MATHEMATICAL PROBLEMS OF NONLINEARITY

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Vibrational Stability of Periodic Solutions of the Liouville Equations

E. V. Vetchanin, E. A. Mikishanina

The dynamics of a body with a fixed point, variable moments of inertia and internal rotors are considered. A stability analysis of permanent rotations and periodic solutions of the system is carried out. In some simplest cases the stability analysis is reduced to investigating the stability of the zero solution of Hill's equation. It is shown that by periodically changing the moments of inertia it is possible to stabilize unstable permanent rotations of the system. In addition, stable dynamical regimes can lose stability due to a parametric resonance. It is shown that, as the oscillation frequency of the moments of inertia increases, the dynamics of the system becomes close to an integrable one.

Keywords: Liouville equations, Euler – Poisson equations, Hill's equation, Mathieu equation, parametric resonance, vibrostabilization, Euler – Poincaré case, Joukowski – Volterra case

Introduction

1. As a rule, finite-dimensional mathematical models with a periodic right-hand side arise in descriptions of the motion of various mechanical systems controlled by internal mechanisms which produce parametric excitation. Such models, which describe the motion of the Chaplygin sleigh, were studied, for example, in [2–4, 10, 11]. An analysis of the dynamics of a Roller-Racer with periodic control is presented in [5]. The motion of spherical robots actuated by periodic rotation of rotors was studied in [9, 22] within the framework of the nonholonomic model. The motion of smooth bodies and foils with a sharp edge which is achieved by parametric excitation was dealt with in [15, 16, 25–27].

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Evgeny V. Vetchanin
eugene186@mail.ru
Udmurt State University,
ul. Universitetskaya 1, Izhevsk, 426034 Russia

Evgeniya A. Mikishanina
evaeva_84@mail.ru
Chuvash State University,
Moskovskii prosp. 15, Cheboksary, 428015 Russia

A fairly wide class of problems concerns the control of the orientation of satellites by using moving internal masses and rotors. In this case, the dynamics of the system is described by the Euler–Poisson equations with a gyrostat and variable moments of inertia in the general case (Liouville equations) [19, 23, 24]. The properties of the Euler–Poisson equations were studied, for example, in [8, 13, 17]. The problem of the stability of satellite motion, as well as suppression of chaotic dynamics [20], is important when it comes to controlling the orientation of satellites.

2. In this paper, we study the dynamics of a body with a fixed point and variable moments of inertia (Liouville’s problem). In Section 1 we construct a mathematical model and find first integrals of equations of motion. In Section 2 we show that by periodically changing the moments of inertia it is possible to stabilize unstable permanent rotations. A similar effect can be achieved for an inverted pendulum with a vibrating point of support (Kapitsa’s pendulum) [18]. Moreover, stable permanent rotations and periodic solutions can lose stability because of parametric resonance. In Section 3 we show that the onset of chaos in the system is due to the splitting of separatrices. It is shown that, as the oscillation frequency of the moments of inertia increases, the dynamics of the system becomes close to an integrable one.

1. Mathematical model

1.1. Equations of motion

Consider the motion of a rigid body with a fixed point which contains moving material points and rotors. For the system under consideration we make the following assumptions:

- 1° The center of mass of the system coincides with the fixed point throughout the motion. Thus, the moment of gravity is zero.
- 2° The body is acted upon by no dissipative forces.
- 3° The internal material points perform periodic motions with the same period τ . We also assume that the directions of the principal axes of inertia do not change due to the motion of the internal material points.
- 4° The rotors are axisymmetric, and the axes of rotation coincide with its symmetry axes. Thus, the rotation of the rotors does not change the mass distribution of the system. The angular velocity of the rotors is a τ -periodic function of time.

To describe the motion of the system, we introduce two coordinate systems: a fixed coordinate system $Oxyz$ and a moving coordinate system $Cx_1x_2x_3$, which is attached to the body. We assume that the axes of the moving coordinate system coincide with the principal axes of inertia of the system.

Let α , β and γ denote the unit vectors directed along the axes Ox , Oy and Oz , respectively. Their projections onto the axes of the moving coordinate system form an orthogonal matrix of transition from the fixed coordinate system to the moving one:

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in \text{SO}(3). \quad (1.1)$$

The matrix $\mathbf{Q}$ uniquely defines the configuration of the system. Thus, the configuration space of the system $\mathcal{G}$ is three-dimensional; it is $\text{SO}(3)$. The following kinematic relations hold:

$$\dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \quad \dot{\gamma} = \gamma \times \omega, \quad (1.2)$$

where ω is the angular velocity of the body.

The motion of the system is described by the Euler–Poisson equations [14]:

$$\frac{d}{dt} \frac{\partial T}{\partial \boldsymbol{\omega}} = \frac{\partial T}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega}, \quad (1.3)$$

where T is the kinetic energy of the system which includes the kinetic energy of the body, the moving material points and the rotors.

We write the components of the kinetic energy of the system T :

- the kinetic energy of the body

$$T^b = \frac{1}{2} m^b (\boldsymbol{\omega} \times \boldsymbol{\rho}^b, \boldsymbol{\omega} \times \boldsymbol{\rho}^b) + \frac{1}{2} (\boldsymbol{\omega}, \mathbf{I}^b \boldsymbol{\omega}), \quad (1.4)$$

where m^b is the mass of the body, $\mathbf{I}^b$ is the tensor of inertia of the body, and $\boldsymbol{\rho}^b$ is the radius vector of the center of mass of the body;

- the kinetic energy of the i th material point

$$T_i^p = \frac{1}{2} m_i^p (\dot{\boldsymbol{\rho}}_i^p + \boldsymbol{\omega} \times \boldsymbol{\rho}_i^p, \dot{\boldsymbol{\rho}}_i^p + \boldsymbol{\omega} \times \boldsymbol{\rho}_i^p), \quad (1.5)$$

where m_i^p is the mass of the i th material point and $\boldsymbol{\rho}_i^p$ is the radius vector defining the position of the i th material point;

- the kinetic energy of the i th rotor

$$T_i^r = \frac{1}{2} m_i^r (\boldsymbol{\omega} \times \boldsymbol{\rho}_i^r, \boldsymbol{\omega} \times \boldsymbol{\rho}_i^r) + \frac{1}{2} (\boldsymbol{\omega} + \boldsymbol{\Omega}_i, \mathbf{I}_i^r (\boldsymbol{\omega} + \boldsymbol{\Omega}_i)), \quad (1.6)$$

where m_i^r is the mass of the i th rotor, $\mathbf{I}_i^r$ is the tensor of inertia of the i th rotor, $\boldsymbol{\rho}_i^r$ is the radius vector of the center of mass of the i th rotor, and $\boldsymbol{\Omega}_i$ is the angular velocity of the i th rotor.

Taking the expressions (1.4)–(1.6) into account, the total kinetic energy of the system can, up to the known function of time, be written as

$$\begin{aligned} T &= T^b + \sum_i T_i^p + \sum_i T_i^r = \frac{1}{2} (\boldsymbol{\omega}, \mathbf{I} \boldsymbol{\omega}) + (\boldsymbol{\omega}, \mathbf{k}), \\ \mathbf{I} &= \mathbf{I}^b + m^b \left((\boldsymbol{\rho}^b, \boldsymbol{\rho}^b) \mathbf{E} - \boldsymbol{\rho}^b \otimes \boldsymbol{\rho}^b \right) + \sum_i m_i^p \left((\boldsymbol{\rho}_i^p, \boldsymbol{\rho}_i^p) \mathbf{E} - \boldsymbol{\rho}_i^p \otimes \boldsymbol{\rho}_i^p \right) + \\ &\quad + \sum_i \mathbf{I}_i^r + \sum_i m_i^r \left((\boldsymbol{\rho}_i^r, \boldsymbol{\rho}_i^r) \mathbf{E} - \boldsymbol{\rho}_i^r \otimes \boldsymbol{\rho}_i^r \right), \\ \mathbf{k} &= \sum_i m_i^p \boldsymbol{\rho}_i^p \times \dot{\boldsymbol{\rho}}_i^p + \sum_i \mathbf{I}_i^r \boldsymbol{\Omega}_i. \end{aligned} \quad (1.7)$$

By virtue of Assumption 3° and the choice of the moving coordinate system, the tensor of inertia of the system $\mathbf{I}$ has the diagonal form

$$\mathbf{I}(t) = \text{diag}(I_1(t), I_2(t), I_3(t)). \quad (1.8)$$

As a consequence of Assumptions 3° and 4°, the tensor of inertia of the system $\mathbf{I}$ and of the gyrostatic momentum vector $\mathbf{k}$ are τ -periodic functions of time:

$$\mathbf{I}(t + \tau) = \mathbf{I}(t), \quad \mathbf{k}(t + \tau) = \mathbf{k}(t). \quad (1.9)$$

If we denote the angular momentum by $\mathbf{M} = \frac{\partial T}{\partial \boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} + \mathbf{k}$, then the complete system of equations (1.2), (1.3) takes the form

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \left(\mathbf{I}^{-1}(\mathbf{M} - \mathbf{k}) \right), \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \left(\mathbf{I}^{-1}(\mathbf{M} - \mathbf{k}) \right), \quad (1.10a)$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \left(\mathbf{I}^{-1}(\mathbf{M} - \mathbf{k}) \right),$$

$$\dot{\mathbf{M}} = \mathbf{M} \times \left(\mathbf{I}^{-1}(\mathbf{M} - \mathbf{k}) \right). \quad (1.10b)$$

We note that both the subsystem (1.10b) and the complete system (1.10) possess a standard invariant measure. Thus, the phase space of the system can have no attractors and repellers, either regular or chaotic ones.

1.2. First integrals, reduction, and period-advance mapping

The complete system of equations (1.10) admits six geometric integrals:

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = (\boldsymbol{\beta}, \boldsymbol{\beta}) = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \boldsymbol{\gamma}) = (\boldsymbol{\beta}, \boldsymbol{\gamma}) = 0. \quad (1.11)$$

The subsystem (1.10b) admits the first integral of motion

$$G = (\mathbf{M}, \mathbf{M}) = \text{const}. \quad (1.12)$$

In addition, for $\mathbf{I} = \text{const}$, $\mathbf{k} = \text{const}$ the subsystem (1.10b) admits the energy integral [14]

$$E = \frac{1}{2} \left(\mathbf{M} - \mathbf{k}, \mathbf{I}^{-1}(\mathbf{M} - \mathbf{k}) \right) = \text{const}. \quad (1.13)$$

REMARK 1. In the presence of dynamical symmetry, for example,

$$\mathbf{I}(t) = (I_1(t), I_1(t), I_3(t)), \quad \mathbf{k}(t) = (0, 0, k_3(t)), \quad (1.14)$$

the equations of motion (1.10b) admit an additional first integral (Lagrange's integral)

$$M_3 = \text{const}. \quad (1.15)$$

In this case, the system becomes integrable.

Fixing the level set of the integral (1.12) $G = g$, we obtain a system with $\frac{3}{2}$ degrees of freedom. In explicit form the reduction can be performed by making the change of variables

$$M_1 = \sqrt{g} \cos \varphi \sqrt{1 - z^2}, \quad M_2 = \sqrt{g} \sin \varphi \sqrt{1 - z^2}, \quad M_3 = \sqrt{g} z, \quad (1.16)$$

$$\varphi \in [-\pi, \pi), \quad z \in [-1, 1].$$

Since the functions $\mathbf{I}(t)$ and $\mathbf{k}(t)$ are τ -periodic, an area-preserving two-dimensional period-advance mapping can be constructed on the fixed level set of the integral (1.12):

$$\Pi^2: \mathbb{R}^2(\varphi, z) \rightarrow \mathbb{R}^2(\varphi, z). \quad (1.17)$$

REMARK 2. Without loss of generality the dynamics of the system (1.10b) can be considered on the level set of the integral (1.12) $G = 1$. This can always be achieved by choosing units of measurement, in particular, by performing the renormalization

$$\mathbf{M} \rightarrow \sqrt{g}\mathbf{M}, \quad \mathbf{k} \rightarrow \sqrt{g}\mathbf{k}, \quad t \rightarrow \frac{t}{\sqrt{g}}, \quad \tau \rightarrow \frac{\tau}{\sqrt{g}}. \quad (1.18)$$

2. Problems of vibrostabilization and the vibrational loss of stability

In the case $\mathbf{I} = \text{const}$ and $\mathbf{k} = \text{const}$, the equations of motion (1.10b) admit partial solutions (fixed points of the system) corresponding to the permanent rotations $\mathbf{M} = \text{const}$. The stability of these dynamical regimes with $\mathbf{k} = 0$ (the Euler–Poinso case) and $\mathbf{k} \neq 0$ (the Joukowski–Volterra case) has been studied and the corresponding bifurcation diagrams have been plotted [6, 14].

In the perturbed system ($\mathbf{I} \neq \text{const}$) these permanent rotations remain unchanged for $\mathbf{k} = 0$ and in some particular cases for $\mathbf{k} \neq 0$. In the general case, permanent rotations turn into periodic solutions.

It turns out that the stability of permanent rotations and periodic solutions can change depending on the amplitude and the frequency of oscillations of the moments of inertia. That is, the system exhibits the effects of vibrostabilization and the vibrational loss of stability. In this case, the stability analysis of permanent rotations reduces to investigating the stability of the zero solution of Hill's equation.

Next, to illustrate the examples given below, we assume that two internal masses move symmetrically along the axis of the moving coordinate system Cx_2 , and the tensor of inertia changes according to the law

$$\begin{aligned} \mathbf{I}(t) &= \text{diag}(J_1 + \delta^2 \sin^2 \Omega t, J_2, J_3 + \delta^2 \sin^2 \Omega t), \\ J_1 &= 2, \quad J_2 = 3, \quad J_3 = 4. \end{aligned} \quad (2.1)$$

2.1. Perturbation of the Euler–Poinso case

Consider the motion of the system in the absence of gyrostatic momentum ($\mathbf{k} = 0$). On the fixed level sets of the first integrals (1.12), (1.13) $G = g$, $E = h$ the unperturbed system ($\delta = 0$) admits three pairs of permanent rotations:

$$\mathbf{M}^{(1)} = (\pm\sqrt{g}, 0, 0), \quad \mathbf{M}^{(2)} = (0, \pm\sqrt{g}, 0), \quad \mathbf{M}^{(3)} = (0, 0, \pm\sqrt{g}), \quad (2.2)$$

which correspond to the branches of the bifurcation diagram of the Euler–Poinso case (see Fig. 1). The permanent rotations (2.2) remain unchanged in the case $\delta \neq 0$.

Next, we investigate the stability of the permanent rotation $\mathbf{M}^{(1)} = (\sqrt{g}, 0, 0)$. We linearize the equations of motion (1.10b) in a neighborhood of this solution:

$$\dot{\xi}_1 = 0, \quad (2.3a)$$

$$\dot{\xi}_2 = \sigma(t)\xi_3, \quad \dot{\xi}_3 = \chi(t)\xi_2, \quad (2.3b)$$

$$\sigma(t) = \sqrt{g}(I_1^{-1} - I_3^{-1}), \quad \chi(t) = \sqrt{g}(I_2^{-1} - I_1^{-1}),$$

where ξ_i is the perturbation of the i th component of the vector $\mathbf{M}$.

For stability analysis it suffices to consider the subsystem (2.3b). Indeed, if the rotation is stable, then the perturbations ξ_2 and ξ_3 are bounded functions of time. Then by virtue of the integral (1.12) the perturbation ξ_1 is also bounded.

The system of equations (2.3b) can be reduced to Hill's equation by eliminating ξ_3 and making the change of variable $\xi_2 = \sqrt{\sigma}\eta$:

$$\ddot{\eta} + p(t)\eta = 0, \quad p(t) = \frac{2\ddot{\sigma}\sigma - 3\dot{\sigma}^2}{4\sigma^2} - \chi\sigma. \quad (2.4)$$

Fig. 1. Bifurcation diagram of the Euler–Poincaré case on the plane (g, h) . The straight lines A, B, C correspond to rotations about the axes with the minimal, mean and maximal moment of inertia, respectively. The solid lines correspond to stable rotations, and the dashed line corresponds to unstable rotation. Gray denotes regions where no motion is possible. The tensor of inertia is $\mathbf{I} = (2, 3, 4)$

At a small amplitude of perturbation ($\delta^2 \ll 1$) Eq. (2.4) can be approximately represented in a form similar to that of the Mathieu equation:

$$\frac{d^2\eta}{ds^2} + \omega^2 \left(1 + \delta^2 (A + B \cos s)\right) \eta = 0, \quad (2.5)$$

$$\omega^2 = \frac{(J_3^{-1} - J_1^{-1})(J_2^{-1} - J_1^{-1})}{4\Omega^2}, \quad s = 2\Omega t, \quad (2.6)$$

where A, B are the coefficients depending on the system parameters, and the quantity ω has the meaning of eigenfrequency. It is well known that the zero solution of Eq. (2.5) is unstable at arbitrarily small values of δ if the eigenfrequency of the system is an integer or a half-integer [1], i.e.,

$$\omega = \frac{n}{2}, \quad n \in \mathbb{Z}^+. \quad (2.7)$$

From (2.6) and (2.7) we find the oscillation frequencies Ω of the moments of inertia in the neighborhood of which a parametric resonance is observed in this case:

$$\Omega = \frac{\sqrt{g(J_3^{-1} - J_1^{-1})(J_2^{-1} - J_1^{-1})}}{n}. \quad (2.8)$$

At arbitrary values of δ the function $p(t)$, which appears in Eq. (2.4), has a rather complex form. Therefore, for stability analysis we will use a numerical approach in what follows. We note that, from a computational point of view, it is more convenient to use the system (2.3b).

Since the coefficients of the system (2.3b) are τ -periodic functions of time, the stability of its zero solution is determined by the condition $|\text{tr}\mathbf{X}(\tau)| < 2$, where $\mathbf{X}$ is the matrizer of the system (2.3b) [1, 28]. We recall that the matrizer is the solution of the Cauchy matrix problem

$$\dot{\mathbf{X}} = \begin{pmatrix} 0 & \sigma(t) \\ \chi(t) & 0 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fig. 2. Regions of stability (S) and instability (U) of permanent rotations (a) $\mathbf{M}^{(1)} = (1, 0, 0)$; (b) $\mathbf{M}^{(2)} = (0, 1, 0)$; (c) $\mathbf{M}^{(3)} = (0, 0, 1)$. The moments of inertia change according to (2.1). The markers denote the frequencies in the neighborhood of which a parametric resonance arises. The chart was plotted using a grid consisting of 1000×1000 meshes.

The regions of stability (S) and instability (U) of the solution $\mathbf{M}^{(1)} = (\sqrt{g}, 0, 0)$ on the parameter plane (Ω, δ) are shown in Fig. 2a.

It can be seen from Fig. 2a that the boundary of the stability region has the form of the so-called “Arnol’d tongues”. As the frequency Ω decreases, the thickness of the “tongues” in the neighborhood of $\delta = 0$ diminishes rapidly and an extreme decrease in the step of the computational grid is required to calculate their boundary.

When the boundary of the stability region is crossed, the topology of the corresponding parts of the map (1.17) will change. The permanent rotation under consideration corresponds to the fixed point of the map (1.17) with coordinates $\varphi = 0, z = 0$. The fragments of the map (1.17) in the neighborhood of this point at $\Omega = 0.183$ and various values of δ are shown in Fig. 3.

Fig. 3. Period-advance mappings (1.17) at different values of δ ; the moments of inertia change according to (2.1), $\Omega = 0.183$. When $\delta = 0.4$ and $\delta = 0.5$, the fixed point $\varphi = 0, z = 0$ is elliptic, and when $\delta = 0.45$, it is a saddle.

Linear systems similar to (2.3) can be obtained for any of the permanent rotations (2.2). For each of the pairs (2.2) the linearized equations will be identical up to the time reversal $t \rightarrow -t$. The regions of stability and instability of the permanent rotations $\mathbf{M}^{(2)}$ and $\mathbf{M}^{(3)}$ are shown in Figs. 2b and 2c, respectively.

Thus, *in the absence of gyrostatic momentum the stability of permanent rotations can be changed by periodically changing the principal moments of inertia of the system with a suitable amplitude and frequency. In this case, the loss of stability occurs due to a parametric resonance.*

2.2. Perturbation of the Joukowski – Volterra case

Consider the motion of the system in the presence of constant gyrostatic momentum. In the particular case,

$$\mathbf{k} = (k_1, 0, 0), \quad (2.9)$$

with $\delta = \text{const}$, the system (1.10b) admits two permanent rotations

$$\mathbf{M} = (\pm\sqrt{g}, 0, 0), \quad (2.10)$$

which remain unchanged also in the presence of periodic perturbations of the tensor of inertia ($\delta \neq 0$). The stability analysis of the solutions (2.10) is similar to that presented in Section 2.1, and reduces to investigating the stability of the zero solution of Hill's equation:

$$\ddot{\eta} + p(t)\eta = 0, \quad p(t) = \frac{2\ddot{\sigma}\sigma - 3\dot{\sigma}^2}{4\sigma^2} - \chi\sigma, \\ \sigma(t) = \frac{\pm\sqrt{g} - k_1}{I_1(t)} - \frac{\pm\sqrt{g}}{I_3(t)}, \quad \chi(t) = \frac{\pm\sqrt{g}}{I_2(t)} - \frac{\pm\sqrt{g} - k_1}{I_1(t)}.$$

The stability regions of different permanent rotations of the form (2.10) in the case where the moments of inertia change according to the law (2.1) are shown in Fig. 4. The local restructuring of the map (1.17), which arises at the intersection of the boundary of the stability region, is similar to that shown in Fig. 3.

Fig. 4. Regions of stability (S) and instability (U) of permanent rotations (a) $\mathbf{M} = (1, 0, 0)$, $\mathbf{k} = (0.2, 0, 0)$; (b) $\mathbf{M} = (0, 1, 0)$, $\mathbf{k} = (0, 0.2, 0)$; (c) $\mathbf{M} = (0, 0, 1)$, $\mathbf{k} = (0, 0, -0.2)$. The markers denote the frequencies in the neighborhood of which a parametric resonance arises. The moments of inertia change according to (2.1).

At constant moments of inertia ($\delta = 0$) and an arbitrary $\mathbf{k} = \text{const}$ the permanent rotations of the system (1.10b) are defined by the branches of the bifurcation diagram of the Joukowski–Volterra case [6, 14] (see Fig. 5), which are parameterized as follows [21]:

$$h = \frac{t}{2} \left(\frac{I_1^{-1}k_1^2}{(I_1^{-1} - t)^2} + \frac{I_2^{-1}k_2^2}{(I_2^{-1} - t)^2} + \frac{I_3^{-1}k_3^2}{(I_3^{-1} - t)^2} \right), \\ g = \frac{I_1^{-2}k_1^2}{(I_1^{-1} - t)^2} + \frac{I_2^{-2}k_2^2}{(I_2^{-1} - t)^2} + \frac{I_3^{-2}k_3^2}{(I_3^{-1} - t)^2}. \quad (2.11)$$

It can be seen from the bifurcation diagram that, depending on the level set of the integral (1.12), the unperturbed system (1.10b) will possess two, four or six fixed points, and the phase portrait of the system will change (see Fig. 6).

Fig. 5. Bifurcation diagram of the Joukowski–Volterra case on the plane (g, h) . The solid lines correspond to stable rotations, and the dashed lines indicate unstable rotations. The tensor of inertia and the gyrostatic momentum are: $\mathbf{I} = (2, 3, 4)$, $\mathbf{k} = (0.3, 0.2, 0.4)$.

Fig. 6. Phase portraits of the reduced system on different level sets of the integral (1.12). The tensor of inertia and the gyrostatic momentum are: $\mathbf{I} = (2, 3, 4)$, $\mathbf{k} = (0.3, 0.2, 0.4)$.

When $\delta \neq 0$, the fixed points of the unperturbed system turn into periodic solutions which correspond to the fixed points of the map (1.17). Depending on the amplitude of δ and the oscillation frequency Ω of the moments of inertia, the stability of these periodic solutions can be investigated numerically.

Let us fix the following values of gyrostatic momentum and the level set of the integral (1.12):

$$\mathbf{k} = (0.3, 0.2, 0.4), \quad G = 12. \quad (2.12)$$

In this case, the unperturbed system possesses six fixed points, of which four are stable:

$$\begin{aligned} \mathbf{M}^{(1)} &= (0.536, 0.590, 3.371), & \varphi &= 0.833, & z &= 0.973, \\ \mathbf{M}^{(2)} &= (0.688, 1.295, -3.138), & \varphi &= 1.083, & z &= -0.906, \\ \mathbf{M}^{(3)} &= (3.386, -0.545, -0.486), & \varphi &= -0.160, & z &= -0.140, \\ \mathbf{M}^{(4)} &= (-3.433, -0.317, -0.340), & \varphi &= -3.500, & z &= -0.098, \end{aligned} \quad (2.13)$$

and two are unstable:

$$\begin{aligned} \mathbf{M}^{(5)} &= (0.792, 2.942, -1.648), & \varphi &= 1.308, & z &= -0.476, \\ \mathbf{M}^{(6)} &= (1.030, -3.166, -0.958), & \varphi &= -1.256, & z &= -0.277. \end{aligned} \quad (2.14)$$

The regions of stability and instability of the periodic solutions corresponding to the fixed points $\mathbf{M}^{(1)}$, $\mathbf{M}^{(2)}$, $\mathbf{M}^{(4)}$ of the unperturbed system are shown in Fig. 7.

We note that, when performing the calculations, we could not find values of the frequency Ω and the amplitude δ at which the unstable periodic solution corresponding to the permanent rotation $\mathbf{M}^{(5)}$ of the unperturbed system becomes stable. Moreover, as the amplitude of δ increases, the periodic solutions corresponding to the fixed points $\mathbf{M}^{(3)}$, $\mathbf{M}^{(6)}$ of the unperturbed system disappear. The local changes in the map (1.17) which correspond to this process are presented in Fig. 8, 9.

Fig. 7. Stability regions of periodic solutions of the system (1.10b) which correspond to the fixed points of the unperturbed system: (a) $\mathbf{M}^{(1)}$, (b) $\mathbf{M}^{(2)}$, (c) $\mathbf{M}^{(4)}$. The moments of inertia change according to (2.1).

Fig. 8. The map (1.17) at $\Omega = 3$, $\mathbf{k} = (0.3, 0.2, 0.4)$, $G = 12$ and different values of δ . When $\delta = 0.97$, the system (1.10b) exhibits a stable and an unstable periodic solution. When $\delta = 0.98$, these solutions do not exist.

Fig. 9. The map (1.17) at $\Omega = 2$, $\mathbf{k} = (0.3, 0.2, 0.4)$, $G = 12$ and different values of δ . When $\delta = 0.97$, the system (1.10b) exhibits a stable and an unstable periodic solution. When $\delta = 0.98$, an extensive stochastic layer arises instead of these solutions.

Thus, in contrast to permanent rotations, stable periodic solutions arising in the case of an arbitrary constant gyrostatic momentum can change depending on the law of change of the principal moments of inertia. Stable periodic solutions can lose stability due to parametric resonance. The possibility of vibrostabilization of unstable periodic solutions remains an open question. In addition, some periodic solutions can disappear.

3. The onset of chaos. Separatrix splitting

The equations of motion (1.10b) can exhibit chaotic behavior in the case of variable moments of inertia. Numerical experiments show that the sizes of chaotic regions of the map (1.17) decrease as the oscillation frequency of the moments of inertia Ω increases (see Fig. 10).

Fig. 10. Period-advance mapping for different values of Ω . The moments of inertia change according to the law (2.1), $\delta = 0.5$, $\mathbf{k} = (0.3, 0.2, 0.4)$, $G = 12$.

At sufficiently large values of Ω the behavior of the system becomes close to an integrable one. By averaging over a period it is easy to show that in this case the dynamics of the system (1.10b) is approximately described by equations with constant coefficients:

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \tilde{\mathbf{I}}^{-1}(\mathbf{M} - \mathbf{k}), \\ \tilde{\mathbf{I}}^{-1} &= \left(\langle I_1^{-1}(t) \rangle, \langle I_2^{-1}(t) \rangle, \langle I_3^{-1}(t) \rangle \right). \end{aligned} \quad (3.1)$$

Here $\langle \cdot \rangle$ denotes averaging of the τ -periodic function over a period. In the case of dependences (2.1)

$$\mathbf{I} = \text{diag}(J_1 + \delta^2 \sin^2 \Omega t, J_2, J_3 + \delta^2 \sin^2 \Omega t)$$

the tensor $\tilde{\mathbf{I}}$ takes the form

$$\tilde{\mathbf{I}} = \text{diag} \left(\sqrt{J_1(J_1 + \delta^2)}, J_2, \sqrt{J_3(J_3 + \delta^2)} \right).$$

REMARK 3. We note, that at small frequencies Ω chaos is adiabatic in nature. Issues concerning the adiabatic chaos considered in [12]

Since the system considered possesses an invariant measure, the chaos in it can arise only due to separatrix splitting. A proof of this fact was given in [7]. Here we only illustrate this process by a numerical example (see Fig. 11).

Thus, in the system considered, chaos decays as the oscillation frequency of the moments of inertia Ω increases. As the value of Ω decreases, chaos becomes more intensive.

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Fig. 11. Separatrices of the fixed point of the map (1.17), which corresponds to the permanent rotation $\mathbf{M} = (0, \sqrt{g}, 0)$ for different values of δ and Ω and the law of change of the moments of inertia (2.1). Red denotes unstable manifolds, and blue indicates stable ones. The parameter values are: $\mathbf{k} = 0$, (a) $\delta = 0$, (b) $\delta = 0.5$, $\Omega = 1.5$, (c) $\delta = 0.5$, $\Omega = 1$.

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