NONLINEAR PHYSICS AND MECHANICS

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On the Dynamics of a Gravitational Dipole

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An orbital gravitational dipole is a rectilinear inextensible rod of negligibly small mass which moves in a Newtonian gravitational field and to whose ends two point loads are fastened. The gravitational dipole is mainly designed to produce artificial gravity in a neighborhood of one of the loads. In the nominal operational mode on a circular orbit the gravitational dipole is located along the radius vector of its center of mass relative to the Newtonian center of attraction.

The main purpose of this paper is to investigate nonlinear oscillations of the gravitational dipole in a neighborhood of its nominal mode. The orbit of the center of mass is assumed to be circular or elliptic with small eccentricity. Consideration is given both to planar and arbitrary spatial deviations of the gravitational dipole from its position corresponding to the nominal mode. The analysis is based on the classical Lyapunov and Poincaré methods and the methods of Kolmogorov–Arnold–Moser (KAM) theory. The necessary calculations are performed using computer algorithms. An analytic representation is given for conditionally periodic oscillations. Special attention is paid to the problem of the existence of periodic motions of the gravitational dipole and their Lyapunov stability, formal stability (stability in an arbitrarily high, but finite, nonlinear approximation) and stability for most (in the sense of Lebesgue measure) initial conditions.

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1. Introduction. Equations of motion and their particular solutions

Let l be the length of the rod of the gravitational dipole and let M and m be the masses of its end loads. Assume that l is negligible compared to the characteristic size of the orbit, so that the motion of points M and m relative to their center of mass O has no influence on the orbit of the center of mass itself [1–3]. We assume the orbit to be elliptic with eccentricity e .

Fig. 1. The orbital and associated coordinate systems. Euler angles.

Let $OXYZ$ be an orbital coordinate system (Fig. 1). The axis OX is directed along the radius vector $\mathbf{FO}$ of the center of mass of the gravitational dipole relative to the attracting center F (not shown in Fig. 1), and the axes OY and OZ are parallel, respectively, to the transversal and the normal to the plane of the orbit. The coordinate system $Oxyz$ is rigidly attached to the gravitational dipole. The axis Oz is directed along the rod in the direction from point m to point M . We specify the orientation of the attached coordinate system $Oxyz$ relative to the orbital coordinate system $OXYZ$ by Euler angles ψ , ϑ and φ . For projections of the absolute angular velocity of the rigid trihedron $Oxyz$ onto its axes we have [4] the following expressions:

$$\begin{aligned}\omega_x &= (\dot{\psi} + \dot{\nu})a_{31} + \dot{\vartheta} \cos \varphi, \\ \omega_y &= (\dot{\psi} + \dot{\nu})a_{32} - \dot{\vartheta} \sin \varphi,\end{aligned}\tag{1.1}$$

$$\begin{aligned}\omega_z &= (\dot{\psi} + \dot{\nu})a_{33} + \dot{\varphi}, \\ a_{31} &= \sin \vartheta \sin \varphi, \quad a_{32} = \sin \vartheta \cos \varphi, \quad a_{33} = \cos \vartheta.\end{aligned}\tag{1.2}$$

The dot in (1.1) denotes the time derivative t , and $\dot{\nu}$ is the angular velocity of rotation of the vector $\mathbf{FO}$ (ν is the true anomaly in the Keplerian motion of the center of mass in the elliptic orbit),

$$\dot{\nu} = \frac{\omega_0}{(1 - e^2)^{3/2}} (1 + e \cos \nu)^2,\tag{1.3}$$

where ω_0 is the angular velocity for circular ($e = 0$) motion of the center of mass.

For the moments of inertia of the gravitational dipole relative to the axes Ox , Oy and Oz we have the expressions

$$A = B = \frac{mM}{m + M} l^2, \quad C = 0.\tag{1.4}$$

The direction cosines a_{11} , a_{12} , a_{13} of the vector $\mathbf{FO}$ in the system $Oxyz$ are given by the equations

$$\begin{aligned} a_{11} &= \cos \psi \cos \varphi - \sin \psi \cos \vartheta \sin \varphi, \\ a_{12} &= -\cos \psi \sin \varphi - \sin \psi \cos \vartheta \cos \varphi, \\ a_{13} &= \sin \psi \sin \vartheta. \end{aligned} \quad (1.5)$$

The kinetic energy of the gravitational dipole in its motion relative to the center of mass is calculated by the formula

$$T = \frac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2),$$

which, with Eqs. (1.1), (1.2) and (1.4) taken into account, is written as

$$T = \frac{1}{2}A \left[(\dot{\psi} + \dot{\nu})^2 \sin^2 \vartheta + \dot{\vartheta}^2 \right]. \quad (1.6)$$

The potential energy in the motion of the gravitational dipole relative to the center of mass is given by the equation [1, 2]

$$\Pi = \frac{3}{2} \frac{\omega_0^2}{(1 - e^2)^3} (1 + e \cos \nu)^3 (Aa_{11}^2 + Ba_{12}^2 + Ca_{13}^2).$$

Taking expressions (1.4) and (1.5) into account, we can rewrite this equation as

$$\Pi = -\frac{3}{2} \frac{A\omega_0^2}{(1 - e^2)^3} (1 + e \cos \nu)^3 \sin^2 \psi \sin^2 \vartheta. \quad (1.7)$$

In (1.7), the quantities that are independent of the Euler angles and insignificant for the equations of motion have been neglected.

The equations of motion can be written in the form of a system of ordinary differential fourth-order equations with the Lagrangian $L = T - \Pi$. The precession and nutation angles ψ and ϑ will be generalized coordinates. But in what follows the Hamiltonian form of the equations of motion will be used. The dimensionless momenta p_ψ and p_ϑ corresponding to the angles ψ and ϑ will be given by the equations

$$\frac{\partial L}{\partial \dot{\psi}} = \frac{A\omega_0}{(1 - e^2)^{3/2}} p_\psi, \quad \frac{\partial L}{\partial \dot{\vartheta}} = \frac{A\omega_0}{(1 - e^2)^{3/2}} p_\vartheta, \quad (1.8)$$

and the true anomaly ν (see (1.3)) will be taken as an independent variable. In the variables ψ , ϑ , p_ψ and p_ϑ the equations of motion of the gravitational dipole relative to its center of mass have the form

$$\frac{d\psi}{d\nu} = \frac{\partial G}{\partial p_\psi}, \quad \frac{d\vartheta}{d\nu} = \frac{\partial G}{\partial p_\vartheta}, \quad \frac{dp_\psi}{d\nu} = -\frac{\partial G}{\partial \psi}, \quad \frac{dp_\vartheta}{d\nu} = -\frac{\partial G}{\partial \vartheta}, \quad (1.9)$$

where [4]

$$G = \frac{p_\psi^2}{2(1 + e \cos \nu)^2 \sin^2 \vartheta} + \frac{p_\vartheta^2}{2(1 + e \cos \nu)^2} - p_\psi - \frac{3}{2}(1 + e \cos \nu) \sin^2 \psi \sin^2 \vartheta. \quad (1.10)$$

If the orbit is circular, then Eqs. (1.9) admit the (Jacobi) integral

$$\frac{p_\psi^2}{2 \sin^2 \vartheta} + \frac{p_\vartheta^2}{2} - p_\psi - \frac{3}{2} \sin^2 \psi \sin^2 \vartheta = \text{const.} \quad (1.11)$$

In the case of a circular orbit, Eqs. (1.9) have the particular solution

$$\psi = \frac{\pi}{2}, \quad \vartheta = \frac{\pi}{2}, \quad p_\psi = 1, \quad p_\vartheta = 0, \quad (1.12)$$

which corresponds to the equilibrium position of the gravitational dipole in the orbital coordinate system $OXYZ$ when it is fixed along the axis OX . This equilibrium is Lyapunov stable since in its neighborhood the Jacobi integral (1.11) is a positive definite function [1–4].

Fig. 2. A gravidipole in the plane of an elliptical orbit.

In the case of an orbit of arbitrary eccentricity, Eqs. (1.9) admit a family of particular solutions corresponding to motions in which the gravitational dipole always remains in the plane of the orbit (Fig. 2). For this family $\vartheta = \pi/2$, $p_\vartheta = 0$, and the quantities ψ and p_ψ are defined from the equations

$$\frac{d\psi}{d\nu} = \frac{p_\psi}{(1 + e \cos \nu)^2} - 1, \quad \frac{dp_\psi}{d\nu} = 3(1 + e \cos \nu) \sin \psi \cos \psi. \quad (1.13)$$

The motions described by these equations have previously been examined in sufficient detail (see the monographs [1–4] and references therein). In the case of a circular orbit, they describe pendulum motions of the gravitational dipole: oscillations in a neighborhood of stable equilibrium $\psi = \pm\pi/2$ with an arbitrary amplitude (not exceeding $\pi/2$), unstable equilibrium $\psi = 0$ or π (the gravitational dipole lies along the tangent to the orbit), rotations in the plane of the orbit and motions asymptotic to unstable equilibrium.

In an orbit of small eccentricity ($0 < e \ll 1$), stable equilibrium (1.12) gives rise to the motion $\psi = \psi^*(\nu; e)$, $p_\psi = p_\psi^*(\nu; e)$ which is 2π -periodic in ν (eccentricity oscillations):

$$\vartheta = \frac{\pi}{2}, \quad p_\vartheta = 0, \quad (1.14)$$

$$\psi^* = \frac{\pi}{2} + e \sin \nu - \frac{3}{2}e^2 \sin 2\nu + \dots, \quad p_\psi^* = 1 + 3e \cos \nu + 3e^2 \sin^2 \nu + \dots. \quad (1.15)$$

The dots in (1.15) denote convergent series starting with terms of degree 3 and higher in e . It is shown [5] that eccentricity oscillations are stable with respect to perturbations ψ and p_ψ .

The main purpose of this paper is to investigate nonlinear oscillations of the gravitational dipole in a circular or a weakly elliptic orbit in a neighborhood of an equilibrium point or eccentricity oscillations in the presence of arbitrary small perturbations (both those leaving the gravitational dipole in the plane of the orbit and arbitrary spacial perturbations). Special

attention is also given to the nonlinear problem of constructing and investigating the stability of periodic Lyapunov oscillations arising from stable equilibrium, and to the linear problem of the stability of oscillations of the gravitational dipole in a circular orbit under amplitudes that are not small.

Small linear oscillations of the gravitational dipole in a neighborhood of stable equilibrium in a circular orbit are described in detail in the monograph [6]. Regions of linear instability and stability of planar oscillations of the gravitational dipole of large amplitude (close to $\pi/2$) in a circular orbit are studied in [7].

2. The Hamiltonian function in a neighborhood of the particular solutions (1.12) and (1.14), (1.15)

To explore the motion of the gravitational dipole in a neighborhood of its equilibrium (1.12) in a circular orbit or in a neighborhood of eccentricity oscillations (1.14), (1.15) in an elliptic orbit, we introduce new canonically conjugate variables Q_1 , Q_2 , P_1 and P_2 by setting

$$\vartheta = \frac{\pi}{2} + Q_1, \quad \psi = \psi^* + Q_2, \quad p_\vartheta = P_1, \quad p_\psi = p_\psi^* + P_2. \quad (2.1)$$

The new variables correspond to the Hamiltonian function that is calculated by the formula

$$\Phi = G + \frac{dp_\psi^*}{d\nu} Q_2 - \frac{d\psi^*}{d\nu} P_2, \quad (2.2)$$

where G is the function (1.10), in which ϑ , ψ , p_ϑ and p_ψ have been replaced with their expressions (2.1).

The function (2.2) can be represented by a convergent series in powers of eccentricity e :

$$\Phi = \Phi_0 + e\Phi_1 + e^2\Phi_2 + O(e^3), \quad (2.3)$$

$$\Phi_0 = \frac{(P_2 + \sin^2 Q_1)^2}{2 \cos^2 Q_1} + \frac{1}{2} P_1^2 - \frac{1}{2} \cos^2 Q_1 (1 + 3 \cos^2 Q_2), \quad (2.4)$$

$$\begin{aligned} \Phi_1 = & -\frac{\cos \nu (P_2^2 - \sin^2 Q_1 P_2 - 3 \sin^2 Q_1 + 1)}{\cos^2 Q_1} - \cos \nu P_1^2 + \\ & + \frac{3}{2} \sin \nu (\sin 2Q_2 \cos^2 Q_1 - 2Q_2) - \frac{3}{2} \cos \nu \cos^2 Q_1 \cos^2 Q_2, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \Phi_2 = & \frac{3 \cos^2 \nu P_2^2 - 2 \cos 2\nu \sin^2 Q_1 P_2 + 2 \sin^2 \nu \sin^2 Q_1}{2 \cos^2 Q_1} + \frac{3}{2} \cos^2 \nu P_1^2 - \\ & - \frac{3}{2} \sin 2\nu (\sin 2Q_2 \cos^2 Q_1 - 2Q_2) + \frac{3}{2} \sin^2 \nu \cos^2 Q_1 \cos 2Q_2. \end{aligned} \quad (2.6)$$

3. On the methods and algorithms of investigation

The analysis of the motion of the gravitational dipole is carried out using classical and modern analytical and numerical methods of investigating dynamical systems described by differential Hamilton equations [8–21].

To analyze motion near stable equilibrium of the gravitational dipole or its eccentricity oscillations, use is made of the algorithms of the Deprit–Hori method [4, 22]. This method

considers systems with the Hamiltonian function Ψ which can be represented by a series in powers of ε ($0 < \varepsilon \ll 1$):

$$\Psi = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \psi_m(\mathbf{u}, \mathbf{U}, \nu), \quad (3.1)$$

where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{U} = (U_1, U_2)$ are vectors of generalized coordinates and momenta (in the problem of the dynamics of the gravitational dipole we consider here they are two-dimensional). A search is performed for the canonical univalent near-identity transformation

$$\mathbf{u}, \mathbf{U} \rightarrow \mathbf{v}, \mathbf{V}, \quad (3.2)$$

which transforms the Hamiltonian (3.1) to the new function

$$\Xi = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \chi_m(\mathbf{v}, \mathbf{V}, \nu). \quad (3.3)$$

The generating function W of the transformation is represented as the series

$$W = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} W_{m+1}(\mathbf{u}, \mathbf{U}, \nu). \quad (3.4)$$

To calculate the functions χ_m in the expansion (3.3) and the transformation (3.2), recurrent relations have been obtained in the Deprit–Hori method. We write them out for $m = 0, 1$ and 2 :

$$\chi_0 = \psi_0(\mathbf{v}, \mathbf{V}, \nu), \quad (3.5)$$

$$\chi_1 = \psi_1(\mathbf{v}, \mathbf{V}, \nu) + L_1 \psi_0(\mathbf{v}, \mathbf{V}, \nu) - \frac{\partial W_1(\mathbf{v}, \mathbf{V}, \nu)}{\partial \nu}, \quad (3.6)$$

$$\begin{aligned} \chi_2 = & \psi_2(\mathbf{v}, \mathbf{V}, \nu) + L_1 \psi_1(\mathbf{v}, \mathbf{V}, \nu) + L_1 \chi_1(\mathbf{v}, \mathbf{V}, \nu) + \\ & + L_2 \psi_0(\mathbf{v}, \mathbf{V}, \nu) - \frac{\partial W_2(\mathbf{v}, \mathbf{V}, \nu)}{\partial \nu}, \quad (L_m R = (R, W_m)). \end{aligned} \quad (3.7)$$

The change of variables (3.2) is expressed in terms of the functions $W_m(\mathbf{v}, \mathbf{V}, \nu)$ by the formulae

$$\begin{aligned} u_j = & v_j + \varepsilon \frac{\partial W_1}{\partial V_j} + \frac{1}{2} \varepsilon^2 \left(\frac{\partial W_2}{\partial V_j} + \left(\frac{\partial W_1}{\partial V_j}, W_1 \right) \right) + O(\varepsilon^3), \\ U_j = & V_j - \varepsilon \frac{\partial W_1}{\partial v_j} - \frac{1}{2} \varepsilon^2 \left(\frac{\partial W_2}{\partial v_j} + \left(\frac{\partial W_1}{\partial v_j}, W_1 \right) \right) + O(\varepsilon^3) \quad (j = 1, 2). \end{aligned} \quad (3.8)$$

In (3.6)–(3.8), (f, g) denotes the Poisson bracket of the functions of $\mathbf{v}, \mathbf{V}$ and ν :

$$(f, g) = \sum_{j=1}^2 \left(\frac{\partial f}{\partial v_j} \frac{\partial g}{\partial V_j} - \frac{\partial f}{\partial V_j} \frac{\partial g}{\partial v_j} \right).$$

In the problem of the motion of the gravitational dipole the functions W_1 and W_2 , which are defined from Eqs. (3.6) and (3.7), either do not depend on ν (in the case of a circular orbit) or are 2π -periodic in ν (when the orbit is elliptic). They are chosen so as to simplify as much as possible the functions χ_1 and χ_2 in the transformed Hamiltonian function

$$\Xi = \chi_0 + \varepsilon \chi_1 + \frac{1}{2} \varepsilon^2 \chi_2 + O(\varepsilon^3). \quad (3.9)$$

4. A circular orbit

In the case of a circular orbit the motion of the gravitational dipole relative to its center of mass is described by equations given by the Hamiltonian function Φ_0 from (2.4). Its series expansion contains no terms linear in Q_1, Q_2, P_1, P_2 . Neglecting the constant to which the function Φ_0 is equal in the stable equilibrium position (1.12) and introducing, instead of Q_1, Q_2, P_1, P_2 , new canonically conjugate variables x_1, x_2, X_1, X_2 by the formulae

$$Q_1 = \frac{1}{\sqrt{2}}x_1, \quad Q_2 = \frac{1}{3^{1/4}}x_2, \quad P_1 = \sqrt{2}X_1, \quad P_2 = 3^{1/4}X_2, \quad (4.1)$$

we find that the new variables correspond to the Hamiltonian function Γ which can be represented by the convergent series

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + \cdots, \quad (4.2)$$

where Γ_k is a form of degree k in x_1, x_2, X_1, X_2 , and

$$\Gamma_2 = \frac{1}{2}\sigma_1(x_1^2 + X_1^2) + \frac{1}{2}\sigma_2(x_2^2 + X_2^2), \quad (4.3)$$

$$\Gamma_3 = \frac{3^{1/4}}{2}x_1^2X_2, \quad \Gamma_4 = -\frac{1}{24}x_1^4 - \frac{\sqrt{3}}{4}x_1^2x_2^2 + \frac{\sqrt{3}}{4}x_1^2X_2^2 - \frac{1}{6}x_2^4. \quad (4.4)$$

In (4.3), σ_1 and σ_2 are frequencies of small (linear) oscillations of the gravitational dipole in a neighborhood of its stable equilibrium (1.12) in the circular orbit:

$$\sigma_1 = 2, \quad \sigma_2 = \sqrt{3}. \quad (4.5)$$

Conditionally periodic motions. The quantity $k_1\sigma_1 + k_2\sigma_2$ cannot vanish for any integers k_1 and k_2 that are nonzero simultaneously. Consequently, the problem of nonlinear oscillations near the equilibrium (1.12) is nonresonant.

We reduce the function Γ to its normal Birkhoff form in terms through degree four. To do so, we use the Deprit–Hori method described above. It is more convenient to perform calculations in complex conjugate variables rather than real variables. To this end we make the canonical transformation $x_j, X_j \rightarrow u_j, U_j$ ($j = 1, 2$)

$$u_j = X_j + ix_j, \quad U_j = X_j - ix_j \quad (i \text{ being an imaginary unit}). \quad (4.6)$$

The transformed Hamiltonian function is $\Psi = 2i\Gamma$, where Γ is the function (4.2), in which the variables x_j and X_j are expressed in terms of u_j and U_j in accordance with the transformation (4.6). If we assume that the quantities x_j and X_j are of order ε , then the function Ψ can be written as the series (3.1), in which the functions ψ_m do not depend on ν , and

$$\psi_0 = i\sigma_1 u_1 U_1 + i\sigma_2 u_2 U_2, \quad \psi_1 = 2i\Gamma_3, \quad \psi_2 = 4i\Gamma_4. \quad (4.7)$$

In the last two equations, Γ_3 and Γ_4 are expressed in terms of u_1, u_2, U_1, U_2 .

The transformation (3.2) with the generating function (3.4) reduces the function Ψ to the function (3.3), which does not depend on ν . If we then pass from the complex conjugate variables to real coordinates and momenta z_j and Z_j by the formulae

$$z_j = \frac{1}{2i}(v_j - V_j), \quad Z_j = \frac{1}{2}(v_j + V_j) \quad (j = 1, 2), \quad (4.8)$$

then we obtain the normal form F of the Hamiltonian function in real variables: $F = \Xi/(2i)$, where in the function Ξ the variables v_j and V_j are expressed in terms of z_j and Z_j by formulae (4.8).

The above changes of variables $x_j, X_j \rightarrow u_j, U_j \rightarrow v_j, V_j \rightarrow z_j, Z_j$ were made using computer algorithms of analytical calculations. It is shown that the variables x_j and X_j are expressed in terms of z_j and Z_j by the following formulae:

$$\begin{aligned} x_1 = z_1 - \frac{2}{13}3^{1/4}z_1Z_2 + \frac{8}{39}3^{3/4}z_2Z_1 + \frac{617}{16\,224}z_1^3 + \frac{425}{4056}\sqrt{3}z_1z_2^2 + \\ + \frac{223}{5408}z_1Z_1^2 - \frac{153}{1352}\sqrt{3}z_1Z_2^2 + \frac{7}{13}z_2Z_1Z_2 + O_4, \end{aligned} \quad (4.9)$$

$$\begin{aligned} x_2 = z_2 - \frac{2}{13}3^{1/4}z_1Z_1 - \frac{19}{676}z_1^2z_2 - \frac{1}{4}\sqrt{3}z_1Z_1Z_2 + \frac{5}{144}\sqrt{3}z_2^3 + \\ + \frac{75}{338}z_2Z_1^2 + \frac{1}{16}\sqrt{3}z_2Z_2^2 + O_4, \end{aligned} \quad (4.10)$$

$$\begin{aligned} X_1 = Z_1 - \frac{5}{39}3^{3/4}z_1z_2 + \frac{2}{13}3^{1/4}Z_1Z_2 - \frac{297}{5408}z_1^2Z_1 + \frac{3}{26}z_1z_2Z_2 - \\ - \frac{745}{4056}\sqrt{3}z_2^2Z_1 - \frac{245}{5408}Z_1^3 + \frac{185}{1352}\sqrt{3}Z_1Z_2^2 + O_4, \end{aligned} \quad (4.11)$$

$$\begin{aligned} X_2 = Z_2 - \frac{5}{78}3^{3/4}z_1^2 - \frac{4}{39}3^{3/4}Z_1^2 + \frac{59}{676}z_1^2Z_2 - \frac{15}{52}\sqrt{3}z_1z_2Z_1 - \\ - \frac{5}{48}\sqrt{3}z_2^2Z_2 - \frac{107}{338}Z_1^2Z_2 - \frac{1}{48}\sqrt{3}Z_2^3 + O_4. \end{aligned} \quad (4.12)$$

In (4.9)–(4.12), O_4 denotes the set of terms of degree higher than three in z_1, z_2, Z_1, Z_2 .

If we now pass from the variables z_j and Z_j to symplectic polar coordinates r_j and φ_j by the formulae

$$z_j = \sqrt{2r_j} \sin \varphi_j, \quad Z_j = \sqrt{2r_j} \cos \varphi_j \quad (j = 1, 2), \quad (4.13)$$

then the normalized Hamiltonian function F takes the form

$$F = H + O\left((r_1 + r_2)^{5/2}\right), \quad (4.14)$$

$$H = \sigma_1 r_1 + \sigma_2 r_2 - \frac{9}{52}r_1^2 - \frac{\sqrt{3}}{13}r_1 r_2 - \frac{1}{4}r_2^2. \quad (4.15)$$

The Hamiltonian function (4.15) is nondegenerate since

$$\det \left(\frac{\partial^2 H}{\partial r_i \partial r_j} \right)_{i,j=1}^2 = \frac{105}{676} \neq 0. \quad (4.16)$$

Therefore, according to KAM theory [11–15], in a neighborhood of the equilibrium (1.12) there exist invariant tori that are close to tori of a linearized system. The motion of the gravitational dipole which corresponds to these tori is conditionally periodic with frequencies ω_1 and ω_2 :

$$\omega_1 = \frac{\partial H}{\partial r_1} = \sigma_1 - \frac{9}{26}r_1 - \frac{\sqrt{3}}{13}r_2, \quad \omega_2 = \frac{\partial H}{\partial r_2} = \sigma_2 - \frac{\sqrt{3}}{13}r_1 - \frac{1}{2}r_2. \quad (4.17)$$

For the trajectories $Q_j(\nu)$ and $P_j(\nu)$ starting in the small neighborhood $\sum_{j=1}^2 [Q_j^2 + P_j^2] < \mu$ of the equilibrium (1.12), the relative measure of the initial conditions corresponding to conditionally periodic motions of the gravitational dipole is no less than $1 - O(\mu^{1/4})$.

On periodic Lyapunov motions. Since the ratio of frequencies (4.5) is not an integer, it follows by the Lyapunov theorem of a holomorphic integral [9] that there exist families of short-periodic and long-periodic (in ν) motions of the gravitational dipole with periods close to $2\pi/\sigma_1$ and $2\pi/\sigma_2$.

1. We first consider short-periodic motions. Using the canonical transformation analytic in z_j, Z_j ($j = 1, 2$), from the expansion of the function $F(z_1, z_2, Z_1, Z_2)$ as a series (see the previous section) one can eliminate terms linear in z_2 and Z_2 . And then in the transformed Hamiltonian function (4.14) the quantity $O((r_1 + r_2)^{5/2})$, which can be represented by a series in powers of $r_1^{1/2}$ and $r_2^{1/2}$, will contain no terms linear in $r_2^{1/2}$, and the equations of motion admit particular solutions corresponding to short-periodic oscillations of the gravitational dipole:

$$z_1 = \varepsilon \sin \varphi_1, \quad Z_1 = \varepsilon \cos \varphi_1, \quad z_2 = 0, \quad Z_2 = 0 \quad (0 < \varepsilon \ll 1), \quad (4.18)$$

$$\varphi_1 = \tau + \varphi_1(0), \quad \tau = \Omega_1 \nu, \quad \Omega_1 = 2 - \frac{9}{52} \varepsilon^2 + O(\varepsilon^4). \quad (4.19)$$

Here Ω_1 is the frequency ω_1 from (4.17), which has been calculated for the values $r_1 = \varepsilon^2/2$ and $r_2 = 0$ corresponding to (4.18).

It follows from (4.1) and (4.9)–(4.12) that in the variables Q_1, Q_2, P_1 and P_2 (see (2.1)) the family of solutions (4.18) can be written as

$$\begin{aligned} Q_1 &= \varepsilon \frac{\sqrt{2}}{2} \sin \varphi_1 + \varepsilon^3 \frac{\sqrt{2}}{32448} (630 \sin \varphi_1 + 13 \sin 3\varphi_1) + O(\varepsilon^4), \\ Q_2 &= -\varepsilon^2 \frac{1}{13} \sin 2\varphi_1 + O(\varepsilon^4), \\ P_1 &= \varepsilon \sqrt{2} \cos \varphi_1 - \varepsilon^3 \frac{\sqrt{2}}{5408} (258 \cos \varphi_1 - 13 \cos 3\varphi_1) + O(\varepsilon^4), \\ P_2 &= -\varepsilon^2 \frac{1}{52} (13 + 3 \cos 2\varphi_1) + O(\varepsilon^4). \end{aligned} \quad (4.20)$$

To investigate the orbital stability of periodic motions (4.20), we introduce perturbations η_1 and η_2 by setting

$$r_1 = \frac{1}{2} \varepsilon^2 + \eta_1, \quad r_2 = \eta_2.$$

Substituting these values of r_1 and r_2 into the function (4.14) and passing to the new independent variable τ , we obtain a Hamiltonian function of perturbed motion in the form

$$F = \eta_1 + \lambda \eta_2 + c_{20} \eta_1^2 + c_{11} \eta_1 \eta_2 + c_{02} \eta_2^2 + O((|\eta_1| + \eta_2)^{5/2}), \quad (4.21)$$

where

$$\begin{aligned} \lambda &= \frac{\sqrt{3}}{2} + \frac{5\sqrt{3}}{208} \varepsilon^2 + O(\varepsilon^4), & c_{20} &= -\frac{9}{104} - \frac{81}{10816} \varepsilon^2 + O(\varepsilon^4), \\ c_{11} &= -\frac{\sqrt{3}}{26} - \frac{9\sqrt{3}}{2704} \varepsilon^2 + O(\varepsilon^4), & c_{02} &= -\frac{1}{8} - \frac{9}{832} \varepsilon^2 + O(\varepsilon^4). \end{aligned}$$

Set

$$D = c_{20} \lambda^2 - c_{11} \lambda + c_{02}. \quad (4.22)$$

The condition $D \neq 0$ of the isoenergetic nondegeneracy of the function (4.21) is a sufficient condition for orbital stability [13–16]. In the case we consider here,

$$D = -\frac{55}{416} - \frac{645}{43264} \varepsilon^2 + O(\varepsilon^4),$$

therefore, at sufficiently small ε the short-periodic motions of the gravitational dipole are orbitally stable.

2. The family of long-periodic motions is treated in a similar way. For this family,

$$z_1 = 0, \quad Z_1 = 0, \quad z_2 = \varepsilon \sin \varphi_2, \quad Z_2 = \varepsilon \cos \varphi_2, \quad (4.23)$$

$$\varphi_2 = \tau + \varphi_2(0), \quad \tau = \Omega_2 \nu, \quad \Omega_2 = \sqrt{3} - \frac{1}{4}\varepsilon^2 + O(\varepsilon^4). \quad (4.24)$$

In the variables Q_j and P_j we have

$$\begin{aligned} Q_1 &= 0, \quad P_1 = 0, \\ Q_2 &= \varepsilon \frac{3^{3/4}}{3} \sin \varphi_2 + \varepsilon^3 \frac{3^{1/4}}{144} (6 \sin \varphi_2 + \sin 3\varphi_2) + O(\varepsilon^4), \\ P_2 &= \varepsilon 3^{1/4} \cos \varphi_2 - \varepsilon^3 \frac{3^{3/4}}{48} (2 \cos \varphi_2 - \cos 3\varphi_2) + O(\varepsilon^4). \end{aligned} \quad (4.25)$$

The long-periodic motions are oscillations of the gravitational dipole in the plane of the orbit.

In analyzing their orbital stability we obtain a Hamiltonian function of perturbed motion in the form (4.21), such that

$$\begin{aligned} \lambda &= \frac{2\sqrt{3}}{3} + \frac{5}{39}\varepsilon^2 + O(\varepsilon^4), & c_{20} &= -\frac{\sqrt{3}}{12} - \frac{1}{48}\varepsilon^2 + O(\varepsilon^4), \\ c_{11} &= -\frac{1}{13} - \frac{\sqrt{3}}{156}\varepsilon^2 + O(\varepsilon^4), & c_{02} &= -\frac{3\sqrt{3}}{52} - \frac{3}{208}\varepsilon^2 + O(\varepsilon^4). \end{aligned}$$

From this and Eq. (4.22) we find

$$D = -\frac{55\sqrt{3}}{468} - \frac{505}{8112}\varepsilon^2 + O(\varepsilon^4),$$

therefore, at small ε the planar oscillations of the gravitational dipole are orbitally stable.

Planar oscillations of arbitrary amplitude. Consider the problem of the stability of planar oscillations without assuming their amplitudes to be small. We restrict ourselves to analysis in the first (linear) approximation relative to small spatial perturbations.

For ease of calculations we make in the Hamiltonian function (2.4), which defines the equations of motion of the gravitational dipole in a circular orbit, a canonical (with valence $4\sqrt{3}/3$) change of variables $Q_1, P_1, Q_2, P_2 \rightarrow Q, P, q, p$ by the formulae

$$Q_1 = \frac{1}{2}Q, \quad P_1 = \frac{\sqrt{3}}{2}P, \quad Q_2 = \frac{1}{2}q, \quad P_2 = \frac{\sqrt{3}}{2}p$$

and, instead of ν , we take the quantity $u = \sqrt{3}\nu$ as an independent variable. The new variables correspond to the function Φ_0 which can be represented by the convergent series

$$\Phi_0 = \varphi_0 + \varphi_2 + \dots \quad (4.26)$$

Here

$$\varphi_0 = \frac{1}{2}p^2 - \cos q, \quad \varphi_2 = \frac{1}{2}P^2 + \frac{1}{2}fQ^2, \quad (4.27)$$

$$f = \frac{1}{4}p^2 + \frac{\sqrt{3}}{3}p + \frac{1}{2}\cos q + \frac{5}{6}. \quad (4.28)$$

The dots in (4.26) denote a series in powers of Q which starts with terms of degree four, the coefficients of the series being functions of q and p .

The planar oscillations of the gravitational dipole under study are described by equations with the Hamiltonian function φ_0 . They are pendulum-like oscillations that are expressed in terms of elliptic Jacobi functions. Let $q = 0$ at the initial time, and let α be the amplitude of oscillations ($0 < \alpha < \pi$). Then

$$q = 2 \arcsin[k \operatorname{sn}(u, k)], \quad p = 2k \operatorname{cn}(u, k) \quad (k = \sin(\alpha/2)). \quad (4.29)$$

Under the oscillations (4.29) the function φ_0 is constant,

$$\varphi_0 = 2k^2 - 1. \quad (4.30)$$

The linearized equations of perturbed motion are given by the function φ_2 . If we take the quantity $w = \pi u / (2K(k))$ as an independent variable, then these equations can be written as

$$\frac{dQ}{dw} = \frac{2K(k)}{\pi} P, \quad \frac{dP}{dw} = -\frac{2K(k)}{\pi} f Q. \quad (4.31)$$

Here f is the function (4.28). It is 2π -periodic in w and, with Eqs. (4.29) and (4.30) taken into account, can be represented as

$$f = 2k^2 \operatorname{cn}^2\left(\frac{2K(k)}{\pi} w, k\right) + \frac{2\sqrt{3}}{3} k \operatorname{cn}\left(\frac{2K(k)}{\pi} w, k\right) - k^2 + \frac{4}{3}. \quad (4.32)$$

If $k = 0$, the function f is equal to $4/3$. Since $4/3$ is not equal to the square of a half-integer, parametric resonance is impossible at small k and stability [8] takes place. This conclusion is in agreement with the previous section, where the stability of small planar oscillations of small amplitude is shown not only in the linearized problem of stability, but also in its rigorous nonlinear version.

Under amplitudes that are not small, stability was investigated using numerical calculations. To this end, a calculation was made of the coefficient a depending on k in the characteristic equation of the monodromy matrix of the system (4.31)

$$\varrho^2 - 2a\varrho + 1 = 0. \quad (4.33)$$

If $|a| > 1$, then the oscillations are unstable (in the rigorous version of the stability problem), and if $|a| < 1$, then linear stability [8] takes place.

A graph of the function $a(k)$ is presented in Fig. 3 for values of k from the interval $(0, 0.99)$. If $0 < k < k_* = 0.84860807\dots$, then $-1 < a < 0.563638594\dots$. The value $k = k_*$ corresponds (see (4.29)) to the value $\alpha = \alpha_* = 2.026697153\dots$. Thus, oscillations of the gravitational dipole with an amplitude not exceeding $Q_2^* = \alpha_*/2 \simeq 58^\circ$ are linearly stable.

Under amplitudes lying in the interval $Q_2^* < Q_2 < Q_2^{**} \simeq 75^\circ$, which correspond to the values of k from the narrow interval

$$k_* < k < k_{**} = 0.968158697\dots, \quad (4.34)$$

the inequality $a < -1$ holds. Here the planar oscillations of the gravitational dipole are unstable. When $k > k_{**}$, the first instability interval (4.34) will be followed by the stability interval in which $-1 < a < 1$, and then by the second instability interval in which $a > 1$:

$$0.992381028\dots < k < 0.997869232\dots \quad (4.35)$$

Fig. 3. Graph of the function $a(k)$ for $0 < k < 0.99$.

As k increases further, the intervals of linear stability and intervals of instability alternate with each other and become increasingly narrow. For example, the three instability intervals following the interval (4.35) are as follows:

$$0.9994\dots < k < 0.9998\dots, \quad 0.99996\dots < k < 0.99999\dots, \quad 0.999997 < k < 0.999999\dots$$

No calculations have been made for values of k that are larger than 0.999999.

5. The elliptic orbit of small eccentricity

Let us make a transformation $Q_j, P_j \rightarrow y_j, Y_j$ in the Hamiltonian function (2.3) using formulae similar to (4.1):

$$Q_1 = \frac{1}{\sqrt{2}}y_1, \quad Q_2 = \frac{1}{3^{1/4}}y_2, \quad P_1 = \sqrt{2}Y_1, \quad P_2 = 3^{1/4}Y_2 \quad (5.1)$$

and expand it as a series in powers of y_j and Y_j through terms of degree four in y_j and Y_j . The coefficients of the series will be functions of e and ν . Assuming y_j and Y_j to be quantities of order e and using the Deprit–Hori method, we construct a 2π -periodic change of variables $y_j, Y_j \rightarrow z_j, Z_j$ which reduces the Hamiltonian function to its normal form. The calculations involved will be more cumbersome than those in the case of a circular orbit, since the change of variables will explicitly depend on ν . Computer calculations have shown that the normalizing change of variables can be represented as

$$y_j = x_j + \delta_j, \quad Y_j = X_j + \Delta_j \quad (j = 1, 2), \quad (5.2)$$

where x_j and X_j are given by Eqs. (4.9)–(4.12), in which O_4 is the set of terms of degree higher than three in e , and δ_j and Δ_j are calculated by the following formulae:

$$\delta_1 = e \left[\sin \nu \left(\frac{2}{33} 3^{3/4} z_1 z_2 + \frac{4}{11} 3^{1/4} Z_1 Z_2 \right) - \cos \nu \left(z_1 - \frac{2}{13} 3^{1/4} z_1 Z_2 + \frac{140}{429} 3^{3/4} z_2 Z_1 \right) \right] +$$

$$+e^2 \left[\frac{1}{4}(1 + 3 \cos 2\nu)z_1 - \frac{1}{2} \sin 2\nu Z_1 \right], \quad (5.3)$$

$$\begin{aligned} \delta_2 = e \left[\sin \nu \left(-\frac{4}{11}\sqrt{3}Z_2 + \frac{59}{286}3^{1/4}z_1^2 + \frac{4}{13}3^{3/4}z_2^2 + \frac{42}{143}3^{1/4}Z_1^2 + \frac{9}{13}3^{3/4}Z_2^2 \right) - \right. \\ \left. - \cos \nu \left(\frac{9}{11}z_2 - \frac{18}{143}3^{1/4}z_1Z_1 + \frac{6}{13}3^{1/4}z_2Z_2 \right) \right] - \\ - e^2 \left[\frac{3}{44} \left(\frac{1}{11} - \cos 2\nu \right) z_2 - \frac{27}{44}\sqrt{3} \sin 2\nu Z_2 \right], \quad (5.4) \end{aligned}$$

$$\begin{aligned} \Delta_1 = e \left[\sin \nu \left(\frac{1}{2}z_1 - \frac{50}{143}3^{1/4}z_1Z_2 + \frac{6}{143}3^{3/4}z_2Z_1 \right) + \right. \\ \left. + \cos \nu \left(Z_1 + \frac{10}{429}3^{3/4}z_1z_2 + \frac{2}{13}3^{1/4}Z_1Z_2 \right) \right] + \\ + e^2 \left[\frac{1}{4} \sin 2\nu z_1 + \frac{1}{4}(1 - \cos 2\nu)Z_1 \right], \quad (5.5) \end{aligned}$$

$$\begin{aligned} \Delta_2 = e \left[\sin \nu \left(\frac{7}{11}\sqrt{3}z_2 - \frac{68}{429}3^{3/4}z_1Z_1 - \frac{8}{13}3^{3/4}z_2Z_2 \right) + \right. \\ \left. + \cos \nu \left(\frac{9}{11}Z_2 + \frac{10}{429}3^{3/4}z_1^2 + \frac{10}{13}3^{1/4}z_2^2 - \frac{10}{429}3^{3/4}Z_1^2 + \frac{3}{13}3^{1/4}Z_2^2 \right) \right] - \\ - e^2 \left[\frac{1}{44}\sqrt{3} \sin 2\nu z_2 + \frac{3}{484}(1 - 99 \cos 2\nu)Z_2 \right]. \quad (5.6) \end{aligned}$$

If we introduce symplectic polar coordinates by formulae similar to (4.13):

$$\begin{aligned} z_j = \gamma_j \sqrt{2r_j} \sin \varphi_j, \quad Z_j = \gamma_j^{-1} \sqrt{2r_j} \cos \varphi_j \quad (j = 1, 2), \\ \gamma_1 = \left(\frac{2 + e^2}{2 - e^2} \right)^{1/4}, \quad \gamma_2 = \left(\frac{22 + 15e^2}{22 - 31e^2} \right)^{1/4}, \end{aligned} \quad (5.7)$$

then the normalized Hamiltonian function can be written as $F = H + O(e^5)$, where H is calculated by formula (4.15), in which σ_1 and σ_2 have been replaced with λ_1 and λ_2 which are functions of e :

$$\lambda_1 = \sqrt{4 - e^4}, \quad \lambda_2 = \frac{1}{22} \sqrt{3(22 + 15e^2)(22 - 31e^2)}. \quad (5.8)$$

1. The part of the function H that is quadratic in r_1 and r_2 is sign definite. Therefore, for a system with the Hamiltonian function F there exists a sign-definite integral that can be represented by a (possible divergent) series, i. e., the formal stability [17–19] of equilibrium $Q_j = P_j = 0$ of the system with the Hamiltonian function (2.3) takes place. This implies that the eccentricity oscillations of the gravitational dipole in the presence of spatial perturbations will be Lyapunov stable when terms up to an arbitrarily high (finite) degree in Q_j , P_j ($j = 1, 2$) are taken into account in the series expansion of the function (2.3).

2. Using KAM theory, these conclusions on stability can be significantly supplemented and reinforced. To do so, we note that the Hamiltonian function H is nondegenerate (see (4.16)). Therefore [11–16], eccentricity oscillations in the presence of spatial perturbations are metrically stable. For all values of ν , most trajectories of the system with the Hamiltonian function F are described by conditionally periodic functions with three frequencies $\omega_1 = \partial H / \partial r_1$, $\omega_2 = \partial H / \partial r_2$

and $\omega_3 = 1$. The set of initial conditions which do not correspond to these trajectories has a relative measure tending to zero as eccentricity decreases.

Trajectories starting in this set of small measure can move far away from their initial points. But this possible motion away from the origin will be extremely slow. This follows from the fact that, since the part of the function H that is quadratic in r_1 and r_2 is sign definite, it satisfies what is called the steepness conditions [20]. And if one sets $r_1(0) + r_2(0) = \varepsilon \sim e^3$, then it can be shown [20, 21] that there exist positive numbers c and d such that $\sum_j^2 |r_j(\nu) - r_j(0)| < \varepsilon^c$ for $0 < \nu < \exp(\varepsilon^{-d})$.

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