Some Trajectories of a Point in the Potential of a Fixed Ring and Center

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The problem of three-dimensional motion of a passively gravitating point in the potential created by a homogeneous thin fixed ring and a point located in the center of the ring is considered. Motion of the point allows two first integrals. In the paper equilibrium points and invariant manifolds of the phase space of the system are found. Motions in them are analyzed. Bifurcations in the phase plane corresponding to the motion in the equatorial plane are shown.

Keywords: celestial mechanics, axisymmetric potential, center, ring, phase portrait, phase space, first integrals, bifurcations

1. Introduction

In problems of the modeling of a star motion in the gravitational field of a galaxy various axisymmetric model objects, such as the Kuzmin disk [1, 2] are used. At the same time, such models, despite their relative geometric simplicity, often provide a rather complex potentials. For example, the potential of a uniform thin disk is expressed in the form of elliptic integrals of the first, second and third kinds. The potential of the Kuzmin disk, despite its simplicity, is not smooth in the plane of the disk.

In this paper the following system is proposed: a passively gravitating point (a star or a spacecraft, for example) moves in an axisymmetric potential created by a thin homogeneous ring and a fixed material point which coincides with the center of the ring. The potential created of this system is much simpler: it is expressed only via an elliptic integral of the first kind. It is assumed that such a model can (to some approximation) describe a field created by a heavy attracting center and an axisymmetric distributed mass around this center. The second possible feature of this model is that it can be used as an approximation of a circular restricted three-body problem in the case where the mass of a body moving in a circle is evenly distributed in the orbit (as is assumed by the averaging method).
2. Formulation of the problem

We consider a system consisting of a material thin ring with mass $1 - \mu$, where $0 \leqslant \mu \leqslant 1$, the radius of the disk is equal to one and a point $O$ located in the center of the ring with mass $\mu$. The ring and point $O$ are assumed to be fixed in absolute space. We arrange the Cartesian coordinate system $Oxyz$ in such a way that its origin coincides with the point $O$ and the plane $Oxy$ coincides with the plane of the ring (Fig. 1). The motion of the passively gravitating free point $A$ in three-dimensional space under the action of Newtonian gravitational forces created by the ring and the point $O$ is discussed.

![Fig. 1. The ring with mass 1 - μ, point O with mass μ and a passively gravitating point A.](image)

The potential created by the ring and the point $O$ is defined by the formula [3–5]:

$$\Pi = -\frac{1 - \mu}{r} - \frac{2\mu K(k)}{\pi p},$$

where $r$ is the distance from the points $O$ to $A$, $K(k)$ is the complete elliptic integral of the first kind with the module $k = \sqrt{1 - q^2/p^2}$, $q$ and $p$ are the minimum and maximum distances from the point $A$ to the ring, respectively, $0 \leqslant k < 1$. The gravitational constant is equal to one. Let’s move from the Cartesian to a cylindrical coordinate system:

$$(x, y, z) \to (\rho, \varphi, z): x = \rho \cos \varphi, y = \rho \sin \varphi, z = z.$$

Then the values above are expressed by the formulas:

$$r = \sqrt{\rho^2 + z^2}, \quad q = \sqrt{(\rho - 1)^2 + z^2}, \quad p = \sqrt{(\rho + 1)^2 + z^2},$$

$$K(k) = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 \alpha}}.$$

Due to the symmetry of the mass distribution of the ring with respect to the $z$-axis, the potential doesn’t depend on the angle $\varphi$: $\Pi = \Pi(\rho, z)$.

3. Equations of motion and first integrals

The Lagrangian of the point $A$

$$L = \frac{1}{2} (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2) - \Pi(\rho, z)$$

doesn’t depend on the coordinate $\varphi$ and time. Consequently, the equations of motion admit two first integrals: a cyclic integral that makes sense of the projection of the angular momentum of point $A$ onto the $z$-axis, and the energy integral:

$$c = \rho^2 \dot{\varphi}, \quad h = \frac{1}{2} (\dot{\rho}^2 + \dot{z}^2) + \Pi(\rho, z), \quad (3.1)$$
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where \( \tilde{\Pi}(\rho, z) = \Pi + \frac{c^2}{2\rho^2} \) is the effective potential. It is possible to represent the Lagrange equations of the point \( A \) in the form

\[
\ddot{\rho} = -\frac{\partial \tilde{\Pi}}{\partial \rho}, \quad \ddot{z} = -\frac{\partial \tilde{\Pi}}{\partial z}.
\] (3.2)

The configuration manifold of this system is the meridional plane \((\rho, z)\) rotating with an angular velocity \( \dot{\phi} \) around the \( z \)-axis. The figure of \( \tilde{\Pi}(\rho, z) \) is shown in Fig. 2.

![Fig. 2. Effective potential \( \tilde{\Pi}(\rho, z) \) with \( \mu = 0.8 \) and \( c = 0.1 \).](image)

Note important properties of the system (3.2).

**Proposition 1.** Effective potential has the symmetry with respect to the variable \( z \):

\( \tilde{\Pi}(\rho, -z) = \tilde{\Pi}(\rho, z) \).

Let’s denote the ring plane \( Oxy \) as an equatorial plane.

**Proposition 2.** If the point \( A \) doesn’t move in the equatorial plane, its coordinate \( z \) will oscillate around zero.

This proposition follows from the second equation of (3.2) and the following formula:

\[
\frac{\partial \tilde{\Pi}}{\partial z} = z \left( 1 - \frac{\mu}{r^3} + \frac{2\mu E(k)}{\pi pq^2} \right), \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha} d\alpha,
\]

where \( E(k) \) is the complete elliptic integral of the second kind. The expression in brackets is always greater than zero, therefore \( \ddot{z} \) and \( z \) have opposite signs.

**Proposition 3.** Equilibrium points of the system (3.2) can be located only in the equatorial plane.

Let’s now describe invariant manifolds of the phase space of the system (3.2):

1. \( \rho = \dot{\rho} = 0 \): point \( A \) moves along the \( z \)-axis. Since at point \( O \) the potential energy reaches infinity, such trajectories make sense only if \( h > 0 \) and speed \( \dot{z} \) has the same sign as the coordinate \( z \). Then the motion of the point \( A \) is infinite.
2. \( c = 0, \rho \neq 0 \): point \( A \) moves in the fixed meridional plane. In this case the problem reduces to the problem of three fixed centers: the center \( O \) with mass \( 1 - \mu \) and two points with masses \( \mu/2 \) located on a straight line passing through the center \( O \) at a unit distance.

3. \( z = \dot{z} = 0 \): point \( A \) moves in the equatorial plane.

Consider in detail the third case as the most interesting.

4. Motion in the equatorial plane

In the equatorial plane the motion of the point \( A \) is described by the equation

\[ \ddot{\rho} = -\frac{d\tilde{\Pi}(\rho, 0)}{d\rho}, \quad \tilde{\Pi}(\rho, 0) = -\frac{1 - \mu}{\rho} - \frac{2\mu K(k)}{\pi(\rho + 1)} + \frac{c^2}{2\rho^2}. \] (4.1)

Let us find stationary points of this equation. Since the right-hand side must be equal to zero, we get the following condition:

\[ R(\rho) = \frac{\mu \rho^2}{\pi} \left( \frac{E(k)}{\rho - 1} + \frac{K(k)}{\rho + 1} \right) + (1 - \mu)\rho = c^2. \]

Stationary points are determined by the intersection of the function \( R(\rho) \) and the horizontal line \( c^2 \geq 0 \). If \( \mu \neq 0 \), the function \( R(\rho) \) has the local minimum and maximum at points \( \rho_1 < 1 < \rho_2 \), respectively (Fig. 3).

Depending on the values of parameters \( \mu \) and \( c \), the following cases are possible:

1. \( \mu = 0 \). The mass of the ring is equal to zero and the problem reduces to the Kepler problem. There is one solution \( \rho = c^2 > 0 \).
2. \( 0 < \mu < 1 \). The function \( R(\rho) \) has a local maximum at \( \rho_1 \) and minimum at \( \rho_2 \) and besides \( 0 < R(\rho_1) < R(\rho_2) \) as shown in Fig. 3. If

   2.1 \( c^2 < R(\rho_1) \), then there are two solutions \( \rho_{1l} \) and \( \rho_{1r} \) lying to the left and right of \( \rho_1 \):
   \[ \rho_{1l} < \rho_1 < \rho_{1r} < 1; \]
   2.2 \( R(\rho_1) < c^2 < R(\rho_2) \), then there are no solutions;
   2.3 \( R(\rho_2) < c^2 \), then there are two solutions \( \rho_{2l} \) and \( \rho_{2r} \) lying to the left and right of \( \rho_2 \):
   \[ 1 < \rho_{2l} < \rho_2 < \rho_{2r}. \]

![Fig. 3. Functions R(\rho) and c^2.](image)
3. $\mu = 1$. The mass of the center $O$ is equal to zero. The function $R(\rho)$ has a local maximum at $\rho_1 = 0$ with the value $R(0) = 0$ and a local minimum at $\rho_2 > 1$ with the value $R(\rho_2) > 0$. Since $c^2 \geq 0$, we get solutions similar to cases 2.2 and 2.3.

This bifurcations could be represented on the phase plane: Figure 4 shows different phase portraits depending on the parameter $c$ (everywhere $\mu = 0.5$ is taken). For small values of $c$ (up to about $c = 0.5$) in the $\rho < 1$ region there are two equilibrium points, as shown in Fig. 4a). This phase portrait corresponds to case 2.1: solution $\rho_{1l}$ corresponds to the center, and $\rho_{1r}$ corresponds to the saddle. After reaching the value $c = 0.5$, the phase portrait is rearranged and the equilibrium positions disappear as shown in Fig. 4b). This portrait corresponds to case 2.2. Starting approximately from $c \approx 1.4$, there is another restructuring of the phase portrait, which corresponds to case 2.3. New stable $\rho_{2r}$ and unstable $\rho_{2l}$ equilibrium points in the $\rho > 1$ region appear as shown in Fig. 4c). If we further increase the value of the parameter $c$, the stable equilibrium point will move to the right.

Fig. 4. Phase portraits at the plane $(\rho, \dot{\rho})$ corresponding to the motion of a point $A$ in the equatorial plane with $\mu = 0.5$ and a) $c = 0.4$, b) $c = 1$, c) $c = 1.5$. 
Returning to the case of three-dimensional motion of the point \( A \) we can prove one of the particular cases of motion. From the first formula of (3.1) we get \( \rho = \text{const} \) and the next proposition follows from the fact that equilibrium points of the system (3.2) could exist only in the equatorial plane.

**Proposition 4.** The motion of the point \( A \) with the constant angular velocity of the meridional plane \( \dot{\varphi} \) can be realized only in the equatorial plane.

5. Conclusions

In this paper, the dynamics of a passively gravitating point in an axisymmetric gravitational Newtonian field created by the fixed ring and the center point mass has been studied. The equations of motion admit the integral \( c = \rho^2 \dot{\varphi} \), which allows one to reduce this problem to the problem of motion of a point along the surface \((\rho, z)\) with the shape \( \bar{\Pi}(\rho, z) \).

Invariant manifolds of the phase space have been found. The focus has been on the case of the motion in the equatorial plane. Increasing the integral \( c \) from zero to infinity, the potential energy shape is changed, which leads to bifurcations: first the disappearance of two equilibrium positions (center and saddle), and then, after some interval in which there are no equilibrium points, their birth in another area. The dynamics in the equatorial plane is regular because the energy integral exists.

In further work it will be interesting to construct a bifurcation diagram of the motion in the equatorial plane. The main direction of further research is to complete the classification of motions in the three-dimensional case, construction of a bifurcation diagram, determination of possible characteristic modes of motion: periodic and chaotic modes, motion with the critical values of the energy integral [8].

References