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Dynamics of a System of Two Simple Self-Excited Oscillators with Delayed Step-by-Step Feedback

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This paper studies the dynamics of a system of two coupled self-excited oscillators of first order with on-off delayed feedback using numerical and analytical methods. Regions of “fast” and “long” synchronization are identified in the parameter space, and the problem of synchronization on an unstable cycle is examined. For small coupling coefficients it is shown by analytical methods that the dynamics of the initial system is determined by the dynamics of a special one-dimensional map.

Keywords: stability, dynamics, relaxation cycles, irregular oscillations

Introduction

Phenomena of synchronization of interacting dynamical systems have been the focus of a lot of recent research. Whereas originally these phenomena were of purely theoretical interest, they later found practical application. Research done in recent years has brought about a new understanding of the role of synchronization (including the synchronization of complex, chaotic regimes) in nature, and prospects have been outlined for using them to create new information technologies. There is a large number of analytical and experimental studies of chaotic synchronization in diverse dynamical systems [1–27]. Despite the fact that a number of works present rather general results characterizing the laws of the onset of chaotic synchronization, further research into this phenomenon is both of theoretical and practical interest.

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In this paper, we investigate the phenomenon of synchronization in a system of two coupled simple self-excited oscillators of first order with nonlinear on-off delayed feedback. Such systems are widely used in a number of concrete applications, for example, in electrical engineering [4, 6, 9, 10].

The paper consists of three sections. Section 1 investigates the dynamics of the basic mathematical model of a self-excited oscillator of the above-mentioned class. It is shown that the equation has the only stable cycle and a countable set of unstable cycles. Results of numerical analysis are presented which imply that all solutions (with initial conditions satisfying the condition of the type of nondegeneracy) tend to a stable cycle as $t \rightarrow \infty$. It is of great importance to estimate the time of convergence of solutions to a cycle, depending on the delay value and the degree of complexity of the initial conditions. Part of the results of this section is presented in [5, 7].

Section 2 is the main one. It first formulates the synchronization criterion for the “system of first approximation”, the role of which is played by a discrete system of equations. It turns out that, for the initial problem of synchronization in the system of two coupled oscillators, this criterion is of great importance. When this criterion is fulfilled, a “fast” synchronization of oscillations occurs, and its violation leads to a “long” synchronization. For the latter case, a graph of the synchronization time versus the delay parameter is constructed.

As the coefficients of coupling between the oscillators decrease, the structure of solutions can become more complex; therefore, Section 3 addresses the problem of the dynamics of two weakly coupled equations. A finite-dimensional map whose dynamics describes the behavior of the solutions of the initial system is constructed.

1. Dynamics of the first-order equation

Consider the differential equation

$$\dot{x} + x = f(x(t - T)), \quad (1.1)$$

where $T > 0$ is the delay time and $f(s)$ is a function of relay type:

$$f(s) = \begin{cases} 1, & s < g \\ 0, & s \geq g \end{cases}, \quad 0 < g < 1. \quad (1.2)$$

As phase space we fix the space $C_{[-T,0]}$. We note right away that this equation has no equilibrium points.

1.1. The simplest cycle

We call the solution $x(t)$ of Eq. (1.1) *slowly oscillating* if for all t larger than some value the distances between the neighboring roots of the equation $x(t) = g$ are larger than T .

We fix a set $S \subset C_{[-T,0]}$ such that functions $\varphi(s)$ from this set satisfy the conditions $\varphi(0) = g, \varphi(s) > g$ if $s \in [-T, 0)$.

Let $x(t, \varphi)$ denote the solution of Eq. (1.1) with the initial condition $\varphi(s) \in S$ given for $t = 0$. We have the following simple result.

Lemma 1. *The solution $x(t, \varphi)$ is defined for all $t > 0$, is slowly oscillating and does not depend on the choice of the function $\varphi(s) \in S$.*

Let $x_0(t, T)$ denote this solution for $t \geq 0$. We formulate the main result.

Theorem 1. Equation (1.1) has an asymptotically orbitally stable, slowly oscillating periodic solution.

Proof. The condition of slow oscillation of $x_0(t, T)$ implies the inequality $x_0(t, T) > g$ for $t \in [-T, 0]$, and for $t \in (0, T)$ we have $x_0(t, T) = g \exp(-t)$. As t increases, starting from the value $t = T$, the function $x_0(t, T)$ first increases and, after the time interval t_1 , takes the value g :

$$x_0(T + t_1, T) = g.$$

For $t \in [T, T + t_1]$ for $x_0(t, T)$ we have the equation

$$\dot{x} + x = 1,$$

therefore, for such t

$$x_0(t, T) = (x_0(T, T) - 1) \exp(-(t - T)) + 1, \quad (1.3)$$

and hence

$$t_1 = \ln \frac{1 - g \exp(-T)}{1 - g}. \quad (1.4)$$

Let t_2 be the second positive root of the equation $x_0(t, T) = g$. Since the formula (1.3) remains valid also for $t \in [T, T + t_1]$, we have the following equation in the interval $[T + t_1, t_2]$:

$$t_2 = t_1 + T + \ln \frac{x_0(t_1 + T, T)}{g}. \quad (1.5)$$

This immediately implies periodicity of the function $x_0(t, T)$ with period t_2 . Let us consider the question of the stability of this solution.

We fix arbitrarily $\delta > 0$ and consider the time interval $\left[\frac{t_1}{2}, T + \frac{t_1}{2}\right]$ of length T . Let the functions $\psi\left(s + T + \frac{t_1}{2}\right) \in C_{[-T, 0]}$ be such that $\left|\psi\left(s + T + \frac{t_1}{2}\right)\right| \leq \delta$.

We choose the value of δ to be so small that the following inequality holds:

$$\psi\left(s + T + \frac{t_1}{2}\right) + x_0\left(s + T + \frac{t_1}{2}, T\right) < g.$$

We recall that for such s we have the relation

$$x_0\left(s + T + \frac{t_1}{2}, T\right) < g.$$

Let $x_\psi(t, T)$ stand for the solution (1.1) with the initial condition

$$\psi\left(s + T + \frac{t_1}{2}\right) + x_0\left(s + T + \frac{t_1}{2}, T\right),$$

given for $t = \frac{t_1}{2}$. Let $T + t_1(\psi, \delta)$ denote the first root of the equation $x_\psi(t, T) = g$ for $t > T + \frac{t_1}{2}$. It follows from the general properties of solutions (1.1) that the equality $t_1(\psi, \delta) - t_1 = o(1)$, which is asymptotic as $\delta \rightarrow 0$, is satisfied uniformly relative to the choice of $\psi\left(s + T + \frac{t_1}{2}\right)$. If $t > T + t_1(\psi, \delta)$, the solution $x_\psi(t, T)$ is identically equal to $x_0(t + t_1(\psi, \delta) - t_1, T)$. This implies

not only the asymptotic orbital stability of the cycle $x_0(t, T)$, but also the superexponential orbital stability of this solution. The period t_2 of this solution is defined by Eqs. (1.4) and (1.5). This proves the theorem. Figure 1a shows a graph of the solution $x_0(t, T)$ for $T = 1$ and $g = 0.3$.

Let us present asymptotic formulae for the periodic solution. We first consider the case

$$0 < g \ll 1. \quad (1.6)$$

Then the formulae (1.4) and (1.5) take the form

$$t_1 = T + o(1), \quad t_2 = O(|\ln(g)|).$$

An example of the solution $x(t)$ for $T = 1$ and $g = 0.01$ is shown in Fig. 1b.

We note that the case where parameter g is close to 1 reduces to case (1.6) if we make the transformation $x \rightarrow 1 - x$ in Eq. (1.1).

Fig. 1. a) A graph of the solution $x_0(t, T)$ for $T = 1$ and $g = 0.3$; b) an example of the solution $x(t)$ for $T = 1$ and $g = 0.01$; c) an example of a solution for $\varepsilon = 0.02$ and $g = 0.3$.

In what follows we will need asymptotics of the periodic solution for $T \rightarrow \infty$. We make the following transformations in (1.1):

$$t \rightarrow Tt, \quad x(Tt) \rightarrow x(t). \quad (1.7)$$

Setting $\varepsilon = T^{-1}$, we arrive at the equation

$$\varepsilon \dot{x} + x = f(x(t-1)). \quad (1.8)$$

Using the formulae (1.3)–(1.5), we find that, as $\varepsilon \rightarrow 0$, the simplest cycle of Eq. (1.8) is close to a step function, which alternately takes two values, 0 and 1, in the time intervals of length $1 + o(1)$, and has period $2 + o(1)$. An example of a solution for $\varepsilon = 0.02$ and $g = 0.3$ is presented in Fig. 1c.

1.2. Fast oscillating periodic solutions

Above we have established the existence of the stable periodic solution $x_0(t, T)$, which slowly oscillates near the straight line $x = g$. In this section we will show that there is a countable number of unstable periodic solutions oscillating fast near this straight line.

Consider the set of initial functions

$$C(\tau_1, \tau_2) = \left\{ \begin{aligned} &\varphi(s, \tau) \in C_{[-T, 0]}, \quad 0 < \tau_1, \tau_2 < 1, \quad \tau_1 + \tau_2 < 1, \\ &\varphi(-T + T\tau_1) = \varphi(-T + T(\tau_1 + \tau_2)) = \varphi(0) = g, \\ &\varphi(s) > g \text{ for } s \in [-T, -T + T\tau_1] \cup (-T + T(\tau_1 + \tau_2), 0), \\ &\varphi(s) < g \text{ for } s \in (-T + T\tau_1, -T + T(\tau_1 + \tau_2)) \end{aligned} \right\}.$$

We note that the solution $x(t, \tau)$ of Eq. (1.1) with the initial conditions $x(s, \tau) \in C(\tau_1, \tau_2)$ depends only on $\tau = (\tau_1, \tau_2)$ and does not depend on the choice of an element of the set $C(\tau_1, \tau_2)$. An example of the function $\varphi(s, \tau)$ is shown in Fig. 2.

For $x(t, \tau)$ we have

$$\begin{aligned} x(t, \tau) &= g \exp(-t), \text{ for } t \in [0, T\tau_1], \\ x(t, \tau) &= (x(T\tau_1, \tau) - 1) \exp(-(t - T\tau_1)) + 1, \text{ for } t \in (T\tau_1, T(\tau_1 + \tau_2)], \\ x(t, \tau) &= x(T(\tau_1 + \tau_2), \tau) \exp(-(t - T(\tau_1 + \tau_2))), \text{ for } t \in (T(\tau_1 + \tau_2), T]. \end{aligned}$$

Fig. 2. An example of the function $\varphi(s, \tau)$.

Under the condition $x(T(\tau_1 + \tau_2), \tau) < g$ the solution $x(t, \tau)$ will coincide after some time with the slowly oscillating periodic solution $x_0(t + \text{const}, T)$. Let

$$x(T(\tau_1 + \tau_2), \tau) > g. \quad (1.9)$$

As before, we let t_1, t_2 denote the first and the second positive roots of the equation $x(t, \tau) = g$. Then we obtain the equations

$$\begin{aligned} t_1 &= T\tau_1 + \ln(1 - g \exp(-T\tau_1)) - \ln(1 - g), \\ t_2 &= T(\tau_1 + \tau_2) + \ln x(T(\tau_1 + \tau_2), \tau) - \ln g. \end{aligned}$$

If $t_2 \geq T$, then $x(t, \tau)$ will coincide after some time with the solution $x_0(t + \text{const}, T)$. Let

$$t_2 < T. \quad (1.10)$$

Consider the Poincaré operator

$$\Pi(\varphi(s, \tau)) = x(t_2 + s, \tau).$$

Under conditions (1.9) and (1.10) this operator transforms the set $C(\tau_1, \tau_2)$ into $C(\bar{\tau}_1, \bar{\tau}_2)$, where

$$\bar{\tau}_1 = 1 - t_2 T^{-1}, \quad \bar{\tau}_2 = t_1 T^{-1}. \quad (1.11)$$

In a similar manner one constructs $2m$ -dimensional ($m = 2, 3, \dots$) maps which describe the behavior of solutions with $2m$ intersections with the straight line $x = g$ over time intervals of length T . The dynamics of such maps defines the behavior of solutions of Eq. (1.1) for $t \rightarrow \infty$ with initial conditions from the chosen special sets.

We show that each of such maps has a fixed point and note that a fixed point corresponds to a periodic solution of Eq. (1.1).

Let us fix an arbitrary $z > 0$ and consider the function $x_0(t, z)$. Let $P(z)$ denote the period of this function. For each integer $m = 0, 1, \dots$ the function $x_0(t, z)$ is a periodic solution of the equation

$$\dot{x} + x = f(x(t - z - mP(z))).$$

Consider the equation in z :

$$T = z + mP(z).$$

Since the function $P(z)$ increases monotonically and $P(z) \rightarrow 0$ as $z \rightarrow +0$, this equation has a unique solution z_m for each m . This implies that each of the functions $x_0(t, z_m)$ ($m = 0, 1, \dots$) is a periodic solution of Eq. (1.1). In the interval $(-T, 0)$ the number of roots of the equation $x_0(t, z_m) = g$ is equal to $2m$.

The fixed point of the map (1.11), which corresponds to the solution $x_0(t, z_1)$, is easily found from the above formulae for $x_0(t, z)$. We note that the periodic solutions $x_0(t, z_m)$ are unstable for $m \geq 1$. Numerical analysis shows that all solutions of (1.1) (except for $x_0(t, z_0)$) will coincide after some time with $x_0(t + \text{const}, T)$.

For illustration, we fix on the “phase plane” τ_1, τ_2 of the map (1.11) arbitrarily a point (τ_1, τ_2) ($0 < \tau_1, \tau_2 < 1, \tau_1 + \tau_2 < 1$), and perform, according to (1.11), for $\varepsilon = 0.1$, first 20 (Fig. 3a) and then 100 iterations (Fig. 3b), and for $\varepsilon = 0.02$, accordingly, 100 (Fig. 3c), and 100 000 (Fig. 3d) iterations. Dark color on this plane denotes those points for which inequalities (1.9) and (1.10) cease to be valid, i.e., the corresponding solution loses the initial structure and coincides with $x_0(t + \text{const}, T)$. We note that, as T increases slightly, the number of iterations necessary for all points to be colored dark increases sharply (this is discussed in more detail in Section 1.3).

Fig. 3. a) $\varepsilon = 0.1$, 20 iterations; b) $\varepsilon = 0.1$, 100 iterations; c) $\varepsilon = 0.02$, 100 iterations; d) $\varepsilon = 0.02$, 10^5 iterations.

1.3. Estimation of the time of convergence to the simplest cycle

Equation (1.1) has apparently no stable solutions except for the simplest cycle $x_0(t, T)$. The results of numerical experiments show that, from some time L on, all solutions coincide with the simplest cycle. This raises two questions:

1. How does L depend on the parameter T ?
2. How does L depend on initial conditions?

Figure 4a shows the dependence of time L on $T = \varepsilon^{-1}$ for solutions with initial functions from $C(\tau_1, \tau_2)$. Standard numerical methods show that the dependence is exponential.

To keep track of the dependence of L on initial conditions, we consider solutions with initial functions from $C(\tau_1, \tau_2)$ (see Fig. 2). Let us fix arbitrarily a value of τ_1 and vary τ_2 from 0 to $1 - \tau_1$. Figure 4b shows the dependence of L on τ_2 for $\tau_1 = 0.14$ and $\tau_1 = 0.17$.

Thus, the only stable regime of Eq. (1.1) is the simplest cycle, but the time when the solutions pass through a “small” neighborhood of this cycle greatly depends on the smallness of the parameter ε and on the degree of “complexity” of the initial conditions.

Fig. 4. Graphs showing the dependence of L a) on the parameter $T = \varepsilon^{-1}$ under different initial conditions; b) on the parameter τ_2 at $T = 50$, $\tau_1 = 0.14$ and $\tau_1 = 0.17$.

2. Dynamics of a system of two coupled equations

In this section we turn to a study of the dynamics of two identical equations of the form (1.1) with different types of coupling between them. We examine two types of coupling: diffusion coupling and coupling via nonlinearity of $f(x)$. Let us consider each of them separately.

2.1. Dynamics of a system of equations with diffusion coupling

Suppose we are given a system of differential equations

$$\begin{aligned}\dot{x} + x &= f(x(t - T)) + d_1(y - x), \\ \dot{y} + y &= f(y(t - T)) + d_2(x - y),\end{aligned}\tag{2.1}$$

where the diffusion coupling coefficients d_1, d_2 are nonnegative, the function $f(s)$ has the form (1.2), and the delay time is assumed to be large:

$$T \gg 1.\tag{2.2}$$

Condition (2.2) is the main restriction under which we investigate here the synchronization of solutions of the system (2.1).

Rescaling time as in (1.7) and denoting $\varepsilon = T^{-1}$, we find that the system of equations (2.1) takes the form

$$\begin{aligned}\varepsilon \dot{x} + x &= f(x(t - 1)) + d_1(y - x), \\ \varepsilon \dot{y} + y &= f(y(t - 1)) + d_2(x - y).\end{aligned}\tag{2.3}$$

For $\varepsilon = 0$ we have the following system of two coupled maps:

$$\begin{aligned} x(t) &= f(x(t-1)) + d_1(y(t) - x(t)), \\ y(t) &= f(y(t-1)) + d_2(x(t) - y(t)). \end{aligned} \quad (2.4)$$

We first consider the question of synchronization for the system (2.4).

Since the function f takes only two values, 0 and 1, it follows that, solving the system (2.4) for x and y at some t , we obtain 4 different cases:

$$\begin{aligned} 1. \quad & \begin{pmatrix} f(x(t-1)) \\ f(y(t-1)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ 2. \quad & \begin{pmatrix} f(x(t-1)) \\ f(y(t-1)) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Delta^{-1} \begin{pmatrix} 1 + d_2 \\ d_2 \end{pmatrix}, \\ 3. \quad & \begin{pmatrix} f(x(t-1)) \\ f(y(t-1)) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Delta^{-1} \begin{pmatrix} d_1 \\ 1 + d_1 \end{pmatrix}, \\ 4. \quad & \begin{pmatrix} f(x(t-1)) \\ f(y(t-1)) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where

$$\Delta = \det \begin{pmatrix} 1 + d_1 & -d_1 \\ -d_2 & 1 + d_2 \end{pmatrix} = 1 + d_1 + d_2.$$

It follows from the above formulae that synchronization takes place if

$$\min\{g, 1 - g\} < \Delta^{-1} \max\{d_1, d_2\}, \quad (2.5)$$

which may be regarded as a condition for the coupling coefficients d_1 and d_2 .

Coming back to the system of differential equations, we note that in the general case the dynamics of the system with $\varepsilon = 0$ and $\varepsilon > 0$ are fairly different, but, as numerical experiments show, if inequality (2.5) (conditions for synchronization at $\varepsilon = 0$) is satisfied in the system (2.3), then a “fast” synchronization occurs in a relatively short time, which does not increase with decreasing ε . In the numerical calculations this time does not exceed 10.

Even if condition (2.5) is not fulfilled, synchronization does take place, but the time after which it occurs is much larger; as $\varepsilon \rightarrow 0$, it increases indefinitely. Figure 5 shows the dependence of the synchronization time on ε^{-1} for different values of the parameters d_1 , d_2 and $g = 0.3$. We note that this dependence turns out to be linear (if condition (2.5) is fulfilled, the graph is parallel to the abscissa axis).

It is important to note that the synchronization in the system (2.3) occurs much faster than the simplest cycle sets in (see, e.g., Figs. 4a and 5a).

2.2. Dynamics of a system of equations coupled via a nonlinear function

In the case of a nonlinear coupling the following system of equations serves as a mathematical model:

$$\begin{aligned} \dot{x} + x &= f[x(t-T) + d_1(y(t-T) - x(t-T))], \\ \dot{y} + y &= f[y(t-T) + d_2(x(t-T) - y(t-T))], \end{aligned} \quad (2.6)$$

Fig. 5. Graphs showing the dependence of the synchronization time on ε^{-1} for different values of the parameters d_1, d_2 and $g = 0.3$.

where the coupling coefficients d_1, d_2 satisfy the restriction $0 \leq d_1, d_2 \leq 1$ and the function $f(s)$ has the form (1.2).

The synchronization of solutions of the system (2.6) is examined under condition (2.2). After standard transformations we arrive at the system

$$\begin{aligned} \varepsilon \dot{x} + x &= f[x(t-1) + d_1(y(t-1) - x(t-1))], \\ \varepsilon \dot{y} + y &= f[y(t-1) + d_2(x(t-1) - y(t-1))]. \end{aligned} \quad (2.7)$$

The condition for synchronization of the system (2.7), which is degenerate at $\varepsilon = 0$, is that the following inequality is satisfied:

$$\min\{g, 1 - g\} < \max\{d_1, d_2\}. \quad (2.8)$$

If $0 < \varepsilon \ll 1$, then, as in the previous case, with condition (2.8) satisfied, a “fast” synchronization occurs during a time which does not increase as ε decreases. If condition (2.8)

is not satisfied, then synchronization occurs in a much larger time which increases indefinitely as $\varepsilon \rightarrow 0$. Figure 5b shows the dependence of the synchronization time on $T = \varepsilon^{-1}$ for different values of the parameters d_1 , d_2 and $g = 0.3$. We note that this dependence also turns out to be linear (if condition (2.8) is satisfied, the graph is parallel to the abscissa axis).

2.3. Estimation of the parameter of synchronization of an unstable cycle

The problem of synchronization of unstable cycles arises in the study of methods for data processing and transfer (see, e.g., [1]).

It was shown above that Eq. (1.1) has a countable number of unstable cycles $x_m(t) = x_0(t, z_m)$ ($m = 1, 2, \dots$). Let us fix arbitrarily $m \geq 1$. The system of equations (2.1) has an unstable periodic solution $x(t) = y(t) = x_m(t)$. Consider separately the second equation of the system for $x(t) = x_m(t)$:

$$\dot{y} + y = f(y(t - T)) + d(x_m(t) - y). \quad (2.9)$$

This equation has a periodic solution

$$y_m(t) = x_m(t). \quad (2.10)$$

It is obvious that this solution is unstable for small d . Using analytical methods, it is rather simple to establish the existence of a value of d_0 for which, with $d > d_0$, the periodic solution (2.10) is asymptotically stable. However, these methods provide only a rough estimate of d_0 . Therefore, it becomes necessary to numerically find the value of d_0 depending on the parameters m and T . In addition, the problem arises of describing the domain of attraction of the periodic solution (2.10) for $d > d_0$. Such problems are called *problems of synchronization on an unstable cycle*.

Let us formulate the main results of our numerical investigations.

When T , g , ε , m are fixed, the value d_0 is the larger, the more complex the initial conditions are (i.e., the larger the number of intersections with the straight line $y = g$ is in the interval $(-T, 0)$).

Let $d_{\max}$ denote a threshold value of d_0 such that, when $d \geq d_{\max}$, the solution (2.10) is globally stable for any initial conditions. It is established that $d_{\max} = d_0$ in the case where $y(t)$ has no intersections with the straight line $y = g$ in the interval $(-T, 0)$ (i.e., the initial condition corresponds to the simplest cycle). Table 1 presents the results of calculation of $d_{\max}$ for Eq. (2.9), depending on T and m , for $g = 0.3$. Table 2 presents similar data for the equation

$$\dot{y} + y = f[y(t - T) + d(x_m(t - T) - y(t - T))].$$

Table 1

	$m = 1$	$m = 2$
$T = 10$	0.5301	0.8171
$T = 20$	0.4335	0.4902

Table 2

	$m = 1$	$m = 2$
$T = 10$	0.3290	0.4143
$T = 20$	0.3011	0.3166

The general conclusion is that $d_{\max}$ decreases with increasing delay time T , and increases with increasing m .

We note that, if the solution (2.10) is unstable, then $y(t)$ tends to a more complex, nearly periodic solution. The form of this solution for $m = 2$ is shown in Fig. 6.

Fig. 6. A graph of the solution $y(t)$ in the case of instability of the solution $x_m(t)$.

3. Dynamics of a system of equations for small values of the coupling coefficients

Numerical results show that, as the coupling coefficients d_1, d_2 decrease, the structure of solutions of the systems (2.1) and (2.6) becomes more complex. Therefore, it is of interest to examine in detail the dynamics of two weakly coupled oscillators of the form (1.1). In this section we present analytical results concerning the dynamics of such systems under the additional condition (1.6), when the parameter g is small.

These results explain a number of complex effects found in the numerical analysis of systems with a weak coupling.

We note right away that the results obtained can be extended to the case where g is close to 1, since it is reduced to the previous case by the transformation $\bar{x} = 1 - x, \bar{y} = 1 - y$.

3.1. Dynamics of the system (2.1)

In the system (2.1) we make the transformation $x \rightarrow gx, y \rightarrow gy$. This gives

$$\begin{cases} \dot{x} + x = \lambda \Phi(x(t-T)) + d_1(y-x), \\ \dot{y} + y = \lambda \Phi(y(t-T)) + d_2(x-y), \end{cases} \quad (3.1)$$

where $\lambda = g^{-1}$,

$$\Phi(s) = \begin{cases} 1, & \text{for } s < 1, \\ 0, & \text{for } s \geq 1. \end{cases}$$

We note that the system (3.1) has a homogeneous cycle

$$y(t) = x(t) = \lambda x_0(t, T). \quad (3.2)$$

To investigate the dynamics of the system (3.1), we first set $z = y(0) - x(0)$ and consider the set $C(z)$ (which depends on z as a parameter) of such pairs of initial functions $\varphi(s), \psi(s) \in C_{[-T, 0]}$

for which $\varphi(s), \psi(s) \geq 1$ with $s \in [-T, 0]$ and

$$\begin{cases} \varphi(0) = 1 - z, \\ \psi(0) = 1, \end{cases} \quad \text{for } z < 0, \quad \begin{cases} \varphi(0) = 1, \\ \psi(0) = 1 + z, \end{cases} \quad \text{for } z \geq 0.$$

Let $x(t), y(t)$ be solutions to the system (3.1) with initial functions $\varphi(s)$ and $\psi(s)$, respectively, and let $(\varphi(s), \psi(s)) \in C(z)$. We note that x and y do not depend on the choice of an element from $C(z)$.

By successively examining the system (2.1) in the intervals $[0, T]$, $[T, 2T]$, $[2T, 3T]$, and $\dots$, one can obtain an explicit form of the functions $x(t)$ and $y(t)$. We note that, from some time $t = t^*$ on, these functions take asymptotically large values (of order l). Thus, in an asymptotically large time interval adjacent to the point $t_m = t^* + T$, these functions are solutions to the system of linear equations

$$\begin{cases} \dot{x} + x = d_1(y - x), \\ \dot{y} + y = d_2(x - y). \end{cases} \quad (3.3)$$

Let t_{0x}, t_{0y} be the first roots of the equations $x(t) = 1$ and $y(t) = 1$ for $t > t_m$. It follows from the above that $t_{0x} = O(\ln \lambda)$, $t_{0y} = O(\ln \lambda)$. Set $t_0 = \min\{t_{0x}, t_{0y}\}$. Then the Poincaré succession operator $\Pi(\varphi(s), \psi(s)) = (x(t_0 + s), y(t_0 + s))$ transforms the set of initial conditions $C(z)$ into $C(\bar{z})$, where $\bar{z}$ depends only on z . Let $g(z)$ denote the dependence of $\bar{z}$ on z . The trajectory of the map $\bar{z} = g(z)$ defines iterations of the set of initial conditions $C(z)$, and the question of the dynamics of solutions to the system (3.1) reduces to investigating a one-dimensional map $g(z)$. Next, we obtain analytical expressions for it.

Since $t_{0x} - t_{0y} = o(1)$, it follows that $\bar{z} = g(z) = o(1)$. This yields the following result.

Theorem 2. *At any fixed (i.e., independent of λ) values of d_1, d_2 and at sufficiently large λ , the functions $x(t)$ and $y(t)$ tend to the cycle (3.2) as $t \rightarrow \infty$.*

The most interesting results pertain to the case where the coefficients d_1, d_2 are small. Depending on the degree of their smallness, there are two cases in which the dynamics of the system are essentially different. In one of them, the coefficients d_j are of order $O(|\ln \lambda|^{-1})$, and in the other, of order $d_j = O(\lambda^{-1})$. Consider each of these cases separately.

First let $d_j = O(|\ln \lambda|^{-1})$, $j = 1, 2$, i.e.,

$$d_1 = \frac{\tilde{d}_1}{\ln \lambda}, \quad d_2 = \frac{\tilde{d}_2}{\ln \lambda}. \quad (3.4)$$

Let us introduce an auxiliary function $q(d, z)$:

$$q(d, z) = (1 + d)(X - Y)\sigma[(1 + d)X - (X - Y)(1 + d\sigma)]^{-1},$$

where $\sigma = \exp(d)$,

$$\begin{aligned} X &= 1 - \exp(-T), \\ Y &= \begin{cases} 0, & 1 + |z| \geq \exp(T), \\ 1 - (1 + |z|)\exp(-T), & 0 \leq 1 + |z| < \exp(T). \end{cases} \end{aligned} \quad (3.5)$$

The basic statement is that the function $g(z)$, which appears in the map $\bar{z} = g(z)$ (whose dynamics defines the behavior of the solutions $x(t)$ and $y(t)$ as $t \rightarrow \infty$), up to $o(1)$ as $\lambda \rightarrow \infty$, has the form

$$g(z) = \begin{cases} -q(d, z) & \text{for } z \geq 0, \\ q(d^{-1}, z) & \text{for } z < 0, \end{cases} \quad \text{where } d = \frac{d_2}{d_1}. \quad (3.6)$$

To support the formulae (3.5) and (3.6), we consider the case $z > 0$ (the line of reasoning for the case $z < 0$ is analogous). Let the numbers X and Y be the main part of asymptotics of $x(t_m)$ and $y(t_m)$ for $\lambda \rightarrow \infty$, i.e.,

$$x(t_m) = \lambda[X + o(1)], \quad y(t_m) = \lambda[Y + o(1)].$$

Let us find expressions for X and Y . We note that, when $t \in [0, T]$, the functions $x(t)$ and $y(t)$ satisfy the system (3.3), and hence for $\lambda \rightarrow \infty$ we have

$$x(t) = \exp(-t) + o(1), \quad y(t) = (1 + z) \exp(-t) + o(1).$$

An important fact that underlies further constructions is that, for any $z > 0$, there exists $r > 0$ such that in the interval $[T, T + r]$ the functions $x(t)$, $y(t)$ satisfy the system of differential equations

$$\begin{cases} \dot{x} + x = \lambda + d_1(y - x), \\ \dot{y} + y = d_2(x - y). \end{cases}$$

This leads us to the conclusion that, at sufficiently large λ for each r_1 such that $0 < r_1 \leq r$, the following conditions hold:

$$\begin{aligned} x(r_1 + T) &= O(\lambda), \\ y(r_1 + T) &= O(\lambda(\ln \lambda)^{-1}). \end{aligned}$$

Hence, one can set $t_m = 2T + o(1)$.

For $t \in [T, 2T]$ we have $x(t) = \lambda[1 - \exp(-t + T) + o(1)]$, whence we obtain the equation

$$X = 1 - \exp(-T).$$

The behavior of $y(t)$ in the interval $[0, T]$ is dominant for Y . Let the condition $(1 + z) \geq \exp(T)$ hold. Then for $t \in [T, 2T]$ $y(t) = O(\lambda(\ln \lambda)^{-1})$, and hence $Y = 0$. But if $0 \leq 1 + z < \exp(T)$, then $y(t) = 1$ for $t = \ln(1 + z) + o(1)$. Therefore,

$$y(t) = \begin{cases} O(\lambda(\ln \lambda)^{-1}) & \text{for } t \in [T, \ln(1 + z) + T], \\ \lambda(1 - \exp(-(t - T - \ln(1 + z)))) + o(1) & \text{for } t \in [\ln(1 + z) + T, 2T] \end{cases}$$

and

$$Y = 1 - (z + 1) \exp(-T).$$

Since $X \geq Y$, it follows that $t_{x0} > t_{y0}$. Consequently, the following formula holds for t_0 :

$$t_0 = t_m + \ln \lambda + \ln \left[X - \frac{X - Y}{1 + d} (1 + d) \right] + o(1).$$

Then

$$x(t_0) = 1 + \frac{(1 + d)(X - Y)d}{(1 + d)X - (X - Y)(1 + d)} + o(1).$$

This implies that, up to $o(1)$ (as $\lambda \rightarrow \infty$), the function $g(z)$ has the form (3.6).

Analyzing the map $g(z)$, we obtain the following result:

Theorem 3. *Let*

$$\frac{d}{\exp(T) - 1} < 1 \quad (> 1).$$

Then the zero equilibrium point of the map $g(z)$ is asymptotically stable (unstable). This equilibrium point corresponds to the stable (unstable) homogeneous cycle of the system (2.1).

Theorem 4. *Let*

$$|g(\exp(T))| \frac{d^{-1} + 1}{1 - d} \geq \exp(T)$$

and

$$|g(\exp(T))| \frac{d + 1}{1 - d} \geq \exp(T).$$

Then the map $g(z)$ has a superstable cycle of period 2: $(g(\exp(T)), g(-\exp(T)))$, and there are no cycles of other periods. This cycle of the map $g(z)$ corresponds to the stable inhomogeneous cycle of the system (2.1).

Figure 7 shows graphs of the map $g(z)$ for some values of the parameters T , $\tilde{d}_1$, $\tilde{d}_2$.

In the previous constructions we have used the fact that both coupling coefficients, d_1 and d_2 , are different from zero. Therefore, the case where one of the coefficients is zero needs to be examined separately. Let

$$d_1 = 0 \quad d_2 = \frac{\tilde{d}_2}{\ln \lambda}.$$

Then for $z \geq 0$ the map (3.6) has the form

$$g(z) = -\frac{(X - Y) \exp(-\tilde{d}_2)}{X - (X - Y) \exp(-\tilde{d}_2)}, \quad (3.7)$$

where X and Y are defined from the formulae (3.5).

Fig. 7. Graphs of the map $g(z)$ for some values of the parameters T , $\tilde{d}_1$, $\tilde{d}_2$.

But if $z < 0$, then

$$g(z) = -\frac{|z| \exp(-\tilde{d}_2)}{1 + |z|}. \quad (3.8)$$

From the formulae (3.7) and (3.8) one can conclude that the zero equilibrium point of the map $g(z)$, which corresponds to the homogeneous cycle (3.2), is globally stable for any d .

In the case where the coupling coefficients are ($\tilde{d}_1 = \tilde{d}_2 = d$), the zero equilibrium point of the map $g(z)$ is asymptotically stable for $d > -\frac{1}{2} \ln(\exp(T) - 1)$ and unstable for $d < -\frac{1}{2} \ln(\exp(T) - 1)$. When $d \leq \frac{1}{2} \ln(2 \exp(-T) + 1)$, the map $g(z)$ has a superstable cycle of period 2.

It turns out that the phase portrait of the initial system can undergo a dramatic structural change when the coupling coefficients d_1, d_2 become of order $O(\lambda^{-1})$. Let us examine this case.

Assume that

$$d_1 = \frac{\tilde{d}_1}{\lambda}, \quad d_2 = \frac{\tilde{d}_2}{\lambda}. \quad (3.9)$$

Then for $t \in [t_m, t_0]$ we have

$$\begin{aligned} x(t) &= \lambda[X + o(1)] \exp(-(t - t_m)), \\ y(t) &= \lambda[Y + o(1)] \exp(-(t - t_m)), \end{aligned}$$

where t_0 is the first root of the equation $y(t) = 1$ if $X \geq Y$ or of the equation $x(t) = 1$ if $X < Y$. Hence, up to $o(1)$ as $\lambda \rightarrow \infty$, we obtain a formula for the map g :

$$g(z) = \begin{cases} (-XY^{-1} + 1) \operatorname{sign} z, & X \geq Y, \\ (YX^{-1} - 1) \operatorname{sign} z, & X < Y. \end{cases} \quad (3.10)$$

Let us find expressions for X and Y . In the case $z \geq 0$ (for $z < 0$ the line of reasoning is analogous).

When $t \in [0, t_1 + T]$, the solutions $x(t)$ and $y(t)$ satisfy the system of equations

$$\begin{cases} \dot{x} + x = \lambda \Phi(x(t - T)) + d_1(y - x), \\ \dot{y} + y = \tilde{d}_2(x - y), \end{cases} \quad (3.11)$$

where t_1 is the first root of the equation $y(t) = 1$ for which $\dot{y}(t_1) < 0$. Note that the behavior of $x(t)$ in the interval $[0, t_1 + T]$ is similar to that in the case where conditions (3.4) are satisfied. Using (3.11), we conclude that

$$y(t) = \begin{cases} (1 + z) \exp(-t) + o(1), & 0 \leq t \leq T, \\ (y(T) - \tilde{d}_2) \exp(-(t - T)) + \\ \quad + \tilde{d}_2(1 - (t - T) \exp(-(t - T))) + o(1), & T < t \leq M, \\ \exp(-(t - 2T))[\tilde{d}_2(1 - \exp(-T))(t - 2T) + \\ \quad + y(2T)] + o(1), & M < t \leq t_1 + T, \end{cases}$$

where $M = \min\{2T, t_1 + T\}$.

Let $t_1, t_2, \dots$ denote the roots (numbered in the increasing order) of the equation

$$y(t) = 1,$$

which belong to the interval $[t_1, t_1 + T]$. Since in this interval the function $y(t)$ has no more than one maximum and no more than one minimum, then the number of such roots cannot be larger than three.

Let n roots be found. It is convenient to set $t_{n+1} = t_1 + T$. If the condition $0 < y(t) \leq 1$ holds for some i ($i = \overline{1, n}$) in the interval $t \in [t_i, t_{i+1}]$, then for $t \in [t_i + T, t_{i+1} + T]$ the function $y(t)$ is a solution to the equation

$$\dot{y} + y = \lambda[1 + o(1)].$$

Thus,

$$y(t) = \lambda + (y(t_i + T) - \lambda) \exp(-(t - t_i - T)) + o(\lambda). \quad (3.12)$$

But if the inequality $y(t) \geq 1$ is satisfied for some i ($i = \overline{1, n}$) in the interval $t \in [t_i, t_{i+1}]$, then for $t \in [t_i + T, t_{i+1} + T]$ we obtain the equation

$$y(t) = y(t_i + T) \exp(-(t - t_i - T)) + o(\lambda). \quad (3.13)$$

Thus, in the interval $t \in [t_1 + 2T, t_0]$ the functions $x(t)$ and $y(t)$ satisfy Eq. (3.3). Therefore, we set $t_m = t_1 + 2T$. Using (3.12) and (3.13), we obtain the following resulting relations for X and Y in (3.10):

$$\begin{aligned} y(t_1 + 2T) &= \lambda(Y + o(1)), \\ X &= (1 - \exp(-T)) \exp(-t_1). \end{aligned}$$

3.2. Dynamics of the system (2.6)

We make the transformations $u = (1 - d_1)x + yd_1$, $v = d_2x + (1 - d_2)y$ and $x = gu$, $y = gv$ in (2.6):

$$\begin{aligned} \dot{x} + x &= \lambda[(1 - d_1)\Phi(x(t - T)) + d_1\Phi(y(t - T))], \\ \dot{y} + y &= \lambda[d_2\Phi(x(t - T)) + (1 - d_2)\Phi(y(t - T))]. \end{aligned} \quad (3.14)$$

Let us construct a map $g(z)$ that defines iterations of the set of initial conditions $C(z)$. In all cases, up to $o(1)$, the function $g(z)$ has the form

$$g(z) = \begin{cases} (-XY^{-1} + 1)\text{sign } z, & X \geq Y, \\ (YX^{-1} - 1)\text{sign } z, & X < Y. \end{cases} \quad (3.15)$$

We now find expressions for X and Y .

Let $0 < d_1, d_2 \leq 1$ be arbitrary fixed numbers. The values of X and Y depend on the behavior of the functions $x(t)$ and $y(t)$ in the interval $[0, T]$. For the solutions $x(t)$ and $y(t)$ with $t \in [0, T]$ we obtain the equations

$$\begin{aligned} x(t) &= \exp(-t), & y(t) &= (1 + |z|) \exp(-t), \\ y(t) &= (1 + z) \exp(-t), & & \text{for } z \geq 0; & & \text{for } z < 0. \end{aligned}$$

In the interval $[T, 2T]$ the corresponding formulae for $z \leq 0$ have the form

$$\begin{cases} X = (1 - d_1)(1 - \exp(-T)), \\ Y = d_2(1 - \exp(-T)), \end{cases} \quad 1 + |z| \geq \exp(T),$$

$$\begin{cases} X = 1 + ((1 - d_1)(1 - (1 + |z|)^{-1}) - 1) \exp(-T)(1 + |z|), \\ Y = 1 + (d_2(1 - (1 + |z|)^{-1}) - 1) \exp(-T)(1 + |z|), \end{cases} \quad 1 \leq 1 + |z| < \exp(T),$$

and for $z < 0$

$$\begin{cases} X = (1 - d_2)(1 - \exp(-T)), \\ Y = d_1(1 - \exp(-T)), \end{cases} \quad 1 + |z| \geq \exp(T),$$

$$\begin{cases} X = 1 + ((1 - d_2)(1 - (1 + |z|)^{-1}) - 1) \exp(-T)(1 + |z|), \\ Y = 1 + (d_1(1 - (1 + |z|)^{-1}) - 1) \exp(-T)(1 + |z|), \end{cases} \quad 1 \leq 1 + |z| < \exp(T).$$

Figure 8 shows graphs of the map $g(z)$ for different values of the parameters T , d_1 , d_2 .

Fig. 8. Graphs of the map $g(z)$ for different values of the parameters T , d_1 , d_2 .

Theorem 5. *Let*

$$\exp(-T)(1 - \exp(-T))^{-1}|1 - d_1 - d_2| < 1 \quad (> 1),$$

then the zero equilibrium point of the map $g(z)$, which corresponds to the homogeneous cycle (3.2) of the system (2.6), is stable (unstable).

Let conditions (3.4) be satisfied. Then for $1 \leq 1 + |z| < \exp(T)$ we obtain

$$\begin{aligned} X &= 1 - \exp(-T), \\ Y &= 1 - (1 + |z|) \exp(-T). \end{aligned}$$

If, however, $1 + |z| > \exp(T)$, then the first iteration of the set of initial conditions $C(z)$ is the set $\tilde{C}(z)$, which differs from $C(z)$ in that

$$\begin{cases} \varphi(0) = -z \ln \lambda, \\ \psi(0) = 1, \end{cases} \quad \text{for } z < 0, \\ \begin{cases} \varphi(0) = 1, \\ \psi(0) = z \ln \lambda, \end{cases} \quad \text{for } z \geq 0.$$

The basic statement is that the Poincaré succession operator $\Pi(\varphi(s), \psi(s)) = (x(t_0 + s), y(t_0 + s))$ transforms the set of initial conditions $\tilde{C}(z)$ into $\tilde{C}(\bar{z})$, where, up to $o(1)$ (as $\lambda \rightarrow \infty$),

$$\bar{z} = \tilde{g}(z) = \begin{cases} -\tilde{d}_2^{-1}, & \text{for } 1 + z \geq \exp(T), \\ -\tilde{d}_1^{-1}, & \text{for } -1 + z \leq -\exp(T). \end{cases}$$

Figure 9 shows graphs of the map $g(z)$ for different values of the parameters $T, \tilde{d}_1, \tilde{d}_2$.

Fig. 9. Graphs of the map $g(z)$ for different values of the parameters $T, \tilde{d}_1, \tilde{d}_2$.

Theorem 6. *Let*

$$\exp(-T)(1 - \exp(-T))^{-1} < 1 \quad (> 1),$$

then the zero equilibrium point of the map $g(z)$, which corresponds to the homogeneous cycle (3.2) of the system (2.6), is stable (unstable).

For sufficiently large λ the system (3.14) has a stable inhomogeneous cycle that satisfies the initial conditions $x(s) = \exp(-s)$, $y(s) = \tilde{d}_2^{-1} \exp(-s)$.

When conditions (3.9) are satisfied, all arguments are similar to those presented above.

Conclusion

In this paper we have considered the dynamics of one of the simplest oscillators with delayed feedback. It is shown that a cycle is its only stable regime, whereas there are infinitely many unstable periodic regimes. Under some conditions (such as nondegeneracy) each solution

tends to a cycle as $t \rightarrow \infty$. However, depending on the degree of complexity of initial conditions, the corresponding solution exhibits complex behavior over a time interval which increases exponentially with increasing delay.

The dynamics of two simple oscillators with two types of coupling is investigated. A criterion is obtained for “fast” and “long” synchronization in a time interval linearly depending on delay. An estimate is given of the synchronization parameter on an unstable cycle. Analytical methods are used to examine the dynamics in the case of a weak coupling between the oscillators. One-dimensional maps whose dynamics defines the behavior of solutions to the initial system are constructed. It is shown that an inhomogeneous stable cycle can exist along with a homogeneous stable cycle.

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