MATHEMATICAL PROBLEMS OF NONLINEARITY

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Antipodal Points and Diameter of a Sphere

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We give an example of a Riemannian manifold homeomorphic to a sphere such that its diameter cannot be realized as a distance between antipodal points. We consider a Berger sphere, i.e., a three-dimensional sphere with Riemannian metric that is compressed along the fibers of the Hopf fibration. We give a condition for a Berger sphere to have the desired property. We use our previous results on a cut locus of Berger spheres obtained by the method from geometric control theory.

Keywords: diameter, SU_2 , Berger sphere, antipodal points, cut locus, geometric control theory

1. Introduction

Let M be a metric space and let s be an isometric involution of M , i.e., $s^2 = \text{id}$. Assume that s has no fixed points.

Definition 1. Points $a, b \in M$ are antipodal if $b = s(a)$.

Definition 2. The diameter of a metric space M with a metric ρ is

$$\text{diam}_\rho M = \sup \{ \rho(x, y) \mid x, y \in M \}.$$

In [3] Yu. G. Nikonorov proved the following theorem.

Theorem 1. Let M be a metric space homeomorphic to a two-dimensional sphere. Assume that it has an isometric involution s without fixed points. Then there exists a point $x \in M$ such that $\text{diam}_\rho M = \rho(x, s(x))$.

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This theorem is helpful in investigating diameters for closed surfaces in $\mathbb{R}^3$ with central symmetry. In particular, Theorem 1 has applications in computational geometry for evaluation of diameter for a surface of a rectangular parallelepiped [4].

In [3] the following question was stated.

Question. *Is the statement of Theorem 1 true for spheres of an arbitrary dimension?*

REMARK 1. It is easy to see that it is true for a one-dimensional sphere.

REMARK 2. The proof of Theorem 1 essentially uses the Jordan theorem (a closed curve without self-intersections on a two-dimensional sphere is the boundary of a domain that is homeomorphic to a disk).

In this paper we show that, generally speaking, the diameter of a three-dimensional sphere cannot be realized as a distance between antipodal points. This means that the generalization of Theorem 1 is wrong.

2. Counterexample for the generalization of Theorem 1

Theorem 2. *There exists a Riemannian metric on a three-dimensional sphere such that its diameter cannot be realized as the distance between antipodal points.*

Proof. Let us consider a Berger sphere [2], i.e., the three-dimensional Lie group

$$\mathrm{SU}_2 = \{q = q_0 + q_1i + q_2j + q_3k \in \mathbb{H} \mid |q| = 1\}$$

endowed with a left-invariant Riemannian metric $\rho(I_1, I_2, I_3)$ with eigenvalues $I_1 = I_2, I_3 > 0$.

The involution

$$s: \mathrm{SU}_2 \rightarrow \mathrm{SU}_2, \quad s(q) = -q, \quad q \in \mathrm{SU}_2,$$

is an isometry without fixed points.

To determine the diameter, one may consider only distances to the unit point $1 \in \mathrm{SU}_2$, due to the left-invariance of the metric $\rho(I_1, I_2, I_3)$.

Consider the geodesic starting from the unit point. Recall that a *cut point* of such geodesic is the point at which the geodesic loses its minimality. The set of cut points for all geodesics starting from the unit point is called a *cut locus*.

The cut locus for the Berger sphere was found in [7] using the methods of geometric control theory [1]. Also in [7] the following parametrization of geodesics depending on the initial momentum $p \in \mathfrak{su}_2^*$ was achieved (we use the Hamiltonian formalism [1, 5]):

$$\begin{cases} q_0(\tau) &= \cos(\tau) \cos(\tau\eta\bar{p}_3) - \bar{p}_3 \sin(\tau) \sin(\tau\eta\bar{p}_3), \\ \begin{pmatrix} q_1(\tau) \\ q_2(\tau) \end{pmatrix} &= \sin(\tau) R_{-\tau\eta\bar{p}_3} \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \end{pmatrix}, \\ q_3(\tau) &= \cos(\tau) \sin(\tau\eta\bar{p}_3) + \bar{p}_3 \sin(\tau) \cos(\tau\eta\bar{p}_3), \end{cases} \quad (2.1)$$

where p is the initial momentum, $\bar{p}_3 = \frac{p}{|p|}$, the parameter $\eta = \frac{I_1}{I_3} - 1$ characterizes the metric, the operator $R_{-\tau\eta\bar{p}_3}$ is the rotation by angle $-\tau\eta\bar{p}_3$ and $\tau = \frac{t}{2I_1}|p|$, where t is an arclength parameter.

It is clear that the diameter equals the maximum value of the cut time (distance to the cut point) as a function of the initial momentum of a geodesic. The diameter of the Berger sphere

Table 1. Diameter of the Berger sphere and its realization

Eigenvalues of the metric	Diameter	Initial momentum	Cut time
$I_1 \leq I_3$	$2\pi\sqrt{I_1}$	$\bar{p}_3 = 0$	$\tau = \pi$
$I_3 < I_1 \leq 2I_3$	$2\pi\sqrt{I_3}$	$\bar{p}_3 = \pm 1$	$\tau = \tau_3(\bar{p}_3)$
$2I_3 < I_1$	$\frac{\pi I_1}{\sqrt{I_1 - I_3}}$	$\bar{p}_3 = \pm \frac{1}{\eta}$	$\tau = \tau_3(\bar{p}_3)$

Table 2. Most distant points from the unit point

Eigenvalues of the metric	The set of most distant points from 1
$I_1 \leq I_3$	$\{-1\}$
$I_3 < I_1 \leq 2I_3$	$\{-1\}$
$2I_3 < I_1$	$S^1 = \left\{ q_0 + q_1 i + q_2 j \mid q_0 = -\frac{1}{\eta} = \frac{I_3}{I_3 - I_1} \right\}$

was computed in [6]. In fact, in [6] the segments of geodesics that realize the diameter were described. These segments are defined by the initial momentum of a geodesic and a cut time. These results are represented in Table 1, where $\tau_3(\bar{p}_3)$ is the first positive root of the equation $q_3(\tau) = 0$.

From Table 1 and parametrization of geodesics (2.1) it is easy to find the set of most distant points from the unit point (see Table 2). Indeed, this is a result of substitution of the initial momentum and the cut time from Table 1 into Eq. (2.1).

Thus, if $2I_3 < I_1$, then the set of most distant points from the unit point does not contain the point -1 which is antipodal to the unit point. ■

REMARK 3. It seems that the problem of describing left-invariant metrics on a three-dimensional sphere such that its diameter can be realized as a distance between antipodal points is difficult. In the case $I_1 \neq I_2$ the parametrization of geodesics cannot be written in terms of elementary functions as in Eq. (2.1). The cut locus is unknown in this case.

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