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Elliptical Billiards in the Minkowski Plane and Extremal Polynomials

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We derive necessary and sufficient conditions for periodic trajectories of billiards within an ellipse in the Minkowski plane in terms of an underlying elliptic curve. Equivalent conditions are derived in terms of polynomial-functional equations as well. The corresponding polynomials are related to the classical extremal polynomials. Similarities and differences with respect to the previously studied Euclidean case are indicated.

Keywords: Minkowski plane, elliptical billiards, elliptic curve, Akhiezer polynomials.

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1. Introduction

The aim of this paper is to develop the connection between extremal polynomials and billiards in the Minkowski plane. Billiards within quadrics in pseudo-Euclidean spaces were studied in [6, 7, 11]. On the other hand, the relationship between the billiards within quadrics in the Euclidean spaces and extremal polynomials has been studied in [8, 9].

The material is organized as follows. In Section 2, we recall the basic notions about the Minkowski plane, confocal families of conics, and billiards. In Section 3, we give a complete description of the periodic billiard trajectories in algebro-geometric terms. In Section 4, we derive a characterization of such trajectories using polynomial equations. In Section 5, we establish the connection between characteristics of periodic billiard trajectories and extremal polynomials. An expanded version of this paper with detailed proofs will be presented in [1].

2. Confocal families of conics and billiards

The Minkowski plane is $\mathbf{R}^2$ with the Minkowski scalar product: $\langle x, y \rangle = x_1 y_1 - x_2 y_2$, see [6, 7] and references therein.

The Minkowski distance between points x, y is $\text{dist}(x, y) = \sqrt{\langle x - y, x - y \rangle}$. Since the scalar product can be negative, notice that the Minkowski distance can have imaginary values as well. In that case, we choose the value of the square root with the positive imaginary part.

Let ℓ be a line in the Minkowski plane, and v its vector. ℓ is called *space-like* if $\langle v, v \rangle > 0$; *time-like* if $\langle v, v \rangle < 0$; and *light-like* if $\langle v, v \rangle = 0$. Two vectors x, y are *orthogonal* in the Minkowski plane if $\langle x, y \rangle = 0$. Note that a light-like vector is orthogonal to itself.

Confocal families. Denote by

$$\mathcal{E}: \frac{x^2}{a} + \frac{y^2}{b} = 1 \quad (2.1)$$

an ellipse in the plane, with a, b being fixed positive numbers.

The associated family of confocal conics is

$$\mathcal{C}_\lambda: \frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1, \quad \lambda \in \mathbf{R}. \quad (2.2)$$

The family is shown in Figure 1. We may distinguish the following three subfamilies in the family (2.2): for $\lambda \in (-b, a)$, conic $\mathcal{C}_\lambda$ is an ellipse; for $\lambda < -b$, conic $\mathcal{C}_\lambda$ is a hyperbola with x -axis as the major one; for $\lambda > a$, it is a hyperbola again, but now its major axis is y -axis. In addition, there are three degenerated quadrics: $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_\infty$ corresponding to y -axis, x -axis, and the line at infinity, respectively.

Each point inside ellipse $\mathcal{E}$ has elliptic coordinates (λ_1, λ_2) , such that

$$-b < \lambda_1 < 0 < \lambda_2 < a.$$

The differential equation of the lines touching a given conic $\mathcal{C}_\gamma$ is

$$\frac{d\lambda_1}{\sqrt{(a-\lambda_1)(b+\lambda_1)(\gamma-\lambda_1)}} + \frac{d\lambda_2}{\sqrt{(a-\lambda_2)(b+\lambda_2)(\gamma-\lambda_2)}} = 0. \quad (2.3)$$

Billiards. Let v be a vector and p a line in the Minkowski plane. Decompose vector v into the sum $v = a + n_p$ of a vector n_p orthogonal to p and a belonging to p . Then the vector $v' = a - n_p$ is the *billiard reflection* of v on p . It is easy to see that v is also the billiard reflection of v' with respect to p . Moreover, since $\langle v, v \rangle = \langle v', v' \rangle$, vectors v, v' are of the same type.

Fig. 1. Family of confocal conics in the Minkowski plane.

Note that $v = v'$ if v is contained in p and $v' = -v$ if it is orthogonal to p . If n_p is light-like, which means that it belongs to p , then the reflection is not defined.

Line ℓ' is the *billiard reflection* of ℓ off ellipse $\mathcal{E}$ if their intersection point $\ell \cap \ell'$ belongs to $\mathcal{E}$ and the vectors of ℓ, ℓ' are reflections of each other with respect to the tangent line of $\mathcal{E}$ at this point.

The lines containing segments of a given billiard trajectory within $\mathcal{E}$ are all of the same type: they are all either space-like, time-like, or light-like. For a detailed explanation, see [11].

Billiard trajectories within ellipses in the Minkowski plane have caustic properties: each segment of a given trajectory will be tangent to the same conic confocal with the boundary, see [6]. More about the Minkowski plane and related integrable systems can be found in [5, 10, 13].

3. Periodic trajectories

The periodic trajectories of elliptical billiards in the Minkowski plane can be characterized in algebro-geometric terms using the underlying elliptic curve. We deal here with the trajectories with non-degenerate caustic. Such trajectories are either space-like or time-like.

Theorem 1. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n -periodic if and only if $nQ_0 \sim nQ_\gamma$ on the elliptic curve:*

$$\mathcal{C}: y^2 = \varepsilon(a - x)(b + x)(\gamma - x), \quad (3.1)$$

with Q_0 being a point of $\mathcal{C}$ corresponding to $x = 0$, and Q_γ the point corresponding to $x = \gamma$, and $\varepsilon = \text{sign } \gamma$.

The proof of the theorem is based on the following. Consider an n -periodic billiard trajectory and denote by n_1 the number of reflections off time-like arcs and n_2 the number of reflections off space-like ones, $n_1 + n_2 = n$. Along a billiard trajectory within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$, the elliptic coordinate λ_1 traces the segment $[\alpha_1, 0]$, and λ_2 the segment $[0, \beta_1]$, where α_1 is the largest negative and β_1 the smallest positive member of the set $\{a, -b, \gamma\}$. Integrating (2.3) along the

trajectory, we get

$$n_1 \int_{\alpha_1}^0 \frac{d\lambda_1}{\sqrt{\varepsilon(a-\lambda_1)(b+\lambda_1)(\gamma-\lambda_1)}} + n_2 \int_{\beta_1}^0 \frac{d\lambda_2}{\sqrt{\varepsilon(a-\lambda_2)(b+\lambda_2)(\gamma-\lambda_2)}} = 0, \quad (3.2)$$

i.e., $n_1(Q_0 - Q_{\alpha_1}) + n_2(Q_0 - Q_{\beta_1}) \sim 0$.

We also derive the following:

Corollary 1. *The period of a closed trajectory with hyperbola as caustic is even.*

Theorem 2. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n -periodic if and only if:*

$$C_2 = 0, \quad \begin{vmatrix} C_2 & C_3 \\ C_3 & C_4 \end{vmatrix} = 0, \quad \begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0, \dots \quad \text{for } n = 3, 5, 7, \dots$$

$$B_3 = 0, \quad \begin{vmatrix} B_3 & B_4 \\ B_4 & B_5 \end{vmatrix} = 0, \quad \begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0, \dots \quad \text{for } n = 4, 6, 8, \dots$$

Here, we have denoted:

$$\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)} = B_0 + B_1x + B_2x^2 + \dots,$$

$$\frac{\sqrt{\varepsilon(a-x)(b+x)(\gamma-x)}}{\gamma-x} = C_0 + C_1x + C_2x^2 + \dots,$$

the Taylor expansions around $x = 0$.

EXAMPLE 1 (3-PERIODIC TRAJECTORIES). There is a 3-periodic trajectory of the billiard within (2.1), with a nondegenerate caustic $\mathcal{C}_\gamma$ in the Minkowski plane if and only if

- the caustic is an ellipse, i.e., $\gamma \in (-b, a)$; and
- by Theorem 2, $C_2 = 0$.

We solve the equation

$$C_2 = \frac{3a^2b^2 + 2ab(a-b)\gamma - (a+b)^2\gamma^2}{8(ab)^{3/2}\gamma^{5/2}} = 0, \quad (3.3)$$

which yields the following two solutions for the parameter γ for the caustic:

$$\gamma_1 = \frac{ab}{(a+b)^2}(a-b+2\sqrt{a^2+ab+b^2}), \quad (3.4)$$

$$\gamma_2 = -\frac{ab}{(a+b)^2}(-a+b+2\sqrt{a^2+ab+b^2}). \quad (3.5)$$

Notice that both caustics $\mathcal{C}_{\gamma_2}$ and $\mathcal{C}_{\gamma_1}$ are ellipses since $-b < \gamma_2 < 0 < \gamma_1 < a$. Below are two examples of trajectories with two different caustics.

Two examples of 3-periodic trajectories are shown in Figure 2.

Fig. 2. A 3-periodic trajectory with an ellipse along the y -axis as caustic ($a = 3$, $b = 2$, $\gamma \approx 2.332$) is shown on the left, while another trajectory with an ellipse along the x -axis as caustic ($a = 7$, $b = 5$, $\gamma \approx -4.589$) is on the right.

4. Polynomial equations

Now we want to express the periodicity conditions for billiard trajectories in the Minkowski plane in terms of polynomial functions equations.

Theorem 3. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n -periodic if and only if there exists a pair of real polynomials p_{d_1} , q_{d_2} of degrees d_1 , d_2 , respectively, which satisfy the following:*

(a) if $n = 2m$ is even, then $d_1 = m$, $d_2 = m - 2$, and

$$p_m^2(s) - s \left(s - \frac{1}{a} \right) \left(s + \frac{1}{b} \right) \left(s - \frac{1}{\gamma} \right) q_{m-2}^2(s) = 1;$$

(b) if $n = 2m + 1$ is odd, then $d_1 = m$, $d_2 = m - 1$, and

$$\left(s - \frac{1}{\gamma} \right) p_m^2(s) - s \left(s - \frac{1}{a} \right) \left(s + \frac{1}{b} \right) q_{m-1}^2(s) = -\text{sign } \gamma.$$

Corollary 2. *If the billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_\gamma$ are n -periodic, then there exist real polynomials $\hat{p}_n$ and $\hat{q}_{n-2}$ of degrees n and $n - 2$, respectively, which satisfy the Pell equation:*

$$\hat{p}_n^2(s) - s \left(s - \frac{1}{a} \right) \left(s + \frac{1}{b} \right) \left(s - \frac{1}{\gamma} \right) \hat{q}_{n-2}^2(s) = 1. \quad (4.1)$$

Proof. For $n = 2m$, take $\hat{p}_n = 2p_m^2 - 1$ and $\hat{q}_{n-2} = 2p_m q_{m-2}$. For $n = 2m + 1$, we set $\hat{p}_n = 2(\gamma s - 1)p_m^2 + \text{sign } \gamma$ and $\hat{q}_{n-2} = 2p_m q_{m-1}$. $\square$

After Corollary 2 and relation (4.1), we see that the Pell equations arise as the functional polynomial conditions for periodicity. Let us recall some important properties of the solutions of Pell's equations.

5. Classical Extremal Polynomials and Caustics

5.1. Fundamental Properties of Extremal Polynomials

From the previous section we know that the Pell equation plays a key role in functional-polynomial formulation of periodicity conditions in the Minkowski plane. The solutions of the Pell equation are so-called extremal polynomials. Denote $\{c_1, c_2, c_3, c_4\} = \{0, 1/a, 1/b, 1/\gamma\}$ with the ordering $c_1 < c_2 < c_3 < c_4$. The polynomials $\hat{p}_n$ are so-called *generalized Chebyshev polynomials* on two intervals $[c_1, c_2] \cup [c_3, c_4]$, with an appropriate normalization. Namely, one can consider the question of finding the monic polynomial of a certain degree n which minimizes the maximum norm on the union of two intervals. Denote such a polynomial as $\hat{P}_n$ and its norm L_n . The fact that polynomial $\hat{p}_n$ is a solution of the Pell equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ is equivalent to the following conditions:

- (i) $\hat{p}_n = \hat{P}_n / \pm L_n$
- (ii) the set $[c_1, c_2] \cup [c_3, c_4]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the sense above.

Chebyshev was the first who considered a similar problem on one interval, and this was how the celebrated Chebyshev polynomials emerged in the XIXth century. Let us recall a fundamental result about generalized Chebyshev polynomials.

Theorem 4 (A corollary of the Krein – Levin – Nudelman Theorem [12]). *A polynomial P_n of degree n satisfies the Pell equation on the union of intervals $[c_1, c_2] \cup [c_3, c_4]$ if and only if there exists an integer n_1 such that the equation holds:*

$$n_1 \int_{c_2}^{c_3} f(x) dx = n \int_{c_4}^{\infty} f(x) dx, \quad f(x) = \left(\sqrt{\prod_{i=1}^4 (x - c_i)} \right)^{-1}. \quad (5.1)$$

The modulus of the polynomial reaches its maximal values L_n at the points $c_i : |P_n(c_i)| = L_n$. In addition, there are exactly $\tau_1 = n - n_1 - 1$ internal extremal points of the interval $[c_3, c_4]$ where $|P_n|$ reaches the value L_n , and there are $\tau_2 = n_1 - 1$ internal extremal points of $[c_1, c_2]$ with the same property.

Definition 1. We call the pair (n, n_1) the partition and (τ_1, τ_2) the signature of the generalized Chebyshev polynomial P_n (see [8, 9]).

Now we are going to formulate and prove the main result of this section, which relates n_1, n_2 the numbers of reflections off relativistic ellipses and off relativistic hyperbolas, respectively, with the partition and the signature of the related solution of the Pell equation.

Theorem 5. *Given a periodic billiard trajectory with period $n = n_1 + n_2$, where n_1 is the number of reflections off relativistic ellipses, n_2 the number of reflections off the relativistic hyperbolas. The partition corresponding to this trajectory is (n, n_1) . The corresponding extremal polynomial $\hat{p}_n$ of degree n has $n_1 - 1$ internal extremal points in the first interval and $n - n_1 - 1 = n_2 - 1$ internal extremal points in the second interval.*

In particular, for $n = 3$, if the caustic $\mathcal{C}_\gamma$ is an ellipse with $\gamma < 0$, then $n_1 = 1$. Such polynomials and corresponding partitions $(3, 1)$ do not arise in the study of Euclidean billiard trajectories. On the other hand, if the caustic $\mathcal{C}_\gamma$ is an ellipse with $\gamma > 0$, we have $n_1 = 2$. The corresponding extremal polynomials $\hat{p}_3$ are shown in Figure 3. We will provide in Proposition 1 the explicit formulae for such polynomials in terms of the general Akhiezer polynomials.

Fig. 3. On the left: the polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 1$, $n_2 = 2$, $\gamma < 0$. On the right: the polynomial $\hat{p}_3$ corresponding to $n = 3$, $n_1 = 2$, $n_2 = 1$, $\gamma > 0$.

5.2. General Akhiezer polynomials on unions of two intervals

Following Akhiezer [2–4], let us consider the union of two intervals $[-1, \alpha] \cup [\beta, 1]$, where

$$\alpha = 1 - 2sn^2\left(\frac{m}{n}K\right), \quad \beta = 2sn^2\left(\frac{n-m}{n}K\right) - 1. \quad (5.2)$$

Define

$$TA_n(x, m, \kappa) = L \left(v^n(u) + \frac{1}{v^n(u)} \right), \quad (5.3)$$

where

$$v(u) = \frac{H\left(u - \frac{m}{n}K\right)}{H\left(u + \frac{m}{n}K\right)},$$

$$x = \frac{sn^2(u)cn^2\left(\frac{m}{n}K\right) + cn^2(u)sn^2\left(\frac{m}{n}K\right)}{sn^2(u) - sn^2\left(\frac{m}{n}K\right)},$$

and

$$L = \frac{1}{2^{n-1}} \left(\frac{\theta(0)\theta_1(0)}{\theta\left(\frac{m}{n}K\right)\theta_1\left(\frac{m}{n}K\right)} \right), \quad \kappa^2 = \frac{2(\beta - \alpha)}{(1 - \alpha)(1 + \beta)}.$$

Akhiezer proved the following result:

Theorem 6 (Akhiezer).

- (a) The function $TA_n(x, m, \kappa)$ is a polynomial of degree n in x with the leading coefficient 1 and the second coefficient equal to $-n\tau_1$, where

$$\tau_1 = -1 + 2 \frac{sn\left(\frac{m}{n}K\right)cn\left(\frac{m}{n}K\right)}{dn\left(\frac{m}{n}K\right)} \left(\frac{1}{sn\left(\frac{2m}{n}K\right)} - \frac{\theta'\left(\frac{m}{n}K\right)}{\theta\left(\frac{m}{n}K\right)} \right).$$

- (b) The maximum of the modulus of TA_n on the union of the two intervals $[-1, \alpha] \cup [\beta, 1]$ is L .
- (c) The function TA_n takes values $\pm L$ with alternating signs at $\mu = n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $\nu = m + 1$ consecutive points of the interval $[\beta, 1]$. In addition,

$$TA_n(\alpha, m, \kappa) = TA_n(\beta, m, \kappa) = (-1)^m L,$$

and for any $x \in (\alpha, \beta)$, we have

$$(-1)^m T A_n(x, m, \kappa) > L.$$

(d) Let F be a polynomial of degree n in x with the leading coefficient 1, such that:

i) $\max |F(x)| = L$ for $x \in [-1, \alpha] \cup [\beta, 1]$;

ii) $F(x)$ takes values $\pm L$ with alternating signs at $n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $m + 1$ consecutive points of the interval $[\beta, 1]$.

Then $F(x) = T A_n(x, m, \kappa)$.

Let us determine the affine transformations when the caustic is an ellipse.

Case $\lambda \in (-b, 0)$. For $h : [-1, \alpha] \cup [\beta, 1] \rightarrow [\gamma^{-1}, -b^{-1}] \cup [0, a^{-1}]$, $h(x) = \hat{a}x + \hat{b}$, we get

$$\hat{a} = \frac{1}{\beta - \alpha} \frac{1}{b}, \quad \hat{b} = \frac{-\beta}{\beta - \alpha} \frac{1}{b}, \quad \frac{1 - \beta}{\beta - \alpha} = \frac{b}{a}.$$

Thus,

$$\lambda = \frac{\beta - 1}{1 + \beta} a = \frac{\alpha - \beta}{\beta + 1} b. \quad (5.4)$$

EXAMPLE 2. Let $n = 3$ and $m = 2$. From (5.2), one gets

$$\alpha = 1 - 2sn^2 \frac{2}{3} K, \quad \beta = 2sn^2 \frac{K}{3} - 1.$$

It follows that

$$\frac{b}{a} = t = \frac{1 - \beta}{\beta - \alpha} = \frac{1 - sn^2 \frac{K}{3}}{sn^2 \frac{2}{3} K + sn^2 \frac{K}{3} - 1}. \quad (5.5)$$

Thus

$$\lambda = b \frac{\alpha - \beta}{\beta + 1} = b \frac{1 - sn^2 \frac{K}{3} - sn^2 \frac{2}{3} K}{sn^2 \frac{K}{3}}. \quad (5.6)$$

From the addition formula we get

$$sn \frac{2}{3} K = sn \left(K - \frac{K}{3} \right) = \frac{sn K cn \frac{-K}{3} dn \frac{-K}{3} + sn \frac{-K}{3} cn K dn K}{1 - \kappa^2 sn^2 \frac{-K}{3} sn^2 K},$$

hence

$$sn^2 \frac{2}{3} K = \frac{1 - sn^2 \frac{K}{3}}{1 - \kappa^2 sn^2 \frac{K}{3}} = \frac{sn^2 \frac{2}{3} K - 1}{\kappa^2 sn^2 \frac{2}{3} K - 1}.$$

Let $sn \frac{K}{3} = Z$. Then

$$\kappa^2 = \frac{2Z - 1}{Z^3(2 - Z)}, \quad \alpha = 2Z^2 - 4Z + 1, \quad \beta = 2Z^2 - 1.$$

Equation (5.5) implies that

$$t = \frac{1 - Z^2}{2Z - 1}. \quad (5.7)$$

Thus, we have two expressions for λ . One is from the Cayley condition (3.5) and the other is from (5.4). We want to show that these two expressions are identical, that is,

$$b \frac{\alpha - \beta}{\beta + 1} = -\frac{ab}{(a+b)^2}(-a+b+2\sqrt{a^2+ab+b^2}). \quad (5.8)$$

In order to do so, we first express both the left-hand side and the right-hand side of the above equation in terms of $t = \frac{b}{a}$ and then transform both sides in terms of Z . We show that the L.H.S. and the R.H.S. yield the same expression:

$$\frac{Z^2 - Z + 1}{2Z - 1},$$

therefore (5.8) holds.

Case $\lambda \in (0, a)$. For $l: [-1, \alpha] \cup [\beta, 1] \rightarrow [-b^{-1}, 0] \cup [a^{-1}, \lambda^{-1}]$, $l(x) = \hat{a}x + \hat{b}$, we get

$$\hat{a} = \frac{1}{\alpha+1} \frac{1}{b}, \quad \hat{b} = \frac{-\alpha}{\alpha+1} \frac{1}{b}, \quad \frac{\alpha+1}{\beta-\alpha} = \frac{a}{b}.$$

Thus,

$$\lambda = \frac{\alpha+1}{1-\alpha} b. \quad (5.9)$$

EXAMPLE 3. Let $n = 3$, and $m = 1$. From (5.2), one gets

$$\begin{aligned} \alpha &= 1 - 2sn^2 \frac{K}{3}, \quad \beta = 2sn^2 \frac{2K}{3} - 1, \\ \frac{b}{a} = t &= \frac{\beta - \alpha}{\alpha + 1} = \frac{sn^2 \frac{2}{3} K + sn^2 \frac{K}{3} - 1}{1 - sn^2 \frac{K}{3}}. \end{aligned} \quad (5.10)$$

Thus,

$$\lambda = \frac{1 - sn^2 \frac{K}{3}}{sn^2 \frac{K}{3}} b.$$

From the addition formula we get

$$sn \frac{2K}{3} = sn \left(K - \frac{K}{3} \right) = \frac{snK cn \frac{-K}{3} dn \frac{-K}{3} + sn \frac{-K}{3} cnK dnK}{1 - \kappa^2 sn^2 \frac{-K}{3} sn^2 K},$$

hence

$$sn^2 \frac{2}{3} K = \frac{1 - sn^2 \frac{K}{3}}{1 - \kappa^2 sn^2 \frac{K}{3}}.$$

Let $sn \frac{K}{3} = Z$. Then

$$\kappa^2 = \frac{2Z - 1}{Z^3(2 - Z)}.$$

Also, $\beta = -2Z^2 + 4Z - 1$, and $\alpha = 1 - 2Z^2$. Equation (5.10) implies that

$$t = -\frac{2Z - 1}{Z^2 - 1}. \quad (5.11)$$

Thus, we have two expressions for λ . One is from the Cayley condition (3.4) and the other is from (5.9). We want to show that these two expressions are identical, that is,

$$b \frac{1+\alpha}{1-\alpha} = \frac{ab}{(a+b)^2} (a-b+2\sqrt{a^2+ab+b^2}). \quad (5.12)$$

In order to do so, we first express both the left-hand side and the right-hand side of the above equation in terms of $t = \frac{b}{a}$. Then we transform both sides in terms of Z and show that the L.H.S. and the R.H.S. yield the same expression:

$$\frac{Z^2 - Z + 1}{2Z - 1},$$

therefore (5.12) holds.

Proposition 1. For $n = 3$ and $\lambda \in (-b, 0)$, the polynomial $\hat{p}_3$ is, up to a nonessential factor, equal to:

$$\hat{p}_3 \sim TA_3 \left(2a \left(1 - sn^2 \frac{K}{3} \right) s + 2sn^2 \frac{K}{3} - 1; 2, \kappa \right).$$

For $n = 3$ and $\lambda \in (0, a)$, the polynomial $\hat{p}_3$ is, up to a nonessential factor, equal to:

$$\hat{p}_3 \sim TA_3 \left(2b \left(1 - sn^2 \frac{K}{3} \right) s + 1 - 2sn^2 \frac{K}{3}; 1, \kappa \right).$$

Now, using the Akhiezer Theorem, part (c), see Theorem 6, one can compare and see that the number of internal extremal points coincides with $n_1 - 1$ and $n_2 - 1$ as proposed in Theorem 5. These numbers match with Fig. 3.

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