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The Effect of the Mutual Gravitational Interactions on the Perihelia Displacement of the Orbits of the Solar System's Planets

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The classical N -body problem in the case when one of the bodies (the Sun) has a much larger mass than the rest of the mutually gravitating bodies is considered. The system of equations in canonical Delaunay variables describing the motion of the system relative to the barycentric coordinate system is derived via the methods of analytical dynamics. The procedure of averaging over the fast angular variables (mean anomalies) leads to the equation describing the evolution of a single Solar system planet's perihelion as the sum of two terms. The first term corresponds to the gravitational disturbances caused by the rest of the planets, as in the case of a motionless Sun. The second appears because the problem is considered in the barycentric coordinate system and the orbits' inclinations are taken into account. This term vanishes if all planets are assumed to be moving in one static plane. This term contributes substantially to the Mercury's and Venus's perihelion evolutions. For the rest of the planet this term is small compared to the first one. For example, for Mercury the values of the two terms in question were calculated to be 528.67 and 39.64 angular seconds per century, respectively.

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1. Formulation of the problem. Equations of motion

Let us consider a model of the Solar system, consisting of $(n+1)$ material points $P_0, P_1, \dots, P_n$. Let us introduce an inertial coordinate system $CXYZ$ with its origin at the system's center of mass. Let $\mathbf{R}_0$ be the radius vector of point P_0 (the Sun), and let $\mathbf{R}_1, \dots, \mathbf{R}_n$ be the radius vectors of planets $P_1, \dots, P_n$, whose coordinates are written in the coordinate system $CXYZ$.

Let us denote the mass of the Sun as m_s and the masses of the planets as M_k ($k = 1, \dots, n$). Introducing the small parameter ε , equal to the ratio of the mass of the heaviest planet (Jupiter) to the mass of the Sun, $\varepsilon = M_5 m_s^{-1}$, we can represent the masses of other planets as $M_k = \varepsilon m_k$, $k = 1, \dots, n$, where $m_k = M_k M_5^{-1} m_s$, $m_k \leq m_s$.

Since the origin of the inertial coordinate system coincides with the barycenter,

$$m_s \mathbf{R}_0 + \varepsilon \sum_{k=1}^n m_k \mathbf{R}_k = 0. \quad (1.1)$$

The kinetic energy equals

$$T = \frac{1}{2} m_s \dot{\mathbf{R}}_0^2 + \frac{1}{2} \sum_{k=1}^n \varepsilon m_k \dot{\mathbf{R}}_k^2. \quad (1.2)$$

Considering (1.1) and (1.2), we can obtain that

$$T = \frac{\varepsilon}{2} \sum_{k=1}^n m_k \dot{\mathbf{R}}_k^2 + \frac{\varepsilon^2}{2m_s} \left(\sum_{k=1}^n m_k \dot{\mathbf{R}}_k \right)^2. \quad (1.3)$$

The potential energy of gravitational forces equals

$$\Pi = - \sum_{k=1}^n \frac{\varepsilon f_0 m_k}{\left[(\mathbf{R}_0 - \mathbf{R}_k)^2 \right]^{1/2}} - \sum_{\substack{i,j=1, \\ i < j}}^n \frac{\varepsilon^2 f m_i m_j}{R_{ij}}, \quad (1.4)$$

where f is the universal gravitational constant, $f_0 = f m_s$, $R_{ij} = |\mathbf{R}_i - \mathbf{R}_j|$.

Considering equality (1.1) and keeping terms of no higher than first order by ε , we can rewrite the expression (1.4) as

$$\Pi = -\varepsilon \sum_{k=1}^n \frac{f_0 m_k}{R_k} - \varepsilon^2 \sum_{\substack{i,j=1, \\ i < j}}^n \frac{f m_i m_j}{R_{ij}} + \varepsilon^2 \sum_{i,k=1}^n \frac{f m_i m_k (\mathbf{R}_i, \mathbf{R}_k)}{R_k^3}, \quad (1.5)$$

where $R_k = |\mathbf{R}_k|$.

The Lagrangian function of the mechanical system under study is defined as

$$L = T - \Pi.$$

Let us write the Lagrangian function as

$$L = \varepsilon L_1 + \varepsilon^2 L_2,$$

where

$$L_1 = \frac{1}{2} \sum_{k=1}^n m_k \dot{\mathbf{R}}_k^2 + \sum_{k=1}^n \frac{f_0 m_k}{R_k}, \quad (1.6)$$

$$L_2 = \frac{1}{2m_s} \left(\sum_{k=1}^n m_k \dot{\mathbf{R}}_k \right)^2 + \sum_{\substack{i,j=1, \\ i < j}}^n \frac{f m_i m_j}{R_{ij}} - \sum_{i,k=1}^n \frac{f m_i m_k (\mathbf{R}_i, \mathbf{R}_k)}{R_k^3}. \quad (1.7)$$

Function L_1 and the corresponding equations describe the Kepler motions of each planet in the central force field along an elliptic trajectory. The function L_2 defines the perturbations of the Kepler motions.

Let us consider the problem of perturbed motion of the planet with the following Lagrangian function:

$$\tilde{L} = L_1 + \varepsilon L_2. \quad (1.8)$$

We shall use the Delaunay canonical variables [1] to describe the motion of the planets. The first step is to transform from generalized coordinates and velocities $(\mathbf{R}_k, \dot{\mathbf{R}}_k)$ to generalized coordinates and impulses $(\mathbf{R}_k, \mathbf{p}_k)$, $k = 1, \dots, n$:

$$\mathbf{p}_k = \nabla_{\dot{\mathbf{R}}_k} \tilde{L} = m_k \dot{\mathbf{R}}_k + \frac{\varepsilon m_k}{m_s} \sum_{i=1}^n m_i \dot{\mathbf{R}}_i, \quad k = 1, \dots, n. \quad (1.9)$$

Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, $\mathbf{R} = (\mathbf{R}_1, \dots, \mathbf{R}_n)$. Then equality (1.9) can be written as

$$\mathbf{p} = \Lambda_1 (E + \varepsilon \Lambda_2) \dot{\mathbf{R}}, \quad (1.10)$$

where

$$\Lambda_1 = \text{diag} \{m_1, \dots, m_n\}, \quad \Lambda_2 = \left(\frac{\delta_{ii} m_j}{m_s} \right)_{i,j=1}^n, \\ E = (\delta_{ij})_{i,j=1}^n, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Keeping terms of no higher than first order by ε , we can obtain from (1.10) that

$$\dot{\mathbf{R}} = [\Lambda_1 (E + \varepsilon \Lambda_2)]^{-1} \mathbf{p} \approx (E - \varepsilon \Lambda_2) \Lambda_1^{-1} \mathbf{p} = \Lambda_1^{-1} \mathbf{p} - \varepsilon \Lambda_2 \Lambda_1^{-1} \mathbf{p},$$

$$\Lambda_1^{-1} = \text{diag} \left\{ \frac{1}{m_1}, \dots, \frac{1}{m_n} \right\}, \quad \Lambda_2 \Lambda_1^{-1} = \left(\frac{\delta_{ii} \delta_{jj}}{m_s} \right)_{i,j=1}^n.$$

Therefore,

$$\dot{\mathbf{R}}_k = \frac{1}{m_k} \mathbf{p}_k - \frac{\varepsilon}{m_s} \sum_{i=1}^n \mathbf{p}_i, \quad k = 1, \dots, n. \quad (1.11)$$

The Hamiltonian function is defined by the following equality [2]:

$$\mathcal{H} = \sum_{k=1}^n (\mathbf{p}_k, \dot{\mathbf{R}}_k) - \tilde{L} \Big|_{\dot{\mathbf{R}} \rightarrow \mathbf{p}}.$$

Considering (1.6)–(1.8) and (1.11), we can obtain

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1, \quad (1.12)$$

$$\begin{aligned} \mathcal{H}_0 &= \sum_{k=1}^n \frac{\mathbf{p}_k^2}{2m_k} - \sum_{k=1}^n \frac{f_0 m_k}{R_k}, \\ \mathcal{H}_1 &= -\frac{1}{2m_s} \sum_{k,i=1}^n (\mathbf{p}_k, \mathbf{p}_i) - \sum_{\substack{i,j=1, \\ i < j}}^n \frac{f m_i m_j}{R_{ij}} + \sum_{k,i=1}^n \frac{f m_i m_k (\mathbf{R}_k, \mathbf{R}_i)}{R_k^3}. \end{aligned} \quad (1.13)$$

At our next step we shall use spherical coordinates R_k, φ_k, ψ_k of vectors $\mathbf{R}_k$ ($k = 1, \dots, n$):

$$\mathbf{R}_k = R_k \boldsymbol{\xi}_k, \boldsymbol{\xi}_k = (\cos \varphi_k \sin \psi_k, \sin \varphi_k \sin \psi_k, \cos \psi_k). \quad (1.14)$$

Then

$$\dot{\mathbf{R}}_k = \dot{R}_k \boldsymbol{\xi}_k + R_k \dot{\varphi}_k \boldsymbol{\eta}_k + R_k \dot{\psi}_k \boldsymbol{\zeta}_k, \quad (1.15)$$

where

$$\boldsymbol{\eta}_k = (-\sin \varphi_k \sin \psi_k, \cos \varphi_k \sin \psi_k, 0), \quad (1.16)$$

$$\boldsymbol{\zeta}_k = (\cos \varphi_k \cos \psi_k, \sin \varphi_k \cos \psi_k, -\sin \psi_k). \quad (1.17)$$

Let us express $\mathbf{p}_k$ in terms of generalized impulses $p_{R_k}, p_{\varphi_k}, p_{\psi_k}$, corresponding to generalized coordinates R_k, φ_k, ψ_k ($k = 1, \dots, n$):

$$\begin{aligned} p_{R_k} &= \frac{\partial \tilde{L}}{\partial \dot{R}_k} = \left(\nabla_{\dot{\mathbf{R}}_k} \tilde{L}, \frac{\partial \dot{\mathbf{R}}_k}{\partial \dot{R}_k} \right) = (\mathbf{p}_k, \boldsymbol{\xi}_k), \\ p_{\varphi_k} &= \frac{\partial \tilde{L}}{\partial \dot{\varphi}_k} = \left(\nabla_{\dot{\mathbf{R}}_k} \tilde{L}, \frac{\partial \dot{\mathbf{R}}_k}{\partial \dot{\varphi}_k} \right) = (\mathbf{p}_k, R_k \boldsymbol{\eta}_k), \\ p_{\psi_k} &= \frac{\partial \tilde{L}}{\partial \dot{\psi}_k} = \left(\nabla_{\dot{\mathbf{R}}_k} \tilde{L}, \frac{\partial \dot{\mathbf{R}}_k}{\partial \dot{\psi}_k} \right) = (\mathbf{p}_k, R_k \boldsymbol{\zeta}_k). \end{aligned} \quad (1.18)$$

Then from (1.14)–(1.18) we get

$$\mathbf{p}_k = p_{R_k} \boldsymbol{\xi}_k + \frac{p_{\varphi_k}}{R_k \sin^2 \psi_k} \boldsymbol{\eta}_k + \frac{p_{\psi_k}}{R_k} \boldsymbol{\zeta}_k. \quad (1.19)$$

Let us transform from generalized impulses and coordinates $p_{R_k}, p_{\varphi_k}, p_{\psi_k}, R_k, \varphi_k, \psi_k$ to Delaunay variables $L_k, G_k, H_k, l_k, g_k, h_k$ ($k = 1, \dots, n$) using the generating function of canonical transformation

$$\begin{aligned} W &= \sum_{k=1}^n W_k, \\ W_k &= W_k [R_k, \varphi_k, \psi_k, L_k, G_k, H_k] = \\ &= H_k \varphi_k + \int \sqrt{G_k^2 - \frac{H_k^2}{\sin^2 \psi_k}} d\psi_k + \int \sqrt{\frac{2f_0 m_k^2}{R_k} - \frac{G_k^2}{R_k^2} - \frac{f_0^2 m_k^4}{L_k^2}} dR_k. \end{aligned} \quad (1.20)$$

The system of equations defining the transformation to Delaunay variables has the following form:

$$\begin{aligned} p_{R_k} &= \frac{\partial W}{\partial R_k}, & p_{\varphi_k} &= \frac{\partial W}{\partial \varphi_k}, & p_{\psi_k} &= \frac{\partial W}{\partial \psi_k}, \\ l_k &= \frac{\partial W}{\partial L_k}, & g_k &= \frac{\partial W}{\partial G_k}, & h_k &= \frac{\partial W}{\partial H_k}. \end{aligned} \quad (1.21)$$

The magnitude of $\mathbf{R}_k$ can be expressed in terms of Delaunay variables as

$$R_k = \frac{G_k^2}{f_0 m_k^2 \lambda_k}, \quad (1.22)$$

where $\lambda_k = 1 + e_k \cos \vartheta_k$, $e_k = \sqrt{1 - \frac{G_k^2}{L_k^2}}$ is the eccentricity of the k th planet's orbit, and $\vartheta_k = \vartheta_k(l_k, L_k, G_k)$ is the true anomaly defined by l_k, L_k, G_k via the relations

$$\cos w_k = \frac{e_k + \cos \vartheta_k}{1 + e_k \cos \vartheta_k}, \quad l_k = w_k - e_k \sin w_k.$$

The unit vectors $\boldsymbol{\xi}_k$ in the inertial coordinate system can be expressed in Delaunay variables as

$$\begin{aligned} \boldsymbol{\xi}_k &= (\xi_{kx}, \xi_{ky}, \xi_{kz}), \\ \xi_{kx} &= \cos(g_k + \vartheta_k) \cos h_k - \sin(g_k + \vartheta_k) \cos i_k \sin h_k, \\ \xi_{ky} &= \cos(g_k + \vartheta_k) \sin h_k + \sin(g_k + \vartheta_k) \cos i_k \cos h_k, \\ \xi_{kz} &= \sin(g_k + \vartheta_k) \sin i_k, \end{aligned} \quad (1.23)$$

where i_k is the k th planet's orbit inclination, $\cos i_k = H_k/G_k$, g_k is the longitude of the perihelion from the ascending node, and h_k is the longitude of the ascending node.

From relations (1.14), (1.16) and (1.23) we can obtain expressions for the coordinates of the vectors $\boldsymbol{\eta}_k$ in terms of Delaunay variables:

$$\boldsymbol{\eta}_k = (-\xi_{ky}, \xi_{kx}, 0). \quad (1.24)$$

The components of $\boldsymbol{\zeta}_k$ are defined by the following relations:

$$(\boldsymbol{\zeta}_k, \boldsymbol{\xi}_k) = 0, \quad (\boldsymbol{\zeta}_k, \boldsymbol{\eta}_k) = 0, \quad (\boldsymbol{\zeta}_k, \boldsymbol{\zeta}_k) = 1, \quad \zeta_{kz} \leq 0. \quad (1.25)$$

Thus, it follows from (1.23)–(1.25) that

$$\boldsymbol{\zeta}_k = \left(\frac{\xi_{kx}\xi_{kz}}{\sqrt{1 - \xi_{kz}^2}}, \frac{\xi_{ky}\xi_{kz}}{\sqrt{1 - \xi_{kz}^2}}, -\sqrt{1 - \xi_{kz}^2} \right). \quad (1.26)$$

The generalized impulses $p_{R_k}, p_{\varphi_k}, p_{\psi_k}$ can be expressed through Delaunay variables from (1.20)–(1.22) as

$$\begin{aligned} p_{R_k} &= \frac{f_0 m_k^2}{G_k} e_k \sin \vartheta_k, & p_{\varphi_k} &= H_k, \\ p_{\psi_k} &= \frac{G_k \sin i_k \cos(g_k + \vartheta_k)}{\sqrt{1 - \sin^2(g_k + \vartheta_k) \sin^2 i_k}}. \end{aligned} \quad (1.27)$$

Using (1.19), (1.23), (1.24), (1.26) and (1.27), we get

$$\mathbf{p}_k = D_k \left\{ e_k \sin \vartheta_k \boldsymbol{\xi}_k + \frac{\lambda_k \cos i_k}{1 - \sin^2(g_k + \vartheta_k) \sin^2 i_k} \boldsymbol{\eta}_k + \frac{\lambda_k \sin i_k \cos(g_k + \vartheta_k)}{\sqrt{1 - \sin^2(g_k + \vartheta_k) \sin^2 i_k}} \boldsymbol{\zeta}_k \right\}, \quad (1.28)$$

$$D_k = \frac{f_0 m_k^2}{G_k}.$$

The unperturbed part of the Hamiltonian (1.12) in terms of Delaunay variables has the following form:

$$\mathcal{H}_0 = - \sum_{k=1}^n \frac{f_0^2 m_k^3}{2L_k^2}. \quad (1.29)$$

The perturbed part of the Hamiltonian (1.12) is defined by (1.13), where vectors $\mathbf{p}_k$ are expressed in terms of Delaunay variables according to (1.28), and $\mathbf{R}_k = R_k \boldsymbol{\xi}_k$ according to (1.22)–(1.23).

The canonical equations of motion have the following form:

$$\begin{aligned} \dot{L}_k &= -\varepsilon \frac{\partial \mathcal{H}_1}{\partial l_k}, & \dot{l}_k &= \omega_k + \varepsilon \frac{\partial \mathcal{H}_1}{\partial L_k}, & \omega_k &= \frac{f_0^2 m_k^3}{L_k^3}, \\ \dot{G}_k &= -\varepsilon \frac{\partial \mathcal{H}_1}{\partial g_k}, & \dot{g}_k &= \varepsilon \frac{\partial \mathcal{H}_1}{\partial G_k}, \\ \dot{H}_k &= -\varepsilon \frac{\partial \mathcal{H}_1}{\partial h_k}, & \dot{h}_k &= \varepsilon \frac{\partial \mathcal{H}_1}{\partial H_k}, & k &= 1, \dots, n. \end{aligned}$$

The right-hand sides of Eqs. (1.30) depend on canonical Delaunay variables and contain n fast angular variables $l_1, \dots, l_n$, by which the function $\mathcal{H}_1$ is 2π -periodic. The rest of the variables are slow. Let us assume that there are no resonances in the system, i.e., $k_1 \omega_1 + \dots + k_n \omega_n \neq 0$ for any nontrivial set of integer numbers $k_1, \dots, k_n$, and apply the method of averaging by fast variables. The operation of averaging consists in calculating the following integral:

$$\langle (*) \rangle_l = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} (*) dl_1 \dots dl_n = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} (*) \frac{\partial l_1}{\partial \vartheta_1} \dots \frac{\partial l_n}{\partial \vartheta_n} d\vartheta_1 \dots d\vartheta_n,$$

where $\frac{\partial l_k}{\partial \vartheta_k} = \frac{(1 - e_k^2)^{3/2}}{\lambda_k^2}$, $k = 1, \dots, n$.

2. The perihelion longitude evolution of the j th planet

Let us derive the equation describing the evolution of the perihelion longitude for the j th planet, $\varpi_1 = g_1 + h_1$. According to (1.30),

$$\langle \dot{\varpi}_j \rangle = \varepsilon \left\langle \frac{\partial \mathcal{H}_1}{\partial G_j} + \frac{\partial \mathcal{H}_1}{\partial H_j} \right\rangle.$$

Taking into consideration only those summands of $\mathcal{H}_1$ that depend on G_j and H_j , we obtain

$$\begin{aligned} \langle \dot{\omega}_j \rangle = \varepsilon \left\langle \left(\frac{\partial}{\partial G_j} + \frac{\partial}{\partial H_j} \right) \left(-\frac{1}{2m_s} \mathbf{p}_j^2 + \frac{f m_j^2}{R_j} - \frac{1}{m_s} \sum_{\substack{k=1, \\ k \neq j}}^n (\mathbf{p}_j, \mathbf{p}_k) - \right. \right. \\ \left. \left. - \sum_{\substack{k=1, \\ k \neq j}}^n \frac{f m_j m_k}{R_{jk}} + \sum_{\substack{k=1, \\ k \neq j}}^n f m_j m_k (\mathbf{R}_j, \mathbf{R}_k) \left(\frac{1}{R_j^3} + \frac{1}{R_k^3} \right) \right) \right\rangle. \end{aligned} \quad (2.1)$$

Let us note that

$$-\frac{1}{2m_s} \mathbf{p}_j^2 + \frac{f m_j^2}{R_j} = \frac{f_0^2 m_j^4}{2m_s L_j^2}.$$

Therefore, the partial derivatives of this expression by G_j and H_j are equal to zero. Next,

$$\left\langle \frac{\mathbf{R}_k}{R_k^3} \right\rangle = \frac{(1 - e_k^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{\boldsymbol{\xi}_k d\vartheta_k}{\lambda_k^2 R_k^2} = \frac{(1 - e_k^2)^{3/2}}{2\pi} \frac{f_0^2 m_k^4}{G_k^4} \int_0^{2\pi} \boldsymbol{\xi}_k d\vartheta_k = 0.$$

After accounting for all the relations above, relation (2.1) takes the following form:

$$\langle \dot{\omega}_j \rangle = \sum_{\substack{k=1, \\ k \neq j}}^n \omega_{jk}^{(p)} + \sum_{\substack{k=1, \\ k \neq j}}^n \omega_{jk}^{(R)} \quad (2.2)$$

where

$$\begin{aligned} \omega_{jk}^{(p)} &= -\frac{\varepsilon}{m_s} \left\langle \left(\frac{\partial}{\partial G_j} + \frac{\partial}{\partial H_j} \right) (\mathbf{p}_j, \mathbf{p}_k) \right\rangle, \\ \omega_{jk}^{(R)} &= -\varepsilon \left\langle \left(\frac{\partial}{\partial G_j} + \frac{\partial}{\partial H_j} \right) \frac{f m_j m_k}{R_{jk}} \right\rangle. \end{aligned}$$

The coordinates of the impulse vector $\mathbf{p}_k$ (1.28) can be represented as

$$\begin{aligned} p_{kx} &= D_k [M_k \cos h_k - N_k \sin h_k], \\ p_{ky} &= D_k [N_k \cos h_k + M_k \sin h_k], \quad p_{kz} = D_k Q_k, \end{aligned}$$

where

$$\begin{aligned} M_k &= M_k(e_k, g_k, i_k, \vartheta_k) = e_k \sin \vartheta_k \cos(g_k + \vartheta_k) + \\ &+ \frac{\lambda_k \sin(g_k + \vartheta_k) [\sin^2 i_k \cos^2(g_k + \vartheta_k) - \cos^2 i_k]}{1 - \sin^2(g_k + \vartheta_k) \sin^2 i_k}, \\ N_k &= N_k(e_k, g_k, i_k, \vartheta_k) = \left[e_k \sin \vartheta_k \sin(g_k + \vartheta_k) + \right. \\ &+ \left. \frac{\lambda_k \cos(g_k + \vartheta_k) [1 + \sin^2 i_k \sin^2(g_k + \vartheta_k)]}{1 - \sin^2(g_k + \vartheta_k) \sin^2 i_k} \right] \cos i_k, \\ Q_k &= Q_k(e_k, g_k, i_k, \vartheta_k) = [e_k \sin \vartheta_k \sin(g_k + \vartheta_k) - \lambda_k \cos(g_k + \vartheta_k)] \sin i_k. \end{aligned}$$

Let us denote

$$\begin{aligned}\widetilde{M}_k &= \widetilde{M}_k(e_k, g_k, i_k) = \langle M_k(e_k, g_k, i_k, \vartheta_k) \rangle, \\ \widetilde{N}_k &= \widetilde{N}_k(e_k, g_k, i_k) = \langle N_k(e_k, g_k, i_k, \vartheta_k) \rangle, \\ \widetilde{Q}_k &= \widetilde{Q}_k(e_k, g_k, i_k) = \langle Q_k(e_k, g_k, i_k, \vartheta_k) \rangle.\end{aligned}$$

Then

$$\begin{aligned}\langle (\mathbf{p}_j, \mathbf{p}_k) \rangle &= \frac{f_0^2 m_j^2 m_k^2 (1 - e_j^2)^{3/2} (1 - e_k^2)^{3/2}}{G_j G_k} F_{jk}, \\ F_{jk} &= \left(\widetilde{M}_j \widetilde{M}_k + \widetilde{N}_j \widetilde{N}_k \right) \cos(h_j - h_k) + \left(\widetilde{M}_j \widetilde{N}_k - \widetilde{N}_j \widetilde{M}_k \right) \sin(h_j - h_k) + \widetilde{Q}_j \widetilde{Q}_k.\end{aligned}$$

In view of the relations

$$\begin{aligned}\frac{\partial}{\partial G_k} \frac{(1 - e_k^2)^{3/2}}{G_k} &= \frac{2}{G_k^2} (1 - e_k^2)^{3/2}, \quad \frac{\partial e_k}{\partial G_k} = \frac{e_k^2 - 1}{e_k G_k}, \quad \frac{\partial i_k}{\partial G_k} + \frac{\partial i_k}{\partial H_k} = \frac{\cos i_k - 1}{G_k \sin i_k}, \\ G_k &= \sqrt{f_0 m_k^2 a_k (1 - e_k^2)}, \quad k = 1, \dots, n,\end{aligned}$$

where a_k is the semimajor axis of the orbit of the k th planet, the summands $\omega_{jk}^{(p)}$ in (2.2) can be expressed as

$$\omega_{jk}^{(p)} = -\frac{\varepsilon m_k (1 - e_k^2)}{a_j} \sqrt{\frac{f (1 - e_j^2)}{m_s a_k}} \left\{ 2F_{jk} + \frac{\partial F_{jk}}{\partial e_j} \cdot \frac{e_j^2 - 1}{e_j} + \frac{\partial F_{jk}}{\partial i_j} \cdot \frac{\cos i_j - 1}{\sin i_j} \right\}. \quad (2.3)$$

Let us express $\omega_{jk}^{(R)}$ in (2.2) in terms of Delaunay variables. Since

$$R_{jk} = \sqrt{(\mathbf{R}_j - \mathbf{R}_k)^2} = \sqrt{R_j^2 - 2R_j R_k (\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + R_k^2},$$

we have

$$\begin{aligned}\left(\frac{\partial}{\partial G_j} + \frac{\partial}{\partial H_j} \right) \frac{1}{R_{jk}} &= -\frac{1}{(R_j^2 - 2R_j R_k (\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + R_k^2)^{3/2}} \times \\ &\times \left\{ R_j \frac{\partial R_j}{\partial G_j} - \frac{\partial R_j}{\partial G_j} R_k (\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) - R_j R_k \left(\frac{\partial \boldsymbol{\xi}_j}{\partial G_j} + \frac{\partial \boldsymbol{\xi}_j}{\partial H_j}, \boldsymbol{\xi}_k \right) \right\}.\end{aligned} \quad (2.4)$$

Let us modify the right-hand side of (2.4) using equality (1.22) for R_j , R_k and the following expressions for the partial derivatives:

$$\begin{aligned}\frac{\partial R_j}{\partial G_j} &= \frac{G_j \cos \vartheta_j}{f_0 m_j^2 e_j}, \quad \frac{\partial \vartheta_j}{\partial G_j} = -\frac{\sin \vartheta_j (2 + e_j \cos \vartheta_j)}{e_j G_j}, \\ \frac{\partial \boldsymbol{\xi}_j}{\partial G_j} + \frac{\partial \boldsymbol{\xi}_j}{\partial H_j} &= \frac{\partial \boldsymbol{\xi}_j}{\partial \vartheta_j} \frac{\partial \vartheta_j}{\partial G_j} + \frac{\partial \boldsymbol{\xi}_j}{\partial i_j} \left(\frac{\partial i_j}{\partial G_j} + \frac{\partial i_j}{\partial H_j} \right).\end{aligned}$$

In the end we get

$$\omega_{jk}^{(R)} = \frac{\varepsilon m_k (1 - e_j^2)^2}{4\pi^2 e_j a_k^2} \sqrt{\frac{f a_j}{m_s (1 - e_k^2)}} \int_0^{2\pi} \int_0^{2\pi} \Phi_{jk} d\vartheta_j d\vartheta_k, \quad (2.5)$$

where

$$\begin{aligned} \Phi_{jk} = & \left\{ \lambda_j^2 - 2\mu_{jk} \lambda_j \lambda_k (\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + \lambda_k^2 \mu_{1k}^2 \right\}^{-3/2} \times \\ & \times \left\{ \mu_{jk} \lambda_k \cos \vartheta_j - \lambda_j \cos \vartheta_j (\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + \left(\frac{\partial \boldsymbol{\xi}_j}{\partial \vartheta_j}, \boldsymbol{\xi}_k \right) \sin \vartheta_j (1 + \lambda_j) + \left(\frac{\partial \boldsymbol{\xi}_j}{\partial i_j}, \boldsymbol{\xi}_k \right) \frac{(1 - \cos i_j) e_j}{\sin i_j} \right\}, \\ \mu_{jk} = & \frac{G_j^2 m_k^2}{G_k^2 m_j^2} = \frac{a_j (1 - e_j^2)}{a_k (1 - e_k^2)}. \end{aligned}$$

Let us find the numerical values of components $\omega_{jk}^{(R)}$, $\omega_{jk}^{(p)}$ for each of the eight planets in the Solar System for the epoch J2000, using the orbital elements given in [3] and the formulas (2.3) and (2.5). The results of these calculations are presented in Tables 1 and 2. All the calculations are performed using the mathematical system Mathcad.

The term $\omega_j^{(R)} = \sum_{k=1, k \neq j}^n \omega_{jk}^{(R)}$ is responsible for the gravitational disturbances caused by the rest of the planet, as in the case of the motionless Sun. The values obtained match the results of calculations performed by U. Le Verrier in 1859 [4] for Mercury and by G. M. Clemence [5, 6] for Mercury and Earth.

The component $\omega_j^{(p)} = \sum_{k=1, k \neq j}^n \omega_{jk}^{(p)}$ has appeared because the problem is considered in the barycentric coordinate system and the orbit inclinations are taken into account. This component becomes zero under the assumption that all the planets move in one static plane. The term $\omega_j^{(p)}$ influences the Mercury's and Venus's evolution of the perihelion considerably. For the rest of the planets this term is small relative to $\omega_j^{(R)}$.

Table 1. The evolution of orbits' perihelia, component $\omega_j^{(R)}$

Influencing planets	Influenced planets							
	Mercury	Venus	Earth-Moon	Mars	Jupiter	Saturn	Uranus	Neptune
Mercury	—	−150.94	−13.71	0.77	0.01	0.001	0.000	0.00
Venus	275.89	—	345.15	49.40	0.41	0.05	0.004	0.001
Earth-Moon	90.10	−574.49	—	227.61	0.99	0.11	0.01	0.002
Mars	2.46	74.88	97.45	—	0.12	0.03	0.003	0.00
Jupiter	152.77	652.78	694.13	1247.15	—	1626.28	124.72	−0.86
Saturn	7.27	8.18	19.19	66.67	628.38	—	121.16	−13.82
Uranus	0.14	0.28	0.56	1.20	10.74	27.03	—	97.29
Neptune	0.04	0.11	0.18	0.35	2.24	6.33	45.24	—
$\omega_j^{(R)}$	528.67	10.8	1142.95	1593.15	642.89	1659.83	291.14	82.61

Table 2. The evolution of orbits' perihelia, component $\omega_j^{(p)}$

Influencing planets	Influenced planets							
	Mercury	Venus	Earth-Moon	Mars	Jupiter	Saturn	Uranus	Neptune
Mercury	—	5.22	−0.00	0.05	0.005	0.06	−0.001	0.01
Venus	0.11	—	−0.00	0.01	0.001	0.01	−0.00	0.001
Earth-Moon	−0.00	−0.00	—	−0.00	−0.00	−0.00	0.00	−0.00
Mars	0.04	0.20	−0.00	—	0.00	0.00	−0.00	0.00
Jupiter	5.65	30.37	−0.00	0.28	—	0.34	−0.01	0.03
Saturn	33.94	182.36	−0.001	1.67	0.17	—	−0.05	0.18
Uranus	−0.12	−0.67	0.00	−0.01	−0.001	−0.01	—	−0.001
Neptune	0.02	0.13	−0.00	0.001	0.00	0.001	−0.00	—
$\omega_j^{(p)}$	39.64	217.61	−0.001	2.00	0.18	0.40	−0.06	0.22

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