of bifurcation curves in terms of the parameters a and b , see Fig. 2.

REMARK 2. The reader should be warned that, at $\delta = \infty$ ($b = 0$), there is a degeneracy such that the flow is similar to that shown in the panel (g) of Fig. 2, but rotated through 90 degrees.

Table 3. Eigenvalues associated with the critical points

set of critical points	$\det A$	eigenvalue: λ
$\mathcal{R}$	$-48b^2h^2(\delta - 1)$	$\pm 4\sqrt{3} b h\sqrt{\delta - 1}$
$\mathcal{M}$	$48b^2h^2(\delta + 2)$	$\pm 4\sqrt{3} b h\sqrt{-(\delta + 2)}$
$\mathcal{C}$ $\mathcal{C}$ with $b = 0$	$16b^2h^2(\delta - 1)(\delta + 2)$ $64h^2a^2$	$\pm 4 b h\sqrt{-(\delta - 1)(\delta + 2)}$ $\pm 8 a /hi$

Fig. 2. Flow evolution of the reduced Hamiltonian (3.1) in the local chart (U_1, ψ_1) . The red dots correspond to stable critical points (centers) and the blue points are unstable points (saddles). The thick brown line in panel (g) corresponds to a continuous set of critical points.

4. Reconstruction of the dynamics of the AGK-Hamiltonian from the averaged system

This section is devoted to obtaining conditions for nondegeneracy of the Hamiltonian (2.20) at the critical points that generate families of periodic orbits and KAM tori described in terms of the parameters.

4.1. Periodic orbits

Let $p(t, \varepsilon) = (x_1(t, \varepsilon), x_2(t, \varepsilon), y_1(t, \varepsilon), y_2(t, \varepsilon))$ be a solution of the system related to the Hamiltonian (2.2) and let $p^* = \Pi^{-1}(\mathcal{O}_j)$ denote the periodic solution associated with the critical point $\mathcal{O}_j$. Now we state our main result on the periodic solutions associated with the Hamiltonian (2.2).

Theorem 1. *For the system associated with the Hamiltonian (2.2), there are at most 6 $T(\varepsilon)$ -periodic solutions $p(t, \varepsilon)$ such that $p(t, 0) = p^*$ and $T(0) = 2\pi$. More precisely:*

1. *The periodic solutions generated by the critical points in $\mathcal{R}$ are linearly stable when $\delta < 1$ and unstable in the Lyapunov sense when $\delta > 1$.*
2. *The periodic solutions generated by the critical points in $\mathcal{M}$ are linearly stable when $\delta > -2$ and unstable in the Lyapunov sense when $\delta < -2$.*
3. *The periodic solutions generated by the critical points in $\mathcal{C}$ are linearly stable when $\delta < -2$ or $\delta > 1$ and unstable in the Lyapunov sense when $-2 < \delta < 1$.*
4. *If $b = 0$ and $a \neq 0$, there are two linearly stable periodic solutions generated by the critical points in $\mathcal{C}$.*

The period of the periodic solutions is given by $T(\varepsilon) = 2\pi(1 - \varepsilon^2 T^*) + O(\varepsilon^4)$, where the corrections T^* are listed in Table 4. Also, the characteristic multipliers of the periodic solutions are

$$1, 1, 1 \pm \varepsilon^2 \lambda_1 + O(\varepsilon^4), \quad \text{with } \lambda_1 = \frac{\lambda}{16},$$

where λ is one of the eigenvalues of $A = \mathbb{J}D^2\widehat{\mathcal{H}}$ given in Table 3.

Proof. Reeb's Theorem 5 implies that each critical point will rise to a family of periodic solutions provided that $\det(D^2\widehat{\mathcal{H}}) \neq 0$. In addition, the kind of stability of the periodic solution is inherited by the corresponding stability of the critical point. Hence, Propositions 1 and 2 imply the first part of the theorem. It remains to prove the last part concerning periods of the periodic orbits.

The Hamiltonian (2.20) yields

$$\dot{\ell} = \frac{\partial \mathcal{H}_\varepsilon}{\partial L} = 1 + \varepsilon^2 \frac{\partial \overline{\mathcal{H}}_2(L, Q, P)}{\partial L} + O(\varepsilon^4).$$

Let $\ell(0) = 0$ and $\ell(T) = 2\pi$, then T is the period of the orbit and

$$2\pi = \ell(T) - \ell(0) = \int_0^T \dot{\ell}(t) dt = T + \varepsilon^2 \int_0^T \frac{\partial \overline{\mathcal{H}}_2(L, Q, P)}{\partial L} dt + O(\varepsilon^4).$$

A straightforward use of the implicit function theorem shows that $T(\varepsilon) = 2\pi(1 - \varepsilon^2 T^*) + O(\varepsilon^4)$, where T^* is shown in Table 4. $\square$

Table 4. Corrections of the periods of the periodic solutions

Set of critical points	T^*
$\mathcal{R}$	$6bh\delta$
$\mathcal{M}$	$6bh(\delta + 1)$
$\mathcal{C}$	$2bh(2\delta + 1)$
$\mathcal{C}$ with $b = 0$	$8ah$

REMARK 3. It is worth noting that using the sets of symplectic coordinates (2.7) and (2.8), we can parameterize each periodic solution of the Hamiltonian (2.2) in terms of the angle $\ell = \ell(t)$. More precisely, if $p_j(t, \varepsilon) = (x_1(t, \varepsilon), x_2(t, \varepsilon), y_1(t, \varepsilon), y_2(t, \varepsilon))$ is the periodic solution associated to the critical point $\mathcal{O}_j$, then

$$\begin{aligned}x_1(\ell, \varepsilon) &= x_1^0(\ell) + \varepsilon^2 x_1^2(\ell) + O(\varepsilon^3), \\x_2(\ell, \varepsilon) &= x_2^0(\ell) + \varepsilon^2 x_2^2(\ell) + O(\varepsilon^3), \\y_1(\ell, \varepsilon) &= y_1^0(\ell) + \varepsilon^2 y_1^2(\ell) + O(\varepsilon^3), \\y_2(\ell, \varepsilon) &= y_2^0(\ell) + \varepsilon^2 y_2^2(\ell) + O(\varepsilon^3),\end{aligned}$$

where the expressions for the functions $x_k^j(\ell)$ and $y_k^j(\ell)$ appear in Tables 5–7.

Table 5. Expression for the functions $x_k^0(\ell)$ and $y_k^0(\ell)$, $k = 1, 2$, at the critical points

Critical point	$x_1^0(\ell)$	$x_2^0(\ell)$	$y_1^0(\ell)$	$y_2^0(\ell)$
$\mathcal{O}_1$	$\sqrt{2h} \cos \ell$	0	$\sqrt{2h} \sin \ell$	0
$\mathcal{O}_2$	0	$\sqrt{2h} \cos \ell$	0	$\sqrt{2h} \sin \ell$
$\mathcal{O}_3$	$\sqrt{h} \cos \ell$	$-\sqrt{h} \cos \ell$	$\sqrt{h} \sin \ell$	$-\sqrt{h} \sin \ell$
$\mathcal{O}_4$	$\sqrt{h} \cos \ell$	$\sqrt{h} \cos \ell$	$\sqrt{h} \sin \ell$	$\sqrt{h} \sin \ell$
$\mathcal{O}_5$	$\sqrt{h} \cos \ell$	$\sqrt{h} \sin \ell$	$\sqrt{h} \sin \ell$	$-\sqrt{h} \cos \ell$
$\mathcal{O}_6$	$\sqrt{h} \cos \ell$	$-\sqrt{h} \sin \ell$	$\sqrt{h} \sin \ell$	$\sqrt{h} \cos \ell$

Table 6. Expression for the functions $x_k^2(\ell)$, $k = 1, 2$, at the critical points

Critical point	$x_1^2(\ell)$	$x_2^2(\ell)$
$\mathcal{O}_1$	$\frac{ah^{3/2}}{8\sqrt{2}}(7 - 2\cos 2\ell) \cos \ell$	0
$\mathcal{O}_2$	0	$\frac{ah^{3/2}}{8\sqrt{2}}(7 - 2\cos 2\ell) \cos \ell$
$\mathcal{O}_3$	$\frac{(2a+b)h^{3/2}}{32}(7 - 2\cos 2\ell) \cos \ell$	$-\frac{(2a+b)h^{3/2}}{32}(7 - 2\cos 2\ell) \cos \ell$
$\mathcal{O}_4$	$\frac{(2a+b)h^{3/2}}{32}(7 - 2\cos 2\ell) \cos \ell$	$\frac{(2a+b)h^{3/2}}{32}(7 - 2\cos 2\ell) \cos \ell$
$\mathcal{O}_5$	$\frac{h^{3/2}}{32}(8a + b + 2b \cos 2\ell) \cos \ell$	$\frac{h^{3/2}}{32}(8a + b - 2b \cos 2\ell) \sin \ell$
$\mathcal{O}_6$	$\frac{h^{3/2}}{32}(8a + b + 2b \cos 2\ell) \cos \ell$	$-\frac{h^{3/2}}{32}(8a + b - 2b \cos 2\ell) \sin \ell$

Table 7. Expression for the functions $y_k^2(\ell)$, $k = 1, 2$, at the critical points

Critical point	$y_1^2(\ell)$	$y_2^2(\ell)$
$\mathcal{O}_1$	$-\frac{3ah^{3/2}}{8\sqrt{2}}(3 + 2\cos 2\ell)\sin \ell$	0
$\mathcal{O}_2$	0	$-\frac{3ah^{3/2}}{8\sqrt{2}}(3 + 2\cos 2\ell)\sin \ell$
$\mathcal{O}_3$	$-\frac{3(2a+b)h^{3/2}}{32}(3 + 2\cos 2\ell)\sin \ell$	$\frac{3(2a+b)h^{3/2}}{32}(3 + 2\cos 2\ell)\sin \ell$
$\mathcal{O}_4$	$-\frac{3(2a+b)h^{3/2}}{32}(3 + 2\cos 2\ell)\sin \ell$	$-\frac{3(2a+b)h^{3/2}}{32}(3 + 2\cos 2\ell)\sin \ell$
$\mathcal{O}_5$	$\frac{h^{3/2}}{32}(-8a + b + 6b\cos 2\ell)\sin \ell$	$\frac{h^{3/2}}{32}(8a - b + 6b\cos 2\ell)\cos \ell$
$\mathcal{O}_6$	$\frac{h^{3/2}}{32}(-8a + b + 6b\cos 2\ell)\sin \ell$	$-\frac{h^{3/2}}{32}(8a - b + 6b\cos 2\ell)\cos \ell$

4.2. KAM tori

In this section we apply KAM theory to the Hamiltonian (2.20). In order to avoid confusion with the hypothesis or strength of the KAM theorem, we provide a version due to Han, Li and Yi [10]. The theorems displayed below are similar, but they should be separated due to their subtle differences.

We start by rewriting the reduced Hamiltonian (3.1) in terms of δ , which yields

$$\widehat{\mathcal{H}} = -b(2h - Q^2 - P^2) [3Q^2 - (\delta - 1)P^2]. \quad (4.1)$$

REMARK 4. We have the totally integrable cases for $\delta = -2, 1$ and $b = 0$, therefore, their phase space is foliated by invariant tori.

Theorem 2. *If $\delta \in (-\infty, 1)$, there are invariant KAM tori of dimension two, enclosing the periodic orbits associated with the critical points of the set $\mathcal{R}$. The measure of the complement of their union is of order $O(\varepsilon^3)$.*

Proof. Since the points in the set $\mathcal{R}$ are elliptic in (Q, P) , we use the action-angle coordinates defined by

$$\overline{Q} = \left(\frac{1-\delta}{3}\right)^{1/4} \sqrt{2I_1} \cos \theta, \quad \overline{P} = \left(\frac{3}{1-\delta}\right)^{1/4} \sqrt{2I_1} \sin \theta. \quad (4.2)$$

In these variables, the Hamiltonian (4.1) becomes

$$\widehat{\mathcal{H}} = -4bh\sqrt{3(1-\delta)}I_1 - 2bI_1^2[\delta - 4 + (\delta + 2)\cos 2\theta]. \quad (4.3)$$

The next step is to reverse the operations that made possible the transformation from $\mathcal{H}_\varepsilon$ to $\widehat{\mathcal{H}}$. This is done by identifying h with the action and adding the dropped term $-6aL^2$. We also introduce a change of time scale in the Hamiltonian by multiplying it by $\varepsilon^2/16$ and adding the dropped term L . We end up with

$$\mathcal{H}_\varepsilon(L, I_1, \ell, \theta) = L + \varepsilon^2 \bar{h}_1(I_1, \theta) + O(\varepsilon^4), \quad (4.4)$$

where

$$\bar{h}_1(I_1, \theta) = -\frac{3b\delta}{16}L^2 - \frac{b\sqrt{3(1-\delta)}}{4}LI_1 - \frac{1}{8}bI_1^2(\delta - 4 + (\delta + 2)\cos 2\theta).$$

Since $\bar{h}_1$ in (4.4) depends on the angle θ , we cannot directly apply a KAM theorem to obtain 2D-tori. However, since the action I_1 close to equilibria is small, we have that I_1 is bigger than I_1^2 . Then, in a small neighborhood of the equilibria, we can replace I_1 by $\varepsilon \bar{I}_1$ in the Hamiltonian (4.4). This means that we are approaching $I_1 = 0$, that is, the equilibrium point. So that $\bar{h}_1$ can be split into two orders in the parameter ε to give

$$\mathcal{H}_\varepsilon(L, \bar{I}_1, \ell, \theta) = L - \varepsilon^2 \frac{3}{16} b \delta L^2 - \varepsilon^3 \frac{\sqrt{3}}{4} b \sqrt{1 - \delta} L \bar{I}_1 + O(\varepsilon^4). \quad (4.5)$$

This Hamiltonian has the form (A.1) as needed in Theorem 7 of the Appendix, with $h_0 = L$, $h_1 = -\frac{3b\delta}{16} L^2$, and $h_2 = -\frac{b\sqrt{3(1-\delta)}}{4} L \bar{I}_1$. Furthermore, with its notation, $n = 2$, $\alpha = 2$, $m_1 = 2$, $m_2 = 3$, $n_0 = n_1 = 1$, $n_2 = 2$, $I^{n_0} = I^{n_1} = L$, $I^{n_2} = (L, \bar{I}_1)$, $\bar{I}^{n_0} = \bar{I}^{n_1} = L$ and $\bar{I}^{n_2} = \bar{I}_1$, while the frequency vector becomes

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = \left(\frac{\partial h_0}{\partial L}, \frac{\partial h_1}{\partial L}, \frac{\partial h_2}{\partial \bar{I}_1} \right),$$

and the 3×3 matrix with columns Ω , $\partial\Omega/\partial L$ and $\partial\Omega/\partial\bar{I}_1$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3b\delta}{8} L & -\frac{3b\delta}{8} & 0 \\ -\frac{b\sqrt{3(1-\delta)}}{4} L & -\frac{b\sqrt{3(1-\delta)}}{4} & 0 \end{bmatrix}.$$

Since this matrix has rank 2 for all values of $\delta \in (-\infty, 1)$, the result follows. Moreover, taking into account the Remark 5 in the Appendix, we obtain $\beta = \sum_{i=1}^2 m_i(n_i - n_{i-1}) = 3$ and $s = 1$. Now, since the perturbation is of order $O(\varepsilon^4)$ and $\varepsilon^{s\beta+\eta} = \varepsilon^{3+\eta} > \varepsilon^4$ for $0 < \eta < 1/5$, the measure of the set where we cannot guarantee the existence of invariant tori is of order $O(\varepsilon^3)$. $\square$

Theorem 3. *If $\delta \in (-2, +\infty)$, there are invariant KAM tori of dimension two enclosing the points in $\mathcal{M}$. These invariant tori form a majority in the sense that the measure of the complement of their union is of order $O(\varepsilon^3)$.*

Proof. We proceed in a similar way as in the proof of Theorem 2. So, we just point out the differences. First, we shift the origin to the critical points and scale the variables by

$$\bar{Q} = Q \mp \sqrt{h}, \quad \bar{P} = P,$$

where “ $-$ ” applies for $\mathcal{O}_3$ and “ $+$ ” corresponds to $\mathcal{O}_4$. Next, we normalize the quadratic part through the symplectic change $(\bar{Q}, \bar{P}) \rightarrow (I_1, \theta)$ given by

$$\bar{Q} = \left(\frac{\delta + 2}{3} \right)^{1/4} \sqrt{I_1} \cos \theta, \quad \bar{P} = 2 \left(\frac{3}{\delta + 2} \right)^{1/4} \sqrt{I_1} \sin \theta. \quad (4.6)$$

Hence, at both critical points the Hamiltonian (4.1) takes the form

$$\hat{\mathcal{H}} = -3bh^2 + 4bh\sqrt{3(\delta+2)}I_1 \pm \frac{2 \times 3^{1/4} b \sqrt{h} I_1^{3/2} \cos \theta [10 - \delta + 3(\delta - 2) \cos 2\theta]}{(\delta + 2)^{1/4}}, \quad (4.7)$$

where the upper sign applies for $\mathcal{O}_3$ and the lower one for $\mathcal{O}_4$.

At this step, we must undo the operations to pass from $\mathcal{H}_\varepsilon$ to $\widehat{\mathcal{H}}$. First, we identify h with the action L and rescale time by multiplying the Hamiltonian (4.7) by ε . Next, we add $-6aL^2$, rescale time again by multiplying the Hamiltonian by $\varepsilon^2/16$ and add L . Then, after some simplifications the resulting Hamiltonian in action-angle coordinates reads as

$$\mathcal{H}_\varepsilon(L, I_1, \ell, \theta) = L + \varepsilon^2 \bar{h}_1(I, \theta) + O(\varepsilon^4), \quad (4.8)$$

where

$$\bar{h}_1(I, \theta) = -\frac{3b(\delta+1)}{16}L^2 + \frac{b\sqrt{3(\delta+2)}}{4}L I_1 \pm \frac{3^{1/4} b \sqrt{L} I_1^{3/2} \cos \theta [10 - \delta + 3(\delta-2) \cos 2\theta]}{8(\delta+2)^{1/4}}.$$

We note that this function $\bar{h}_1$ depends on angle θ , thus we cannot directly apply a standard KAM theorem to obtain 2D-tori. However, since the action I_1 close to equilibria is small, we have that I_1 is bigger than I_1^2 . Hence, by doing $I_1 \rightarrow \varepsilon \bar{I}_1$ in the Hamiltonian (4.8), we find that, in a small neighborhood of the equilibria, we can split $\bar{h}_1$ into two orders in the parameter ε , namely,

$$\mathcal{H}_\varepsilon(L, \bar{I}_1, \ell, \theta) = L - \varepsilon^2 \frac{3b(\delta+1)}{16}L^2 + \varepsilon^3 \frac{b\sqrt{3(\delta+2)}}{4}L \bar{I}_1 + O(\varepsilon^4). \quad (4.9)$$

In order to apply Theorem 7, we set $h_0 = L$, $h_1 = -\frac{3b(\delta+1)}{16}L^2$, and $h_2 = \frac{b\sqrt{3(\delta+2)}}{4}L \bar{I}_1$. Taking $n = 2$, $\alpha = 2$, $m_1 = 2$, $m_2 = 3$, $n_0 = n_1 = 1$, $n_2 = 2$, $I^{n_0} = I^{n_1} = L$, $I^{n_2} = (L, \bar{I}_1)$, $\bar{I}^{n_0} = \bar{I}^{n_1} = L$, $\bar{I}^{n_2} = (L, \bar{I}_1)$, $\bar{I}^{n_0} = \bar{I}^{n_1} = L$ and $\bar{I}^{n_2} = \bar{I}_1$, we obtain the frequency vector

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = \left(\frac{\partial h_0}{\partial L}, \frac{\partial h_1}{\partial L}, \frac{\partial h_2}{\partial \bar{I}_1} \right),$$

and the 3×3 matrix with columns Ω , $\partial\Omega/\partial L$ and $\partial\Omega/\partial \bar{I}_1$, given by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3b(\delta+1)}{8}L & -\frac{3b(\delta+1)}{8} & 0 \\ \frac{b\sqrt{3(\delta+2)}}{4}L & \frac{b\sqrt{3(\delta+2)}}{4} & 0 \end{bmatrix}.$$

Since the rank of the matrix is 2 for $\delta \in (-2, +\infty)$, the result follows. Moreover, from Remark 5 in the Appendix, we obtain $\beta = \sum_{i=1}^2 m_i(n_i - n_{i-1}) = 3$ and $s = 1$. As a consequence, as the perturbation is of order $O(\varepsilon^{7/2})$ and $\varepsilon^{s\beta+\eta} = \varepsilon^{3+\eta} > \varepsilon^{7/2}$, the set where we cannot guarantee the persistence of invariant tori has a measure of order $O(\varepsilon^3)$. $\square$

Theorem 4. *If $\delta \in (-\infty, -2) \cup (1, \infty)$, there are invariant KAM tori of dimension two, enclosing the $\mathcal{C}$ critical points. These invariant tori form a majority in the sense that the measure of the complement of their union is of order $O(\varepsilon^3)$.*

Proof. As was done before, we apply the following change of coordinates:

$$\bar{Q} = Q, \quad \bar{P} = P \mp \sqrt{h}, \quad (4.10)$$

where the upper sign applies for $\mathcal{O}_5$ and the lower one is used for $\mathcal{O}_6$. Now we normalize the quadratic part through the symplectic change

$$\bar{Q} = 2 \left(\frac{\delta-1}{\delta+2} \right)^{1/4} \sqrt{I_1} \cos \theta, \quad \bar{P} = \left(\frac{\delta+2}{\delta-1} \right)^{1/4} \sqrt{I_1} \sin \theta, \quad (4.11)$$

so, after some algebraic manipulation, we find that the Hamiltonian (4.1) becomes

$$\widehat{\mathcal{H}} = bh^2(\delta-1) - 4bh\sqrt{(\delta-1)(\delta+2)}I_1 \mp 2b\sqrt{h}\left(\frac{\delta-1}{\delta+2}\right)^{1/4}I_1^{3/2}\sin\theta[3(\delta-2)+(\delta-10)\cos 2\theta]. \quad (4.12)$$

Next, proceeding in a similar way as in the proof of Theorem 3, we incorporate the terms associated to L and undo the time scalings, obtaining the Hamiltonian

$$\mathcal{H}_\varepsilon = h_0(L) + \varepsilon^2 \bar{h}_1(I, \theta) + O(\varepsilon^4), \quad (4.13)$$

where

$$\bar{h}_1(I, \theta) = -\frac{b(2\delta+1)}{16}L^2 - \frac{b\sqrt{(\delta-1)(\delta+2)}}{4}LI_1 \mp \frac{b}{8}\left(\frac{\delta-1}{\delta+2}\right)^{1/4}I_1^{3/2}L^{1/2}\sin\theta[3(\delta-2)+(\delta-10)\cos 2\theta].$$

Now we proceed as in the previous proofs, scaling the action variable $I_1 \rightarrow \varepsilon \bar{I}_1$ in a small neighborhood of the equilibrium points such that the Hamiltonian (4.12) becomes

$$\mathcal{H}_\varepsilon(L, \bar{I}_1, \ell, \theta) = L. \quad (4.14)$$

To apply Theorem 7, we take $h_0 = L$, $h_1 = -\frac{b(2\delta+1)}{16}L^2$, and $h_2 = -\frac{b\sqrt{(\delta-1)(\delta+2)}}{4}LI_1$. Here, we have $n = 2$, $\alpha = 2$, $m_1 = 2$, $m_2 = 3$, $n_0 = n_1 = 1$, $n_2 = 2$, $I^{n_0} = I^{n_1} = L$, $I^{n_2} = (L, \bar{I}_1)$, $\bar{I}^{n_0} = \bar{I}^{n_1} = L$, $\bar{I}^{n_2} = (L, \bar{I}_1)$, $\bar{I}^{n_0} = \bar{I}^{n_1} = L$ and $\bar{I}^{n_2} = \bar{I}_1$. We find that the frequency vector is

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = \left(\frac{\partial h_0}{\partial L}, \frac{\partial h_1}{\partial L}, \frac{\partial h_2}{\partial \bar{I}_1} \right).$$

Hence, we deduce that the 3×3 matrix whose columns are Ω , $\partial\Omega/\partial L$, and $\partial\Omega/\partial \bar{I}_1$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{b(2\delta+1)}{8}L & -\frac{b(2\delta+1)}{8} & 0 \\ -\frac{b\sqrt{(\delta-1)(\delta+2)}}{4}L & -\frac{b\sqrt{(\delta-1)(\delta+2)}}{4} & 0 \end{bmatrix}$$

which has rank 2 for $\delta \in (-\infty, -2) \cup (1, +\infty)$. Therefore, by Han, Li, and Yi's theorem (Theorem 7), the first part of the theorem is concluded. Now, by applying Remark 5 in the Appendix, we find that $\beta = 3$, $s = 1$ and, since the perturbation is of order $O(\varepsilon^{7/2})$ and $\varepsilon^{s\beta+\eta} = \varepsilon^{3+\eta} > \varepsilon^{7/2}$, we find that the measure of the set where we cannot ensure the persistence of invariant tori is of order $O(\varepsilon^3)$. $\square$

Appendix

Let (M, Ω) be a symplectic manifold of dimension $2n$ and $\mathcal{H}_0: M \rightarrow \mathbb{R}$ a smooth Hamiltonian which defines a Hamiltonian vector field $Y_0 = (d\mathcal{H}_0)^\#$ with symplectic flow φ_0^t . Let $I \subset \mathbb{R}$ be an interval such that each $h \in I$ is a regular value of $\mathcal{H}_0$ and $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a compact connected circle bundle over a base space $\mathcal{B}(h)$ with projection $\pi: \mathcal{N}_0(h) \rightarrow \mathcal{B}(h)$. So, this is the setting of regular reduction theory. Assume that all the solutions of Y_0 in $\mathcal{N}_0(h)$ are periodic and have periods smoothly depending only on the value of the Hamiltonian; i.e., the period is a smooth function $T = T(h)$.

Let ε be a small parameter and let $\mathcal{H}_1: M \rightarrow \mathbb{R}$ be smooth, $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^\#$, $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$, $\pi: \mathcal{N}_\varepsilon(h) \rightarrow \mathcal{B}(h)$ the projection, and ϕ_ε^t be the flow defined by Y_ε .

Let the average of $\mathcal{H}_1$ be

$$\overline{\mathcal{H}} = \frac{1}{T} \int_0^T \mathcal{H}_1(\phi_0^t) dt.$$

The next result provides sufficient conditions for characterizing the existence of periodic solutions of the Hamiltonian system associated to $\mathcal{H}_\varepsilon$. For more information on this subject we refer the reader to [14, 15, 20].

Theorem 5 (Reeb). *If $\overline{\mathcal{H}}$ has a nondegenerate critical point at $\pi(p) = \overline{p} \in \mathcal{B}(h)$ with $p \in \mathcal{N}_0(h)$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for ε small with $p(0) = p$, $T(0) = T$, and $p(\varepsilon) \in \mathcal{N}_\varepsilon$, and the solution of Y_ε through $p(\varepsilon)$ is $T(\varepsilon)$ -periodic. In addition, if the characteristic exponents of the critical point $\overline{p}$ (that is, the eigenvalues of the matrix $A = \mathbb{J}D^2\overline{\mathcal{H}}(\overline{p})$) are $\lambda_1, \lambda_2, \dots, \lambda_{2n-2}$, then the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are*

$$1, 1, 1 + \varepsilon\lambda_1 T + O(\varepsilon^2), 1 + \varepsilon\lambda_2 T + O(\varepsilon^2), \dots, 1 + \varepsilon\lambda_{2n-2} T + O(\varepsilon^2).$$

Theorem 6. *Let p and $\overline{p}$ be as in the previous theorem. If one, or more, of the characteristic exponents λ_j is real or has nonzero real part, then the periodic solution through $p(\varepsilon)$ is unstable. If the matrix A is strongly stable, then the periodic solution through $p(\varepsilon)$ is elliptic, i.e., linearly stable.*

The proofs of Theorems 5 and 6 appear in [14].

Consider a Hamiltonian system of the form

$$\mathcal{H}_\varepsilon(I, \varphi, \varepsilon) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \dots + \varepsilon^{m_\alpha} h_\alpha(I^{n_\alpha}) + \varepsilon^{m_\alpha+1} p(I, \varphi, \varepsilon), \quad (\text{A.1})$$

where $(I, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables with the standard symplectic structure $dI \wedge d\varphi$, and $\varepsilon > 0$ is a sufficiently small parameter. The Hamiltonian $\mathcal{H}_\varepsilon$ is real analytic, and the parameters α, m, n_i ($i = 0, 1, \dots, \alpha$) and m_j ($j = 1, 2, \dots, \alpha$) are positive integers satisfying $n_0 \leq n_1 \leq \dots \leq n_\alpha = n$, $m_1 \leq m_2 \leq \dots \leq m_\alpha = m$, $I^{n_i} = (I_1, \dots, I_{n_i})$, for $i = 1, 2, \dots, \alpha$, and p depends on ε smoothly.

The Hamiltonian $\mathcal{H}_\varepsilon(I, \varphi, \varepsilon)$ is taken in a bounded closed region $Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$. For each ε the integrable part of $\mathcal{H}_\varepsilon$,

$$X_\varepsilon(I) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \dots + \varepsilon^{m_\alpha} h_\alpha(I^{n_\alpha}),$$

admits a family of invariant n -tori $T_\zeta^\varepsilon = \{\zeta\} \times \mathbb{T}^n$, with linear flows $\{x_0 + \omega^\varepsilon(\zeta)t\}$, where, for each $\zeta \in Z$, $\omega^\varepsilon(\zeta) = \nabla X_\varepsilon(\zeta)$ is the frequency vector of the n -torus T_ζ^ε and ∇ is the gradient operator. When $\omega^\varepsilon(\zeta)$ is nonresonant, the n -torus T_ζ^ε becomes quasi-periodic. We refer to the integrable part X_ε and its associated tori $\{T_\zeta^\varepsilon\}$ as the intermediate Hamiltonian and intermediate tori, respectively.

Let $\overline{I}^{n_i} = (I_{n_{i-1}+1}, \dots, I_{n_i})$, $i = 0, 1, \dots, \alpha$ (where $n_{-1} = 0$, hence $\overline{I}^{n_0} = I^{n_0}$), and define

$$\Omega = (\nabla_{\overline{I}^{n_0}} h_0(I^{n_0}), \dots, \nabla_{\overline{I}^{n_\alpha}} h_{n_\alpha}(I^{n_\alpha})),$$

such that, for each $i = 0, 1, \dots, \alpha$, $\nabla_{\overline{I}^{n_i}}$ denotes the gradient with respect to $\overline{I}^{n_i}$.

We assume the following high-order degeneracy-removing condition of Bruno–Rüssmann type (so named by Han, Li, and Yi), giving credit to Bruno and Rüssmann, who provided weak conditions on the frequencies guaranteeing the persistence of invariant tori, the so-called (A) condition: there is a positive integer s such that

$$\text{Rank}\{\partial^r \Omega(I) : 0 \leq |r| \leq s\} = n, \quad \forall I \in Z.$$

For the usual case of a nearly integrable Hamiltonian system of the type

$$\mathcal{H}_\varepsilon(I, \varphi, \varepsilon) = X(I) + \varepsilon p(I, \varphi, \varepsilon), \quad (I, \varphi) \in Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n, \quad (\text{A.2})$$

condition (A) given above generalizes the classical Kolmogorov nondegenerate condition that $\partial \Omega(I)$ be nonsingular over Z , where $\Omega(I) = \nabla X(I)$; Bruno's nondegenerate condition that $\text{Rank}\{\Omega(I), \partial \Omega\} = n$, for all $I \in Z$; and the weakest nondegenerate condition, guaranteeing such persistence provided by Rüssmann, that $\omega(Z)$ should not lie in any $(n-1)$ -dimensional subspace. Rüssmann's condition is equivalent to condition (A) for systems like (A.2). However, Bruno's or Rüssmann's conditions do not apply to the Hamiltonian (A.1), as it is too degenerate.

The following theorem gives the setting that makes certain the persistence of KAM tori in Hamiltonians like (A.1).

Theorem 7 (Han, Li and Yi). *Assume we are given condition (A) and η with $0 < \eta < 1/5$. Then there exists an $\varepsilon_0 > 0$ and a family of Cantor sets $Z_\varepsilon \subset Z$, $0 < \varepsilon < \varepsilon_0$, with $|Z \setminus Z_\varepsilon| = O(\varepsilon^{\eta/s})$, such that each $\zeta \in Z_\varepsilon$ corresponds to a real analytic, invariant, quasi-periodic n -torus $\overline{T}_\zeta^\varepsilon$ of the Hamiltonian (A.1), which is slightly deformed from the intermediate n -torus T_ζ^ε . Moreover, the family $\{\overline{T}_\zeta^\varepsilon : \zeta \in Z_\varepsilon, 0 < \varepsilon < \varepsilon_0\}$ varies Whitney smoothly.*

See the proof in [10].

REMARK 5. In contrast to the case for a usual nearly integrable Hamiltonian system, the excluding measure for the existence of quasi-periodic invariant tori in the properly degenerate case is of a fairly large order of $\varepsilon^{\eta/s}$ for a pre-fixed small positive constant η , as shown in the theorem above. This is mainly caused by a normal form reduction which pushes the perturbation to an order higher than $\varepsilon^{s\beta+\eta}$ for

$$\beta = \sum_{i=1}^{\alpha} m_i (n_i - n_{i-1}),$$

for which the domain G needs to shrink by order $\varepsilon^{\eta/s}$ in measure. This is necessary for general properly degenerate Hamiltonian systems like (A.1) in order to apply the standard KAM iterations. However, if the perturbation in (A.1) is already of order $O(\varepsilon^{s\beta+\eta})$, then a normal form reduction will not be necessary, and the excluding measure for the existence of quasi-periodic invariant tori can be improved to order ε^β .

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