NONLINEAR PHYSICS AND MECHANICS

MSC 2010: 74J30

The Study of Wave Propagation in a Shell with Soft Nonlinearity and with a Viscous Liquid Inside

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This article is devoted to studying longitudinal deformation waves in physically nonlinear elastic shells with a viscous incompressible fluid inside them. The impact of construction damping on deformation waves in longitudinal and normal directions in a shell, and in the presence of surrounding medium are considered.

The presence of a viscous incompressible fluid inside the shell and the impact of fluid movement inertia on the wave velocity and amplitude are taken into consideration. In the case of a shell filled with a viscous incompressible fluid, it is impossible to study deformation wave models by qualitative analysis methods. This makes it necessary to apply numerical methods. The numerical study of the constructed model is carried out by means of a difference scheme analogous to the Crank–Nicolson scheme for the heat conduction equation. The amplitude and velocity do not change in the absence of surrounding medium impact, construction damping in longitudinal and normal directions, as well as in the absence of fluid impact. The movement occurs in the negative direction, which means that the movement velocity is subsonic. The numerical experiment results coincide with the exact solution, therefore, the difference scheme and the modified Korteweg–de Vries–Burgers equation are adequate.

Keywords: nonlinear waves, elastic cylinder shell, viscous incompressible fluid, Crank–Nicolson difference scheme

Received April 10, 2019

Accepted July 15, 2019

The work was supported by the RFBR grant No. 19-01-00014a.

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1. Introduction

The problem of wave propagation in gas dynamics and elastic shells theory is studied using linearized equations. The disturbance propagation velocity is taken to be constant and equal to the sound propagation velocity in an undisturbed medium. However, a number of cases in spite of small values of dependent variables are defined by the dependence of the disturbance propagation velocity on dependent variables values and are studied on the basis of nonlinear equations. These investigations are carried out by perturbation methods.

The study of the wave process in elastic shells is widely applied in various technical fields. The propagation of deformation waves in elastic, viscoelastic, and nonlinear viscoelastic shells and plates was considered in [7–11]. In those papers, the case of interaction of shells and a viscous incompressible fluid is not considered. References [12–14] consider the case of interaction of shells and a viscous incompressible fluid without taking into account wave phenomena; neither were local terms of inertia influence investigated. Various methods are used to solve related and unrelated problems.

When solving unrelated problems, the motion of a fluid interacting with a rigid body is considered. The stress exerted by the fluid on the rigid body, friction and pressure are determined. Thus, it is assumed that there is no influence of the deformation of the shell on the movement of the fluid [15–19]. The parameters obtained are substituted into the equations of the dynamics of an elastic body, then longitudinal and normal (deflection) displacements are found. Thus, the stress-strain state of the elastic structure, as the aim of the uncoupled problem, is determined.

In the case of a related problem, the equations of the dynamics of an elastic body and a fluid are solved simultaneously, taking into account the boundary conditions on impermeable surfaces. This approach has been applied to the study of hydroelastic vibrations [20], as well as in this article to the study of nonlinear deformation waves of elastic shells containing a viscous incompressible fluid, taking into account the inertia of its motion.

It is impossible to investigate models of deformation waves using methods of qualitative analysis in the case of a shell filled with a viscous incompressible fluid [1]. This makes it necessary to apply numerical methods.

This article examines the influence of structural damping in the longitudinal and normal direction, surrounding elastic medium, viscous incompressible fluid inside the shell as well as the inertia of fluid motion on the wave amplitude and velocity.

In the course of investigation the numerical study of the constructed model was carried out by using a difference scheme for an equation similar to the Crank-Nicholson scheme for the heat equation [21].

2. Defining and resolving relations of the physically nonlinear theory of shells

Let us consider an axisymmetric cylindrical shell. Denote: h_0 is the thickness of the shell; R is the radius of the medium surface; R_1 is the radius of the inner surface; U is the longitudinal elastic displacement; W is deflection directed to the center of curvature.

Longitudinal deformation waves in an infinitely long shell are featured in the long-wave low-frequency model described by the Kirchhoff–Love theory.

A. A. Ilyushin's deformation theory of plasticity [2, 22] connects the components of the stress tensor σ_x , σ_Θ with the components of the strain tensor ε_x , ε_Θ , and the square of the

strain intensity ε_u [3, 23]

$$\begin{aligned}\sigma_x &= \frac{E}{1-\mu_0^2} (\varepsilon_x + \mu_0 \varepsilon_\Theta) \left(1 - \frac{m}{E} \varepsilon_u^2\right); \quad \sigma_\Theta = \frac{E}{1-\mu_0^2} (\varepsilon_\Theta + \mu_0 \varepsilon_x) \left(1 - \frac{m}{E} \varepsilon_u^2\right); \\ \varepsilon_u^2 &= \frac{4}{3} (\mu_1 (\varepsilon_x^2 + \varepsilon_\Theta^2) - \mu_2 \varepsilon_x \varepsilon_\Theta); \\ \mu_1 &= \frac{1}{3} \left[1 + \frac{\mu_0}{(1-\mu_0)^2}\right]; \quad \mu_2 = \frac{1}{3} \left[1 - \frac{2\mu_0}{(1-\mu_0)^2}\right],\end{aligned}\quad (2.1)$$

where E is Young's modulus, m is a material constant determined from tensile or compression experiments, and μ_0 is Poisson's ratio of shell material.

We write down the connection between deformations and elastic displacements in the form of [4]

$$\varepsilon_x = \frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 - z \frac{\partial^2 W}{\partial x^2}; \quad \varepsilon_\Theta = -\frac{W}{R}, \quad (2.2)$$

where x is the longitudinal coordinate along the median surface and z is the normal coordinate in the shell $\left(-\frac{h_0}{2} \leq z \leq \frac{h_0}{2}\right)$.

The square of the intensity of deformations is

$$\begin{aligned}\varepsilon_u^2 &= \frac{4}{3} \left\langle \mu_1 \left\{ \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 - z \frac{\partial^2 W}{\partial x^2} \right]^2 + \frac{W^2}{R^2} \right\} + \right. \\ &\quad \left. + \mu_2 \frac{W}{R} \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 - z \frac{\partial^2 W}{\partial x^2} \right] \right\rangle\end{aligned}\quad (2.3)$$

or

$$\begin{aligned}\varepsilon_u^2 &= \frac{4}{3} \left\langle \mu_1 \left\{ \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 \right]^2 + \left(\frac{W}{R}\right)^2 \right\} + \mu_2 \frac{W}{R} \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 \right] - \right. \\ &\quad \left. - z \left[2\mu_1 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 \right) + \mu_2 \frac{W}{R} \right] \frac{\partial^2 W}{\partial x^2} + z^2 \mu_1 \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \right\rangle.\end{aligned}\quad (2.4)$$

Let us define the forces in the middle surface of the shell and the moment according to the following formulas:

$$N_x = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \sigma_x dz; \quad N_\Theta = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \sigma_\Theta dz; \quad M_x = \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \sigma_x z dz. \quad (2.5)$$

where

$$\begin{aligned}\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \left[1 - \frac{m}{E} \varepsilon_u^2\right] dz &= h_0 \left\langle 1 - \frac{m}{E} \frac{4}{3} \left\{ \mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2\right)^2 + \left(\frac{W}{R}\right)^2 \right] + \right. \right. \\ &\quad \left. \left. + \mu_2 \frac{W}{R} \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 \right] + \frac{h_0^2}{12} \mu_1 \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \right\} \right\rangle; \\ \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} z \left[1 - \frac{m}{E} \varepsilon_u^2\right] dz &= -\frac{m}{E} \frac{4}{3} \left\{ -\frac{h_0^3}{12} \left[2\mu_1 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x}\right)^2 \right) + \mu_2 \frac{W}{R} \right] \frac{\partial^2 W}{\partial x^2} \right\};\end{aligned}\quad (2.6)$$

$$\int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} z^2 \left[1 - \frac{m}{E} \varepsilon_u^2 \right] dz = \frac{h_0^3}{12} \left\langle 1 - \frac{m}{E} \frac{4}{3} \left\{ \mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \right. \right. \\ \left. \left. + \mu_2 \frac{W}{R} \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right] + 3 \frac{h_0^2}{20} \mu_1 \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \right\} \right\rangle.$$

Substituting (2.6) into (2.5), we find

$$\begin{aligned} N_x &= \frac{E h_0}{1 - \mu_0^2} \left\langle \frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \mu_0 \frac{W}{R} - \frac{m}{E} \frac{4}{3} \left\{ \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \mu_0 \frac{W}{R} \right] \times \right. \right. \\ &\quad \times \left[\mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \mu_2 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) \frac{W}{R} \right] + \\ &\quad \left. \left. + \frac{h_0^2}{12} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left[3 \mu_1 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) + (\mu_2 - \mu_1 \mu_0) \frac{W}{R} \right] \right\} \right\rangle, \\ N_\Theta &= \frac{E h_0}{1 - \mu_0^2} \left\langle \mu_0 \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right] - \frac{W}{R} - \frac{m}{E} \frac{4}{3} \left\{ \left[\mu_0 \left(\frac{\partial U}{\partial x} + \right. \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) - \frac{W}{R} \right] \left[\mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \right. \right. \\ &\quad \left. \left. + \mu_2 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) \frac{W}{R} \right] + \right. \\ &\quad \left. \left. + \frac{h_0^2}{12} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left[3 \mu_0 \mu_0 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) + (\mu_1 - \mu_2 \mu_0) \frac{W}{R} \right] \right\} \right\rangle, \\ M_x &= - \frac{E h_0^3}{12 (1 - \mu_0^2)} \frac{\partial^2 W}{\partial x^2} \left\langle 1 - \frac{m}{E} \frac{4}{3} \left\{ 3 \mu_1 \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right]^2 + \right. \right. \\ &\quad \left. \left. + 2 (\mu_2 - \mu_1 \mu_0) \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right] \frac{W}{R} + (\mu_1 - \mu_2 \mu_0) \left(\frac{W}{R} \right)^2 + 3 \frac{h_0^2}{20} \mu_1 \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \right\} \right\rangle. \end{aligned} \quad (2.7)$$

Dynamic equations for shells with structural damping in the longitudinal and normal directions are written as

$$\begin{aligned} \frac{\partial N_x}{\partial x} &= \rho_0 h_0 \frac{\partial^2 U}{\partial t^2} + \varepsilon_1 \frac{1}{l} \rho_0 h_0 \sqrt{\frac{E}{\rho_0 (1 - \mu_0^2)}} \frac{\partial U}{\partial t} - \left[q_x - W \frac{\partial q_x}{\partial r} + U \frac{\partial q_x}{\partial x} \right] \\ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial x} N_x \right) + \frac{1}{R} N_\Theta &= \rho_0 h_0 \frac{\partial^2 W}{\partial t^2} + \varepsilon_2 \frac{l}{R^2} \rho_0 h_0 \sqrt{\frac{E}{\rho_0 (1 - \mu_0^2)}} \frac{\partial W}{\partial t} + \\ &\quad + k_1 \frac{h_0}{R^3} \rho_0 h_0 \frac{E}{\rho_0 (1 - \mu_0^2)} W - \left[q_n - W \frac{\partial q_n}{\partial r} + U \frac{\partial q_n}{\partial x} \right]_R, \end{aligned} \quad (2.8)$$

where t is time, ε_1 and ε_2 are the damping coefficients, ρ_0 is the density of shell material, k_1 is the coefficient of subgrade reaction, q_x and q_n are stresses of the fluid inside the circular section, r and x are the cylindrical coordinates, V_r and V_x are velocity vector projections on the axis of the coordinate system, p is the pressure in the fluid, ρ is the fluid density, ν is the kinematic viscosity coefficient, $\bar{n}$ is the normal to the median shell surface, and $\bar{n}_r$, $\bar{n}_\Theta$, $\bar{i}$ are the basis vectors (r, Θ, x) of the cylindrical coordinate system with its center on the geometric axis.

Substituting (2.1), (2.2), (2.5) into (2.8), we find the equations of dynamics in displacements

$$\begin{aligned}
 & \frac{Eh_0}{1-\mu_0^2} \frac{\partial}{\partial x} \left\langle \frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \mu_0 \frac{W}{R} - \frac{m}{E} \frac{4}{3} \left\{ \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \right. \right. \right. \\
 & \left. \left. \left. - \mu_0 \frac{W}{R} \right] \left[\mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \mu_2 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) \frac{W}{R} \right] + \right. \right. \\
 & \left. \left. + \frac{h_0^2}{12} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left[3\mu_1 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) + (\mu_2 - \mu_1 \mu_0) \frac{W}{R} \right] \right\} \right\rangle = \\
 & = \rho_0 h_0 \frac{\partial^2 U}{\partial t^2} + \varepsilon_1 \frac{1}{l} \rho_0 h_0 \sqrt{\frac{E}{\rho_0 (1-\mu_0^2)}} \frac{\partial U}{\partial t} - \left[q_x - W \frac{\partial q_x}{\partial r} + U \frac{\partial q_x}{\partial x} \right]_R - \\
 & - \frac{Eh_0^3}{12(1-\mu_0^2)} \frac{\partial^2}{\partial x^2} \left\langle \frac{\partial^2 W}{\partial x^2} \left\{ 1 - \frac{m}{E} \frac{4}{3} \left[3\mu_1 \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right]^2 \right] + \right. \right. \\
 & \left. \left. + 2(\mu_2 - \mu_1 \mu_0) \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right] \frac{W}{R} + (\mu_1 - \mu_2 \mu_0) \left(\frac{W}{R} \right)^2 + 3 \frac{h_0^2}{20} \mu_1 \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \right\} \right\rangle + \\
 & + \frac{Eh_0}{1-\mu_0^2} \frac{\partial}{\partial x} \left\langle \frac{\partial W}{\partial x} \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \mu_0 \frac{W}{R} - \frac{m}{E} \frac{4}{3} \left\{ \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 - \right. \right. \right. \right. \\
 & \left. \left. \left. - \mu_0 \frac{W}{R} \right] \left[\mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \mu_2 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) \frac{W}{R} \right] + \right. \right. \\
 & \left. \left. + \frac{h_0^2}{12} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left[3\mu_1 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) + (\mu_2 - \mu_1 \mu_0) \frac{W}{R} \right] \right\} \right\rangle + \\
 & + \frac{Eh_0}{1-\mu_0^2} \frac{1}{R} \left\langle \mu_0 \left[\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right] - \frac{W}{R} - \frac{m}{E} \frac{4}{3} \left\{ \left[\mu_0 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) - \right. \right. \right. \right. \\
 & \left. \left. \left. - \frac{W}{R} \right] \left[\mu_1 \left[\left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right)^2 + \left(\frac{W}{R} \right)^2 \right] + \mu_2 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) \frac{W}{R} \right] + \right. \right. \\
 & \left. \left. + \frac{h_0^2}{12} \left(\frac{\partial^2 W}{\partial x^2} \right)^2 \left[3\mu_1 \mu_0 \left(\frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 \right) + (\mu_1 - \mu_2 \mu_0) \frac{W}{R} \right] \right\} \right\rangle = \\
 & = \rho_0 h_0 \frac{\partial^2 W}{\partial t^2} + \varepsilon_2 \frac{1}{R^2} \rho_0 h_0 \sqrt{\frac{E}{\rho_0 (1-\mu_0^2)}} \frac{\partial W}{\partial t} + \\
 & + k_1 \frac{h_0}{R^3} \rho_0 h_0 \frac{E}{\rho_0 (1-\mu_0^2)} W - \left[q_n - W \frac{\partial q_n}{\partial r} + U \frac{\partial q_n}{\partial x} \right]_R.
 \end{aligned} \tag{2.9}$$

3. The asymptotic method for studying the equations of shells with a fluid

For wave problems, the shell is assumed to be infinite. For longitudinal waves in the shell, dimensionless variables and parameters are introduced. We take the wavelength l for the characteristic length, and u_m, w_m are the characteristic values of the elastic displacements

$$W = w_m u_3, \quad U = u_m u_1, \quad x^* = \frac{x}{l}, \quad t^* = \frac{c_0}{l} t, \quad r^* = \frac{r}{R}. \quad (3.1)$$

$c_0 = \sqrt{\frac{E}{\rho(1-\mu_0)}}$ is the propagation velocity of longitudinal elastic waves in the shell. Set

$$\begin{aligned} \frac{h_0}{R} = \varepsilon \ll 1, \quad \frac{R^2}{l^2} = O(\varepsilon), \quad \frac{w_m}{h_0} = O(1), \quad \frac{\varepsilon_1}{\varepsilon} = O(1), \quad \frac{\varepsilon_2}{\varepsilon} = O(1), \\ \frac{u_m}{l} \frac{R}{h_0} = O(1), \quad \frac{m\varepsilon}{E} = O(1), \quad \frac{h_0^2}{l^2} = \frac{h_0^2}{R^2} \cdot \frac{R^2}{l^2} = \varepsilon^3, \end{aligned} \quad (3.2)$$

where ε is the small parameter of the problem. In these variables, Eqs. (2.9) take the form

$$\begin{aligned} & c_0^2 \rho_0 h_0 \frac{1}{l} \frac{\partial}{\partial x^*} \left\langle \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \mu_0 \frac{w_m}{R} u_3 - \right. \\ & - \frac{m}{E} \frac{4}{3} \left\{ \left[\left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \mu_0 \frac{w_m}{R} u_3 \right] \left[\mu_1 \left[\left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) + \left(\frac{w_m}{R} \right)^2 u_3^2 \right] + \mu_2 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) \frac{w_m}{R} u_3 \right] + \frac{h_0^2}{12} \frac{R^2}{l^4} \frac{w_m^2}{R^2} \left(\frac{\partial^2 u_3}{\partial x^{*2}} \right)^2 \left[3\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) + (\mu_2 - \mu_1 \mu_0) \frac{w_m}{R} u_3 \right] \right\} \right\rangle = \\ & = \frac{c_0^2 \rho_0 h_0}{l^2} u_m \frac{\partial^2 u_1}{\partial t^{*2}} + \varepsilon_1 \frac{c_0^2 \rho_0 h_0}{l^2} u_m \frac{\partial u_1}{\partial t^*} - \left[q_x - \frac{w_m}{R} u_3 \frac{\partial q_x}{\partial r^*} + \frac{u_m}{l} u_1 \frac{\partial q_x}{\partial x^*} \right]; \quad (3.3) \\ & - c_0^2 \rho_0 \frac{h_0}{l} \frac{h_0^2}{12} \frac{1}{l^2} \frac{\partial^2}{\partial x^{*2}} \left\langle \frac{R}{l} \frac{w_m}{R} \frac{\partial^2 u_3}{\partial x^{*2}} \left\{ 1 - \frac{m}{E} \frac{4}{3} \left[3\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) + 2(\mu_2 - \mu_1 \mu_0) \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) \frac{w_m}{R} u_3 + (\mu_1 - \mu_2 \mu_0) \left(\frac{w_m}{R} \right)^2 + \right. \right. \right. \\ & \left. \left. \left. + 3 \frac{h_0^2}{20} \frac{R^4}{l^4} \left(\frac{w_m}{R} \right)^2 \mu_1 \left(\frac{\partial^2 u_3}{\partial x^{*2}} \right)^2 \right] \right\} + c_0^2 \rho_0 h_0 \frac{1}{l} \frac{\partial}{\partial x^*} \left\langle \frac{R}{l} \frac{w_m}{R} \frac{\partial u_3}{\partial x^*} \left[\left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \mu_0 \frac{w_m}{R} u_3 - \frac{m}{E} \frac{4}{3} \left\{ \left[\left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \mu_0 \frac{w_m}{R} u_3 \right] \left[\mu_1 \left[\left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 + \left(\frac{w_m}{R} \right)^2 u_3^2 \Big] + \mu_2 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \\
& + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right) \frac{w_m}{R} u_3 \Big] + \frac{h_0^2}{12} \frac{R^2}{l^4} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial^2 u_3}{\partial x^{*2}} \right)^2 \left[3\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \\
& \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right) \right) (\mu_2 - \mu_1 \mu_0) \frac{w_m}{R} u_3^2 \right] \Big] \Big\} + \\
& + c_0^2 \rho_0 h_0 \frac{1}{R} \left\langle \mu_0 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \frac{w_m}{R} u_3 - \right. \\
& - \frac{m}{E} \frac{4}{3} \left\{ \left[\mu_0 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - \frac{w_m}{R} u_3 \right] \left[\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \right. \\
& \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) + \left(\frac{w_m}{R} \right)^2 u_3^2 \right] + \mu_2 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \\
& \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) \frac{w_m}{R} u_3 \right] + \frac{h_0^2}{12} \frac{R^2}{l^4} \frac{w_m^2}{R^2} \left(\frac{\partial^2 u_3}{\partial x^{*2}} \right)^2 \left[3\mu_1 \mu_0 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial x^*} + \right. \right. \\
& \left. \left. + \frac{1}{2} \frac{R^2}{l^2} \left(\frac{w_m}{R} \right)^2 \left(\frac{\partial u_3}{\partial x^*} \right)^2 \right) - (\mu_1 - \mu_2 \mu_0) \frac{w_m}{R} u_3 \right] \Big\} \Big\rangle = \\
& = \rho_0 h_0 \frac{c_0^2}{l^2} w_m \frac{\partial^2 u_3}{\partial t^{*2}} + \varepsilon_2 \rho_0 h_0 \frac{c_0^2}{R^2} w_m \frac{\partial u_3}{\partial t^*} + k_1 \frac{h_0}{R} \rho_0 h_0 \frac{c_0^2}{R^2} w_m u_3 - \\
& - \left[q_n - \frac{w_m}{R} u_3 \frac{\partial q_n}{\partial r^*} + \frac{u_m}{l} u_1 \frac{\partial q_n}{\partial x^*} \right].
\end{aligned}$$

We introduce independent variables in the form

$$\xi = x^* - ct^*, \quad \tau = \varepsilon t^*, \quad (3.4)$$

where τ is fast time and c is the dimensionless unknown volume. In these variables (3.4), leaving the terms of orders ε and ε^2 in Eqs. (3.3) and discarding terms with higher degrees, we obtain equations [5]

$$\begin{aligned}
& \frac{\partial}{\partial \xi} \left\langle \frac{u_m}{l} \frac{\partial u_1}{\partial \xi} - \mu_0 \frac{w_m}{R} u_3 - \frac{m}{E} \frac{4}{3} \left\{ \left(\frac{u_m}{l} \frac{\partial u_1}{\partial \xi} - \mu_0 \frac{w_m}{R} u_3 \right) \left[\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial \xi} \right)^2 + \right. \right. \right. \\
& \left. \left. + \left(\frac{w_m}{R} \right)^2 u_3^2 \right] + \mu_2 \frac{u_m}{l} \frac{\partial u_1}{\partial \xi} \frac{w_m}{R} u_3 \right] \right\} \Big\rangle = \frac{u_m}{l} \left[c^2 \frac{\partial^2 u_1}{\partial \xi^2} - 2\varepsilon c \frac{\partial^2 u_1}{\partial \xi \partial \tau} \right] - \\
& - \frac{u_m}{l} \varepsilon_1 c \frac{\partial u_1}{\partial \xi} - \frac{l}{\rho_0 h_0 c_0^2} q_x; \\
& \mu_0 \frac{u_m}{l} \frac{\partial u_1}{\partial \xi} - \frac{w_m}{R} u_3 - \frac{m}{E} \frac{4}{3} \left(\mu_0 \frac{u_m}{l} \frac{\partial u_1}{\partial \xi} - \frac{w_m}{R} u_3 \right) \left[\mu_1 \left(\frac{u_m}{l} \frac{\partial u_1}{\partial \xi} \right)^2 + \right. \\
& \left. + \left(\frac{w_m}{R} \right)^2 u_3^2 \right] + \mu_2 \frac{u_m}{l} \frac{\partial u_1}{\partial \xi} \frac{w_m}{R} u_3 \Big] = \frac{R^2}{l^2} \frac{w_m}{R} \left[c^2 \frac{\partial^2 u_3}{\partial \xi^2} - 2\varepsilon c \frac{\partial^2 u_3}{\partial \xi \partial \tau} \right] - \\
& - \frac{R^2}{l^2} \frac{w_m}{R} \varepsilon_2 \frac{l^2}{R^2} c \frac{\partial u_3}{\partial \xi} + \frac{w_m}{R} \frac{h_0}{R} k_1 u_3 - \frac{R}{\rho_0 h_0 c_0^2} q_n.
\end{aligned} \quad (3.5)$$

We represent the dependent variables in the form of an asymptotic expansion.

$$u_1 = u_{10} + \varepsilon u_{11} + \dots, \quad u_3 = u_{30} + \varepsilon u_{31} + \dots \quad (3.6)$$

We obtain a system of equations by substituting (4.1) into (3.5) and leaving the terms of order ε .

$$\begin{aligned} \frac{\partial}{\partial \xi} \left\langle \frac{\partial u_{10}}{\partial \xi} - \mu_0 \frac{w_m l}{u_m R} u_{30} \right\rangle &= c^2 \frac{\partial^2 u_{10}}{\partial \xi^2}, \\ \mu_0 \frac{\partial u_{10}}{\partial \xi} - \frac{w_m l}{u_m R} u_{30} &= 0. \end{aligned} \quad (3.7)$$

From this system we get

$$\frac{w_m l}{u_m R} u_{30} = \mu_0 \frac{\partial u_{10}}{\partial \xi}, \quad c^2 = 1 - \mu_0^2. \quad (3.8)$$

Thus, u_{10} is an arbitrary function, and the dimensionless volume $c = (1 - \mu_0^2)^{1/2}$, therefore, the wave velocity equals $\sqrt{\frac{E}{\rho_0}}$, that is, the velocity of the wave in the rod. Since the shell is of infinite length,

$$\xi = \frac{1}{l} \left(x - \sqrt{\frac{E}{\rho_0}} t \right).$$

We get a system of equations in approximation of order ε^2

$$\begin{aligned} &\frac{\partial}{\partial \xi} \left\langle \frac{\partial u_{11}}{\partial \xi} - \mu_0 \frac{w_m l}{u_m R} u_{31} - \frac{m}{E} \frac{4}{3} \left[\mu_1 \left(\frac{\partial u_{10}}{\partial \xi} \right)^3 + \right. \right. \\ &\quad \left. \left. + (\mu_1 - \mu_2 \mu_0) \frac{\partial u_{10}}{\partial \xi} \left(\frac{w_m l}{u_m R} u_{30} \right)^2 + (\mu_2 - \mu_1 \mu_0) \frac{w_m l}{u_m R} u_{30} \left(\frac{\partial u_{10}}{\partial \xi} \right)^2 - \right. \right. \\ &\quad \left. \left. - \mu_1 \mu_0 \left(\frac{w_m l}{u_m R} \right)^3 u_{30}^3 \right] \right\rangle = -2c \frac{\partial^2 u_{10}}{\partial \xi \partial \tau} + c^2 \frac{\partial^2 u_{11}}{\partial \xi^2} - \frac{\varepsilon_1}{\varepsilon} c \frac{\partial u_{10}}{\partial \xi} - \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} q_x; \\ &\mu_0 \frac{\partial u_{11}}{\partial \xi} - \frac{w_m l}{u_m R} u_{31} - \frac{m}{E} \frac{4}{3} \left(\frac{u_m}{l} \right)^2 \left(\mu_0 \frac{\partial u_{10}}{\partial \xi} + \frac{w_m l}{u_m R} u_{30} \right) \left[\mu_0 \left[\left(\frac{\partial u_{10}}{\partial \xi} \right)^3 + \right. \right. \\ &\quad \left. \left. + \left(\frac{w_m l}{u_m R} \right)^2 u_{30}^2 \right] + \mu_2 \frac{w_m l}{u_m R} \frac{\partial u_{10}}{\partial \xi} u_{30} \right] = \\ &= \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{w_m l}{u_m R} c^2 \frac{\partial^2 u_{30}}{\partial \xi^2} - \frac{\varepsilon_2}{\varepsilon} \frac{w_m}{R} \frac{l}{u_m} c \frac{\partial u_{30}}{\partial \xi} + k_1 \frac{1}{\varepsilon} \frac{h_0}{R} \frac{w_m l}{u_m R} u_{30} - \frac{Rl}{\varepsilon u_m \rho_0 h_0 c_0^2} q_n. \end{aligned} \quad (3.9)$$

By substituting relation (4.3) into Eqs. (5.1), we obtain the system

$$\begin{aligned} &\mu_0^2 \frac{\partial^2 u_{11}}{\partial \xi^2} - \mu_0 \frac{w_m l}{u_m R} \frac{\partial u_{31}}{\partial \xi} = \\ &= \frac{m}{E} \frac{4}{3} \left(\frac{u_m}{l} \right)^2 (1 - \mu_0^2) (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) 3 \left(\frac{\partial u_{10}}{\partial \xi} \right)^2 \frac{\partial^2 u_{10}}{\partial \xi^2} - \\ &\quad - 2\sqrt{1 - \mu_0^2} \frac{\partial^2 u_{10}}{\partial \xi \partial \tau} - \frac{\varepsilon_2}{\varepsilon} \sqrt{1 - \mu_0^2} \frac{\partial u_{10}}{\partial \xi} - \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} q_x; \end{aligned} \quad (3.10)$$

$$\begin{aligned} \mu_0^2 \frac{\partial u_{11}}{\partial \xi} - \frac{w_m l}{u_m R} u_{31} &= \frac{1}{\varepsilon} \frac{R^2}{l^2} (1 - \mu_0^2) \mu_0 \frac{\partial^3 u_{10}}{\partial^3 \xi} - \frac{\varepsilon_2}{\varepsilon} \sqrt{1 - \mu_0^2} \mu_0 \frac{\partial^2 u_{10}}{\partial^2 \xi} + \\ &+ k_1 \frac{1}{\varepsilon} \frac{h_0}{R} \mu_0 \frac{\partial u_{10}}{\partial \xi} - \frac{Rl}{\varepsilon u_m \rho_0 h_0 c_0^2} q_n. \end{aligned}$$

By multiplying both sides of the second equation by μ_0 and differentiating with respect to ξ , we get the following equation:

$$\begin{aligned} \mu_0^2 \frac{\partial^2 u_{11}}{\partial^2 \xi} - \mu_0 \frac{w_m l}{u_m R} \frac{\partial u_{31}}{\partial \xi} &= \frac{1}{\varepsilon} \frac{R^2}{l^2} \mu_0^2 (1 - \mu_0^2) \frac{\partial^4 u_{10}}{\partial^4 \xi} - \frac{\varepsilon_2}{\varepsilon} \sqrt{1 - \mu_0^2} \mu_0 \frac{\partial^3 u_{10}}{\partial^3 \xi} + \\ &+ k_1 \frac{1}{\varepsilon} \frac{h_0}{R} \mu_0^2 \frac{\partial^2 u_{10}}{\partial^2 \xi} - \mu_0 \frac{Rl}{\varepsilon u_m \rho_0 h_0 c_0^2} \frac{\partial q_n}{\partial \xi}. \end{aligned} \quad (3.11)$$

The left-hand side of Eqs. (3.10) and (3.11) coincide. Subtracting, term by term, the first equation of system (3.10) from Eq. (3.11), we obtain the resolving equation

$$\begin{aligned} 2\sqrt{1 - \mu_0^2} \frac{\partial^2 u_{10}}{\partial \xi \partial \tau} - 4 \frac{m}{E\varepsilon} \left(\frac{u_m}{l} \right)^2 (1 - \mu_0^2) (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{\partial u_{10}}{\partial \xi} \right)^2 \frac{\partial^2 u_{10}}{\partial^2 \xi} + \\ + \frac{1}{\varepsilon} \frac{R^2}{l^2} \mu_0^2 (1 - \mu_0^2) \frac{\partial^4 u_{10}}{\partial^4 \xi} - \frac{\varepsilon_2}{\varepsilon} \sqrt{1 - \mu_0^2} \mu_0^2 \frac{\partial^3 u_{10}}{\partial^3 \xi} + k_1 \frac{1}{\varepsilon} \frac{h_0}{R} \mu_0^2 \frac{\partial^2 u_{10}}{\partial^2 \xi} + \\ + \frac{\varepsilon_1}{\varepsilon} \sqrt{1 - \mu_0^2} \frac{\partial u_{10}}{\partial \xi} = - \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} \left[q_x - \mu_0 \frac{R}{l} \frac{\partial q_n}{\partial \xi} \right]. \end{aligned} \quad (3.12)$$

We divide both sides of the equation by $2\sqrt{1 - \mu_0^2}$ and get

$$\begin{aligned} \frac{\partial^2 u_{10}}{\partial \xi \partial \tau} - 2 \frac{m}{E\varepsilon} \left(\frac{u_m}{l} \right)^2 \sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{\partial u_{10}}{\partial \xi} \right)^2 \frac{\partial^2 u_{10}}{\partial^2 \xi} + \\ + \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{1}{2} \mu_0^2 \sqrt{1 - \mu_0^2} \frac{\partial^4 u_{10}}{\partial^4 \xi} - \frac{\varepsilon_2}{\varepsilon} \frac{\mu_0^2}{2} \frac{\partial^3 u_{10}}{\partial^3 \xi} + k_1 \frac{\mu_0^2}{2\sqrt{1 - \mu_0^2}} \frac{1}{\varepsilon} \frac{h_0}{R} \mu_0^2 \frac{\partial^2 u_{10}}{\partial^2 \xi} + \\ + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon} \frac{\partial u_{10}}{\partial \xi} \frac{\mu_0^2}{2} = - \frac{1}{2\sqrt{1 - \mu_0^2}} \frac{l^2}{\varepsilon u_m \rho_0 h_0 c_0^2} \left[q_x - \mu_0 \frac{R}{l} \frac{\partial q_n}{\partial \xi} \right]. \end{aligned} \quad (3.13)$$

The left-hand side of the resulting equation corresponds to a modified Korteweg – de Vries – Burgers equation for $\frac{\partial u_{10}}{\partial \xi}$. In the absence of a fluid, the right-hand side of the equation is zero, and then a modified Korteweg – de Vries – Burgers equation is obtained. It is necessary to determine the right-hand side by solving the equations of hydrodynamics.

4. The fluid flow stress acting on the shell surface

The stresses of the fluid layer are defined by the formulas [6]

$$\begin{aligned} q_n &= P_{rr} \cos \left(\widehat{\vec{n}, \vec{n}_r} \right) + P_{rx} \cos \left(\widehat{\vec{n}, \vec{i}} \right); \quad q_x = - \left[P_{rx} \cos \left(\widehat{\vec{n}, \vec{n}_r} \right) + P_{xx} \cos \left(\widehat{\vec{n}, \vec{i}} \right) \right]; \\ P_{rr} &= -p + 2\rho\nu \frac{\partial V_r}{\partial r}; \quad P_{rx} = \rho\nu \left(\frac{\partial V_x}{\partial r} + \frac{\partial V_r}{\partial x} \right); \quad P_{xx} = -p + 2\rho\nu \frac{\partial V_x}{\partial x}. \end{aligned} \quad (4.1)$$

If we take down the stress on the unperturbed surface of the shell, we can assume that $\bar{n} = \bar{n}_r$ and $\cos(\widehat{\bar{n}, \bar{n}_r}) = 1$, $\cos(\widehat{\bar{n}, \bar{i}}) = 0$.

The equation of motion of a viscous incompressible fluid and the equation of continuity in a cylindrical coordinate system (r, Θ, x) in the case of axisymmetric flow are written in the form of [6]

$$\begin{aligned} \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_x \frac{\partial V_r}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial r} &= \nu \left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial x^2} - \frac{V_r}{r^2} \right), \\ \frac{\partial V_x}{\partial t} + V_r \frac{\partial V_x}{\partial r} + V_x \frac{\partial V_x}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= \nu \left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r} \frac{\partial V_x}{\partial r} + \frac{\partial^2 V_x}{\partial x^2} \right), \\ \frac{\partial V_r}{\partial t} + \frac{V_r}{r} + \frac{\partial V_x}{\partial x} &= 0. \end{aligned} \quad (4.2)$$

The fluid adhesion conditions in Lagrange's approach are satisfied at the boundary with the shell.

$$\frac{\partial U}{\partial t} = V_x + U \frac{\partial V_x}{\partial x} - W \frac{\partial V_x}{\partial r}; \quad -\frac{\partial W}{\partial t} = V_r + U \frac{\partial V_r}{\partial x} - W \frac{\partial V_r}{\partial r}. \quad (4.3)$$

We introduce dimensionless variables and parameters

$$\begin{aligned} V_r &= w_m \frac{c_0}{l} v_r; \quad V_x = w_m \frac{c_0}{R_1} v_x; \quad r^* = \frac{r}{R_1}; \quad t^* = \frac{c_0}{l} t; \quad x^* = \frac{x}{l}; \\ p &= \frac{\rho \nu c_0 l w_m}{R_1^3} P + p_0; \quad \frac{R_1}{l} = \psi = O(\varepsilon^{\frac{1}{2}}); \quad \lambda = \frac{w_m}{R_1} = O(\varepsilon); \quad Re = \psi \frac{R_1 c_0}{\nu}. \end{aligned} \quad (4.4)$$

By substituting (4.4) into Eq. (4.2) and the boundary condition (4.3), we obtain the equations and boundary conditions for the dimensionless components of the fluid velocity and pressure. By decomposing pressure and velocity components in powers of a small parameter λ

$$P = P^0 + \lambda P^1 + \dots, \quad v_x = v_x^0 + \lambda v_x^1 + \dots, \quad v_r = v_r^0 + \lambda v_r^1 + \dots, \quad (4.5)$$

for the expansion first terms, we get the equations

$$\begin{aligned} \frac{\partial P^0}{\partial r^*} &= 0; \quad Re \frac{\partial v_x^0}{\partial t^*} + \frac{\partial P^0}{\partial x^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial v_x^0}{\partial r^*} \right); \\ \frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* v_r^0) + \frac{\partial v_x^0}{\partial x^*} &= 0 \end{aligned} \quad (4.6)$$

and boundary conditions of the form

$$\begin{aligned} v_r^0 &= -\frac{\partial u_3}{\partial t^*}; \quad v_x^0 = \frac{u_m R_1}{w_m l} \frac{\partial u_1}{\partial t^*} \quad \text{if } r^* = 1; \\ r^* \frac{\partial v_r^0}{\partial r^*} &= 0; \quad r^* \frac{\partial v_x^0}{\partial r^*} = 0 \quad \text{if } r^* = 0. \end{aligned} \quad (4.7)$$

Now we define the stresses of the fluid on the shell in these variables. With an accuracy of order λ, ψ we have

$$q_x = -\lambda \frac{\nu}{R_1 c_0} \rho c_0^2 \frac{\partial v_x}{\partial r^*} \Big|_{r^*=1}; \quad q_n = -p_0 - \frac{\lambda}{\psi} \frac{\nu}{R_1 c_0} \rho_0 c_0^2 P. \quad (4.8)$$

By assuming the harmonic dependence of pressure, fluid velocity and elastic displacements of the shell on time, we find the exact solution of Eq. (4.6)

$$\begin{aligned}
 P = \int \left\{ 2Re \left[(\alpha - 1) \frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \alpha \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right] + \right. \\
 \left. + 16\gamma \left[\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \int \frac{\partial u_3}{\partial \tau} d\xi \right] \right\} d\xi; \\
 \frac{\partial v_x}{\partial r^*} \Big|_{r^*=1} = Re(\alpha - 1) \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right) + 8\gamma \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial u_1}{\partial \tau} - \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right),
 \end{aligned} \quad (4.9)$$

where

$$\begin{aligned}
 \alpha - 1 = \frac{2\sqrt{Re} q_0 - 4s_0}{Re p_0 - 4\sqrt{Re} q_0 + 4s_0}; \quad \alpha = \frac{Re p_0 - 2\sqrt{Re} q_0}{Re p_0 - 4\sqrt{Re} q_0 + 4s_0}, \\
 16\gamma = \frac{4\sqrt{Re} r_0}{Re p_0 - 4\sqrt{Re} q_0 + 4s_0}.
 \end{aligned}$$

Here

$$\begin{aligned}
 p_0 = ber^2 \sqrt{Re} + bei^2 \sqrt{Re}, \quad q_0 = ber \sqrt{Re} + bei' \sqrt{Re} - ber' \sqrt{Re} bei \sqrt{Re}, \\
 r^0 = ber \sqrt{Re} bei' \sqrt{Re} + bei \sqrt{Re} bei' \sqrt{Re}, \quad s_0 = ber'^2 \sqrt{Re} + bei'^2 \sqrt{Re}.
 \end{aligned}$$

ber and *bei* are zero-order Kelvin functions, and the prime denotes a derivative.

Note that

$$\begin{aligned}
 \alpha = \frac{4}{3}, \quad \gamma = 1 \quad \text{when} \quad Re \rightarrow 0, \\
 \alpha = 1, \quad \alpha - 1 = \sqrt{\frac{2}{Re}}, \quad \gamma = 2\frac{\sqrt{Re}}{8} \quad \text{when} \quad Re > 20.
 \end{aligned}$$

Giving up the assumption of the flow parameters harmony in time in the problem under consideration, we apply the iteration method to the problem (4.6). At the first iteration step, we discard the local inertia term ($Re \ll 1$) and obtain

$$\begin{aligned}
 P = 16 \int \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial u_1}{\partial \tau} - \int \frac{\partial u_3}{\partial \tau} d\xi \right) d\xi, \\
 v_x = \frac{1}{4} (r^{*2} - 1) 16 \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial u_1}{\partial \tau} - \int \frac{\partial u_3}{\partial \tau} d\xi \right) + \frac{u_m \psi}{w_m} \frac{\partial u_1}{\partial \tau}.
 \end{aligned} \quad (4.10)$$

In the second iteration step, by substituting the found solution into the local inertia term, we get

$$\begin{aligned}
 P = \int \left\{ 2Re \left[\frac{1}{6} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \frac{4}{3} \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right] + 16 \left[\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \int \frac{\partial u_3}{\partial \tau} d\xi \right] \right\} d\xi; \\
 \frac{\partial v_x}{\partial r^*} \Big|_{r^*=1} = Re \frac{1}{3} \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial^2 u_1}{\partial \tau^2} - \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right) + 8 \left(\frac{1}{2} \frac{u_m \psi}{w_m} \frac{\partial u_1}{\partial \tau} - \int \frac{\partial^2 u_3}{\partial \tau^2} d\xi \right).
 \end{aligned} \quad (4.11)$$

Formulas (4.11) coincide with the exact solution (4.9) for the harmonic laws of environmental parameters changes with $Re \ll 1$, which allows using these formulas for both nonharmonic parameters changes and for nonlinear equations of shell dynamics.

It should be noted that the convergence of the iteration method was proved in [6]. Taking into account that variables (3.4) are introduced, and using the relation $c = \sqrt{1 - \mu_0^2}$, we obtain, with an accuracy of order ε ,

$$P = \sqrt{1 - \mu_0^2} \left\{ 8 \left(2 \int u_{30} d\xi - \frac{u_m R_1}{w_m l} u_{10} \right) - \frac{1}{3} Re \left(8u_{30} - \frac{u_m R_1}{w_m l} u_{10} \frac{\partial u_{10}}{\partial \xi} \right) \sqrt{1 - \mu_0^2} \right\}, \quad (4.12)$$

where

$$\left. \frac{\partial v_x}{\partial r^*} \right|_{r^*=1} = \sqrt{1 - \mu_0^2} \left\{ 4 \left(u_{30} - \frac{u_m R_1}{w_m l} \frac{\partial u_{10}}{\partial \xi} \right) - \frac{1}{6} Re \left(2 \frac{\partial u_{30}}{\partial \xi} - \frac{u_m R_1}{w_m l} \frac{\partial^2 u_{10}}{\partial \xi^2} \right) \sqrt{1 - \mu_0^2} \right\}. \quad (4.13)$$

As $\frac{u_m l}{w_m R_1} u_{30} = \mu_0 \frac{\partial u_{10}}{\partial \xi}$, we get

$$q_x - \mu_0 \frac{R}{l} \frac{\partial q_n}{\partial \xi} = \frac{\nu}{R_1 c_0} \rho c_0^2 4 \sqrt{1 - \mu_0^2} \frac{u_m}{l} \left[1 - 2\mu_0 \frac{R}{R_1} \right]^2 \frac{\partial u_{10}}{\partial \xi} - \frac{R_1}{l} \rho c_0^2 \frac{1}{6} (1 - \mu_0^2) \frac{u_m}{l} \left[\left(1 - 2\mu_0 \frac{R}{R_1} \right)^2 + 3 \left(2\mu_0 \frac{R}{R_1} \right)^2 \frac{\partial^2 u_{10}}{\partial \xi^2} \right]. \quad (4.14)$$

Therefore, we have the equation

$$\begin{aligned} & \frac{\partial^2 u_{10}}{\partial \xi \partial \tau} + \frac{1}{\varepsilon} \frac{R^2}{l^2} \frac{\mu_0^2 \sqrt{1 - \mu_0^2}}{2} \frac{\partial^4 u_{10}}{\partial \xi^4} - \\ & - 2 \frac{m}{E \varepsilon} \sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{u_m}{l} \right)^2 \left(\frac{\partial u_{10}}{\partial \xi} \right)^2 \frac{\partial^2 u_{10}}{\partial \xi^2} + \\ & + k_1 \frac{1}{\varepsilon} \frac{h_0}{R} \frac{\mu_0^2}{2 \sqrt{1 - \mu_0^2}} \frac{\partial^2 u_{10}}{\partial \xi^2} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon} \frac{\partial u_{10}}{\partial \xi} - \frac{1}{2} \frac{\varepsilon_2}{\varepsilon} \frac{\partial^3 u_{10}}{\partial \xi^3} \mu_0^2 = \\ & = - \frac{1}{2} \frac{l}{\varepsilon \rho_0 h_0} \left\{ \frac{\nu}{R_1 c_0} \rho^4 \left[1 - 2\mu_0 \frac{R}{R_1} \right]^2 \frac{\partial u_{10}}{\partial \xi} \right. \\ & \left. - \frac{R_1}{l} \rho \frac{1}{6} \sqrt{1 - \mu_0^2} \left[\left(1 - 2\mu_0 \frac{R}{R_1} \right)^2 + 3 \left(2\mu_0 \frac{R}{R_1} \right)^2 \right]^2 \frac{\partial^2 u_{10}}{\partial \xi^2} \right\}. \end{aligned} \quad (4.15)$$

The resulting equation generalizes the modified Korteweg–de Vries–Burgers equation for $\frac{\partial u_{10}}{\partial \xi}$.

With an accuracy of order ε (4.15), we can put $R_1 = R$.

Assuming $\frac{\partial u_{10}}{\partial \xi} = \phi$, $\eta = c_1 \xi$, $t = c_2 \tau$, we obtain the generalized modified Korteweg–de Vries–Burgers (MKdV–B) equation

$$\frac{\partial \phi}{\partial t} + (\sigma_2 - \sigma_5 - 6\phi^2) \frac{\partial \phi}{\partial \eta} + \frac{\partial^3 \phi}{\partial \eta^3} - \sigma_3 \frac{\partial^2 \phi}{\partial \eta^2} + (\sigma_4 + \sigma_1) \phi = 0, \quad (4.16)$$

where

$$\begin{aligned} c_1 &= \left[\frac{2}{3} \frac{m}{E} \left(\frac{u_m}{l} \right)^2 \frac{l^2}{R^2} \frac{(\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2)}{\mu_0^2} \right]^{\frac{1}{2}}, \\ c_2 &= c_1 \frac{1}{3} \frac{m}{E \varepsilon} \sqrt{1 - \mu_0^2} (\mu_1 + \mu_2 \mu_0 + \mu_1 \mu_0^2) \left(\frac{u_m}{l} \right)^2, \end{aligned} \quad (4.17)$$

we assume that

$$\begin{aligned} \sigma_1 &= \frac{2}{c_2} (1 - 2\mu_0)^2 \frac{\rho l}{\varepsilon \rho_0 h_0} \frac{\nu}{R_1 c_0}, \quad \sigma_2 = \frac{c_1}{c_2} \frac{\mu_0^2}{\sqrt{1 - \mu_0^2}} \frac{k_1}{2} \frac{1}{\varepsilon} \frac{h_0}{R}, \\ \sigma_3 &= \frac{c_1^2}{c_2} \frac{1}{2} \frac{\varepsilon_2}{\varepsilon} \mu_0^2, \quad \sigma_4 = \frac{1}{c_2} \frac{1}{2} \frac{\varepsilon_1}{\varepsilon}, \\ \sigma_5 &= \frac{c_1}{c_2} \frac{1}{12} \sqrt{1 - \mu_0^2} \frac{\rho l}{\rho_0 h_0 \varepsilon} \frac{R_1}{l} \left[(1 - 2\mu_0)^2 + 3(2\mu_0)^2 \right]. \end{aligned} \quad (4.18)$$

Provided that $(\sigma_4 + \sigma_1) = 0$, in the absence of longitudinal structural damping ($\sigma_4 = 0$) and $\mu_0 = \frac{1}{2}$ for the incompressible material ($\sigma_1 = 0$), the MKdV–B equation

$$\frac{\partial \phi}{\partial t} + (\sigma_2 - \sigma_5 - 6\phi^2) \frac{\partial \phi}{\partial \eta} + \frac{\partial^3 \phi}{\partial \eta^3} - \sigma_3 \frac{\partial^2 \phi}{\partial \eta^2} = 0 \quad (4.19)$$

has an exact solution in the form of a kink-antikink

$$\phi = \pm \frac{1}{6} \sigma_3 \pm k \tanh \left\{ k \left[\eta + \left(2k^2 - (\sigma_2 - \sigma_5) + \frac{1}{6} \sigma_3^2 \right) t \right] \right\}. \quad (4.20)$$

The phase velocity is

$$\frac{\omega}{k} = -2k^2 - \frac{1}{6} \sigma_3^2 - \sigma_5 + \sigma_2.$$

The wave velocity is

$$C = \sqrt{\frac{E}{\rho_0}} \left(1 - \varepsilon \frac{c_2 (2k^2 + \frac{1}{6} \sigma_3^2 + \sigma_5 - \sigma_2)}{c_1 \sqrt{1 - \mu_0^2}} \right).$$

If the numerator of the fraction $2k^2 + \frac{1}{6} \sigma_3^2 - (\sigma_2 - \sigma_5) > 0$, $\left((\sigma_2 - \sigma_5) < 2k^2 + \frac{1}{6} \sigma_3^2 \right)$, then the speed is subsonic. If the numerator of the fraction $2k^2 + \frac{1}{6} \sigma_3^2 - (\sigma_2 - \sigma_5) < 0$, $\left((\sigma_2 - \sigma_5) > 2k^2 + \frac{1}{6} \sigma_3^2 \right)$, then the speed is supersonic. The effect of subgrade reaction σ_2 increases the wave velocity. At the same time, the fluid motion inertia of σ_5 reduces the wave speed. Thus, they act in different directions.

The constructing damping in the longitudinal direction $\sigma_4 > 0$ and the influence of a viscous fluid $\sigma_1 > 0$ affect the amplitude of the wave. Thus, they act in one direction. This effect is investigated by numerically solving the generalized modified Korteweg–de Vries–Burgers equation for $(\sigma_4 + \sigma_1) > 0$.

The wave number k in (4.20) is arbitrary.

5. Computing experiment

For a numerical study of the wave motion model of a physically nonlinear elastic shell with structural damping (energy dissipation) interacting with the surrounding elastic medium under the influence of the fluid, we write Eq. (4.16) in an integral form

$$\oint_{\Omega} \phi d\eta - \left[(\sigma_2 - \sigma_5) \phi - 2\phi^3 - \sigma_3 \frac{\partial \phi}{\partial \eta} + \frac{\partial^2 \phi}{\partial \eta^2} \right] dt + \iint_{\Omega} (\sigma_4 + \sigma_1) \phi dt d\eta = 0 \quad (5.1)$$

for any area Ω . To transfer a discrete formulation, let us compare $u_j^n = \phi(t_n, \eta_j)$ and select the basic contour shown in Figure 1.

Fig. 1. Basic contour for equation (5.1)

Let us add the integral relations

$$\begin{aligned} \int_{m_j}^{m_{j+1}} u_{\eta} d\eta &= u(t, \eta_{j+1}) - u(t, \eta_j), \\ \int_{m_j}^{m_{j+1}} u_{\eta\eta} d\eta &= u_{\eta}(t, \eta_{j+1}) - u_{\eta}(t, \eta_j). \end{aligned} \quad (5.2)$$

We use a trapezoid formula for integration in time and even derivatives in η and an average value formula for odd derivatives in η . We assume $(t_{n+1} - t_n) = \tau$, $(\eta_{j+1} - \eta_j) = h$. We substitute all these expressions into formulas (5.1), (5.2) and, using the Gröbner basis method, we obtain the following difference scheme for Eq. (4.16) similar to the Crank–Nicolson scheme for the heat equation.

$$\begin{aligned} & \frac{u_j^{n+1} - u_j^n}{\tau} + (\sigma_2 - \sigma_5) \frac{(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (u_{j+1}^n - u_{j-1}^n)}{4h} - \\ & - 2 \frac{(u_{j+1}^{3n+1} - u_{j-1}^{3n+1}) + (u_{j+1}^{3n} - u_{j-1}^{3n})}{4h} + \\ & + \frac{(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + (u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n)}{4h^3} - \\ & - \sigma_3 \frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{4h^2} + (\sigma_4 + \sigma_1) \frac{u_j^{n+1} - u_j^n}{2} = 0 \end{aligned} \quad (5.3)$$

with the initial condition in the form of an exact solution (4.20) at $t = 0$, choosing the plus sign, $k = \frac{1}{6}$.

6. The results and conclusions

In the absence of environmental influences, structural damping in the longitudinal, normal directions and the influence of the fluid, the velocity and amplitude of the wave do not change. The movement occurs in the negative direction (Fig. 2). This means that the speed of movement is subsonic. The result of the computational experiment coincides with the exact solution, therefore, the difference scheme and the KdV–B equation are adequate. The dimensionless deformation ϕ , the dimensionless phase variable η , and dimensionless fast time τ are presented in Figs. 2–6.

In the absence of fluid influence, the influence of the surrounding elastic medium (σ_2) leads to an increase in the wave velocity, up to a supersonic one. The presence of damping in the normal direction (σ_3) changes the amplitude by constant volume (leads to the structure of the stretch shock wave $\phi > 0$) and reduces the velocity of the wave. The presence of structural damping in the longitudinal direction (σ_4) leads to a drop in the amplitude of the wave (Fig. 3).

Fig. 2. The absence of environmental influence ($\sigma_2 = 0$), structural damping in both the longitudinal ($\sigma_4 = 0$) and normal ($\sigma_3 = 0$) direction as well as the absence of fluid influence ($\sigma_1 = 0, \sigma_5 = 0$).

Fig. 3. The absence of fluid effect ($\sigma_1 = 0, \sigma_5 = 0$). $\sigma_2 = 1, \sigma_3 = 1, \sigma_4 = 1$.

Fig. 4. The absence of environmental influence ($\sigma_2 = 0$), structural damping both in the longitudinal ($\sigma_4 = 0$) and normal ($\sigma_3 = 0$) direction. $\sigma_1 = 1$, $\sigma_5 = 1$.

Fig. 5. All coefficients $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are nonzero and equal to 1, the stress from the fluid side is greater than the environmental influence ($\sigma_2 < \sigma_5$).

Fig. 6. All coefficients $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are nonzero, the stress from the fluid side is less than the environmental influence ($\sigma_2 > \sigma_5$).

The effect of viscous fluid stress on the shell (σ_1) leads to a drop in the wave amplitude, and the inertia of fluid motion (σ_5) leads to a decrease in the wave velocity (Fig. 4).

The presence of the medium (σ_2) leads to an increase in the wave velocity, while the construction damping in the longitudinal direction (σ_4) leads to a decrease in the wave amplitude. If the inertia of fluid motion (σ_5) is greater than the influence of the medium ($\sigma_2 < \sigma_5$), the wave velocity decreases. The influence of fluid stress leads to a greater decrease in the wave amplitude (Fig. 5).

If the inertia of fluid motion (σ_5) is less than the influence of the medium ($\sigma_5 < \sigma_2$), the increase in the wave velocity due to the presence of the medium (σ_2) reduces (Fig. 6).

In the absence of the fluid influence, the influence of the elastic medium leads to an increase in the velocity of wave motion up to a supersonic one. The presence of construction damping in the normal direction changes the wave amplitude by constant volume (leads to the structure of the stretch shock wave) and reduces the velocity of wave motion. The presence of construction damping in the longitudinal direction reduces the wave amplitude.

The influence of the inertia of fluid movement reduces the deformation wave velocity, while fluid viscous stress on the shell reduces the wave amplitude.

The authors are grateful to the anonymous referees for a careful checking of the details and for helpful comments and suggestions that have improved this paper.

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