MATHEMATICAL PROBLEMS OF NONLINEARITY

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Asymptotic Behavior of Solutions of a System of KdV Type Associated with the Schrödinger Operator with an Energy-Dependent Potential

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In this paper, we study the asymptotic behavior of the solutions of the Cauchy problem for a nonlinear KdV type system associated with the Schrödinger spectral operator with an energy-dependent potential. Using the set of motion integrals for this system, we determine the amplitude of the asymptotic solution in terms of spectral data for the initial condition of the Cauchy problem.

Keywords: nonlinear KdV type system, asymptotic behavior, energy-dependent potential, motion integrals

1. Introduction

The aim of this work is to study the asymptotic behavior of the solutions of the Cauchy problem for a system of KdV type

$$\begin{aligned} -v_t &= -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}uv_x, & -u_t &= \frac{3}{2}uu_x + v_x, \\ u|_{t=0} &= u_0, & v|_{t=0} &= v_0. \end{aligned} \quad (1.1)$$

Here u_0 and v_0 are smooth, quickly decaying functions. As was established in [1] and [2], this system has a Lax pair

$$L = \frac{d^2}{dx^2} - ku - v + k^2, \quad A = k \frac{d}{dx} + \frac{u}{2} \frac{d}{dx} - \frac{1}{4}u_x.$$

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Thus, the system (1.1) is equivalent to the operator equality

$$L_t = [L, A].$$

To study a nonlinear system, it is necessary to study the properties of solutions of the spectral equation with an energy-dependent potential

$$-\psi''_{xx}(x, k, t) + (ku(x, t) + v(x, t))\psi(x, k, t) = k^2\psi(x, k, t), \quad x \in R. \quad (1.2)$$

We will assume that for the initial data of the Cauchy problem of the system (1.1) the spectral equation (1.2) does not have a discrete spectrum. Thus, for asymptotics of the solutions of this system at large times, there will be no soliton terms. In this case, the asymptotic behavior of the solutions of the system has the following form:

$$v(x, t) \simeq \frac{1}{\sqrt{t}} \left(w(z)e^{i\Phi} + \overline{w(z)}e^{-i\Phi} \right), \quad u(x, t) \simeq \frac{2}{\sqrt{t}z} \left(w(z)e^{i\Phi} + \overline{w(z)}e^{-i\Phi} \right),$$

$$z = \frac{x}{t}, \quad \Phi = \frac{z^2}{2}t + B(z) \ln t.$$

In order to find the modulus of the function $w(z)$, we apply the method proposed by [9]. This method involves the use of trace formulas for the operator L and integrals of motion for a nonlinear system. The left side of the trace formula corresponds to the integral of motion for an integrable system. We calculate the value of the integrals of motion in terms of the function $w(z)$ and compare it with the right side of the trace formula, which is given in terms of the data of the inverse problem. This calculation gives an explicit formula for $|w(z)|$ in terms of spectral data corresponding to the initial data of the Cauchy problem of a nonlinear system.

Highlights of spectral theory for the operator (1.2) were studied also in [7] and [8]. In [3], [4] and [7] the authors used the Gel'fand–Levitan–Marchenko approach for the inverse problem on the line. In [8] authors used the method of the Riemann–Hilbert problem. Trace formulas for Eq. (1.2) were found in [6].

Section 2 is devoted to the presentation of standard facts from scattering theory for Eq. (1.2). In Section 3 we recall the construction of trace formulas. In Section 4 we write a formal solution to a nonlinear system. And finally, in Section 5, we calculate the modulus of the function $w(z)$.

2. Properties of the data of the inverse problem

In this section we recall some standard facts, cf [8]. We begin with defining the Jost solution of Eq. (1.2). Let us consider its two solutions satisfying the properties

$$f(x, k) = e^{ikx}(1 + o(1)), \quad x \rightarrow +\infty, \quad (2.1)$$

$$g(x, k) = e^{-ikx}(1 + o(1)), \quad x \rightarrow -\infty. \quad (2.2)$$

We also introduce

$$\alpha(x) = \exp \left\{ i \int_x^\infty \tilde{u}(s) ds \right\}, \quad \tilde{u} = \frac{u}{2}, \quad \alpha_0 = \lim_{x \rightarrow -\infty} \alpha(x). \quad (2.3)$$

Since u is a real function, $\alpha_0^{-1} = \overline{\alpha_0}$.

For a real-valued k we have two pairs of linear independent solutions:

$$\{f(x, k), \overline{f(x, k)}\}, \quad \{g(x, k), \overline{g(x, k)}\}.$$

In particular, the functions f and $\overline{f}$ could be written as linear combinations of g and $\overline{g}$

$$f(x, k) = a(k)\overline{g(x, k)} + b(k)g(x, k), \quad (2.4)$$

$$\overline{f(x, k)} = \overline{a(k)}g(x, k) + \overline{b(k)}\overline{g(x, k)}. \quad (2.5)$$

Let us define the Wronskian

$$W[\varphi, \psi](x, k) = \varphi'_x(x, k)\psi(x, k) - \varphi(x, k)\psi'_x(x, k). \quad (2.6)$$

It is easy to see that for solutions of the spectral equation it does not depend on x . Hence, we get the standard formula

$$|a(k)|^2 - |b(k)|^2 = 1. \quad (2.7)$$

It follows from (1.2) that the function f satisfies the following Volterra equation:

$$f(x, k) = e^{ikx} - \int_x^\infty \frac{\sin k(x-y)}{k} (ku(x) + v(x)) f(y, k) dy. \quad (2.8)$$

For the Volterra equation, we can write a convergent iterative series and prove that the function $f(x, k)$ admits analytic continuation to the upper half-plane with respect to the spectral parameter. This easily yields analyticity in the upper half-plane of the function $a(k)$.

Lemma 1. *For $\operatorname{Im} k > 0$ the function $f(x, k)$ satisfies the following asymptotic relations:*

$$f(x, k) = e^{ikx} \exp \left\{ i \int_x^\infty \tilde{u}(s) ds \right\} (1 + o(1)), \quad k \rightarrow \infty. \quad (2.9)$$

Similarly, we find asymptotic properties of the functions $a(k)$ and $b(k)$.

Lemma 2. *The scattering coefficients a and b satisfy the asymptotic behaviour as $k \rightarrow \infty$*

$$\lim_{k \rightarrow \infty} a(k) = \alpha_0^{-1}, \quad \operatorname{Im} k \geq 0,$$

$$\lim_{k \rightarrow \infty} b(k) = 0, \quad \operatorname{Im} k = 0.$$

Lemma 3. *Zeros of the scattering coefficient $a(k)$ in $\mathbf{C}_+$ coincide with the eigenvalues of the operator (1.2).*

The proof of this fact is completely analogous to the proof arising for the standard Schrödinger operator.

Lemma 4. *If the potentials $u(x, t)$ and $v(x, t)$ in the operator (1.2) depend on time and satisfy the nonlinear system (1.1), then the corresponding function $a(k, t)$ does not depend on time.*

A proof of this fact can be found in [8].

3. Trace formulas

Let us rewrite Eq. (1.2)

$$-f''_{xx}(x, k) + (ku(x) + v(x))f(x, k) = k^2 f(x, k)$$

in terms of the function $h(x)$

$$f = \exp \left(ikx - i \int_{-\infty}^x \tilde{u}(t) dt + \int_{-\infty}^x h(t) dt \right), \quad \tilde{u} = \frac{u}{2}.$$

Then

$$\tilde{u}^2 - h^2 - 2ikh + 2i\tilde{u}h + v + i\tilde{u}' - h' = 0. \quad (3.1)$$

Due to asymptotic properties of the function $f(x, k)$ the function $h(x, k)$ has the following asymptotic expansion as $k \rightarrow \infty$:

$$h(x) = \sum_{j=1}^{\infty} \frac{h_j}{k^j}.$$

Substituting it into Eq. (3.1) gives

$$\tilde{u}^2 + v - 2ih_1 + i\tilde{u}' = 0, \quad (3.2)$$

$$-\sum_{p=1}^{l-1} h_p h_{l-p} - 2ih_{l+1} + 2i\tilde{u}h_l - h'_l = 0, \quad l > 0. \quad (3.3)$$

In particular, we obtain

$$\begin{aligned} h_1 &= \frac{1}{2i} (\tilde{u}^2 + v + i\tilde{u}'), \\ h_2 &= \frac{1}{2i} (\tilde{u}^3 + v\tilde{u}) + \tilde{u}\tilde{u}' + \frac{1}{4} (v' + i\tilde{u}''), \\ h_3 &= \frac{1}{2i} (2i\tilde{u}h_2 - h_1^2 - h'_2) = \\ &= \frac{1}{2i} \left(\tilde{u}^4 + v\tilde{u}^2 + \frac{1}{4} (\tilde{u}^2 + v)^2 - h'_2 \right) + \frac{1}{4} (v\tilde{u})' + \frac{5}{12} (\tilde{u}^3)' - \frac{1}{8i} (2\tilde{u}\tilde{u}'' + (\tilde{u}')^2). \end{aligned}$$

On the other hand, due to analytical properties of the function $a(k)$ it follows that

$$a(k) = \prod_1^n \frac{k - k_l}{k - \bar{k}_l} \exp \left(i \int_{-\infty}^{\infty} \tilde{u}(x) dx + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(s)|^2}{s - k} ds \right).$$

Let us consider the asymptotic expansion of $a(k)$ as $k \rightarrow \infty$

$$\ln a(k) = i \int_{-\infty}^{\infty} \tilde{u}(x) dx + \sum_1^{\infty} \frac{a_j}{k^j}.$$

A simple calculation shows that

$$a_j = \frac{-2i}{j} \sum_{l=1}^n \operatorname{Im}(k_l^j) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s^{j-1} ds. \quad (3.4)$$

Note that for $\text{Im}k > 0$

$$\int_{-\infty}^{\infty} h(x) dx = i \int_{-\infty}^{\infty} \tilde{u}(x) dx - \ln a(k). \quad (3.5)$$

From (3.4) and (3.5) it follows that

$$\int_{-\infty}^{\infty} h_j(x) dx = \frac{2i}{j} \sum_{l=1}^n \text{Im}(k_l^j) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s^{j-1} ds. \quad (3.6)$$

These relations are trace formulas for Eq. (1.2). On the other hand, for solutions of the nonlinear system (1.1), the right-hand side of the trace formula is time-independent. Thus, the left side of the trace formula gives the integrals of motion for the nonlinear system (1.1).

In particular, the first three trace formulas have the following form:

$$\int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^2 + v) dx = 2i \sum_{l=1}^n \text{Im}(k_l) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 ds, \quad (3.7)$$

$$\int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^3 + v\tilde{u}) dx = i \sum_{l=1}^n \text{Im}(k_l^2) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s ds, \quad (3.8)$$

$$\int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^4 + v\tilde{u}^2 + \frac{1}{4}((\tilde{u}^2 + v)^2 + (\tilde{u}')^2)) dx = \frac{2}{3}i \sum_{l=1}^n \text{Im}(k_l^3) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s^2 ds. \quad (3.9)$$

4. Formal solution of a KdF type system

For further constructions, we need a formal solution of the KdV type system.

Theorem 1. *A KDF type system has the following formal solution:*

$$\hat{v}(x, t) \simeq \frac{1}{\sqrt{t}} (w(z)e^{i\Phi} + \overline{w(z)}e^{-i\Phi}) + \sum_{p \geq 2, 0 \leq q \leq p, |m| \leq p} \frac{\ln^q t}{t^{p/2}} e^{im\Phi} v_{pqm}(z), \quad (4.1)$$

$$\hat{u}(x, t) \simeq \frac{2}{\sqrt{tz}} (w(z)e^{i\Phi} + \overline{w(z)}e^{-i\Phi}) + \sum_{p \geq 2, 0 \leq q \leq p, |m| \leq p} \frac{\ln^q t}{t^{p/2}} e^{im\Phi} u_{pqm}(z), \quad (4.2)$$

$$z = \frac{x}{t}, \quad \Phi = \frac{z^2}{2}t + B(z) \ln t.$$

All coefficients $v_{pqm}(z)$, $u_{pqm}(z)$ and $B(z)$ can be found in terms of $w(z)$.

To prove this theorem, it suffices to substitute the series (4.1) and (4.2) into the system (1.1) and equate the terms with different powers of $\frac{\ln^q t}{t^{p/2}} e^{im\Phi}$. The resulting system allows us to determine the coefficients $v_{pqm}(z)$, $u_{pqm}(z)$ and $B(z)$ in terms of $w(z)$.

The function $w(z)$ will be considered a smooth rapidly decaying function satisfying the condition for some $\delta > 0$

$$w(z) = 0, \quad |z| < \delta. \quad (4.3)$$

The last condition is necessary in order to exclude from further analysis the complex behavior of the asymptotics of the solution of a KdV type system near of the point $z = 0$.

5. The asymptotics of a solution of the KdV type system

To calculate the amplitude $|w(z)|$ of the asymptotic solution of the system (1.1), we use the method proposed in [9]. We substitute the series (4.1) and (4.2) in the left side of the trace formula (3.6). In this case, we obtain integrals of the form

$$\int_{-\infty}^{\infty} \alpha(z) e^{im\Phi} dx.$$

Using the stationary phase method, it is easy to see that for $m \neq 0$ the integral of the oscillating function will tend to zero as $t \rightarrow \infty$. So for $t \rightarrow \infty$, only nonoscillating terms remain in the integrals of the functions h_j .

Consider the asymptotic behavior of the integrals $\int_{-\infty}^{\infty} h_j(x) dx$ in the trace formula for $t \rightarrow \infty$. Given the formulas (3.7)–(3.9) for the first integrals of motion, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} h_1(x) dx &= \int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^2 + v) dx \simeq \int_{-\infty}^{\infty} \frac{-2i|w(z)|^2}{z^2} dz, \\ \int_{-\infty}^{\infty} h_2(x) dx &= \int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^3 + \tilde{u}v) dx \simeq \int_{-\infty}^{\infty} \frac{-i|w(z)|^2}{z} dz, \\ \int_{-\infty}^{\infty} h_3(x) dx &= \int_{-\infty}^{\infty} \frac{1}{2i} (\tilde{u}^4 + v\tilde{u}^2 + \frac{1}{4}((\tilde{u}^2 + v)^2 + (\tilde{u}')^2)) dx \simeq \int_{-\infty}^{\infty} \frac{-i|w(z)|^2}{2} dz. \end{aligned}$$

Let us study arbitrary l . Using induction on l , for a recurrence relation we obtain

$$h_j = O(t^{-1/2}).$$

The integrals of the order function $O\left(\frac{1}{t^{3/2}}\right)$ will tend to zero at large times, regardless of whether there is an oscillation there. Let us consider the $h_j^0(x)$ part of the functions $h_j(x)$ linear in u and v . By induction, we obtain

$$h_l^0(x) = -\frac{1}{(-2i)^l} \left(v^{(l-1)} + i\tilde{u}^{(l)} \right) \approx \frac{1}{\sqrt{t}} \overline{2w(z)} (-iz)^{l-1} e^{-i\Phi}.$$

Therefore, for large times in the recurrence relation (3.3)

$$h_{l+1} = \frac{1}{2i} \left(-\sum_{p=1}^{l-1} h_p h_{l-p} + 2i\tilde{u} h_l - h_l' \right)$$

we can replace the functions $h_j(x)$ with their linear part $h_j^0(x)$. In addition, the contribution from the sum in the recurrence relation to the integral of motion will be zero, since there will be no nonoscillating terms in the sum. As a result, we arrive at the following answers for the limits of the integrals of motion at large times:

$$\int_{-\infty}^{\infty} h_l(x) dx \simeq \int_{-\infty}^{\infty} \tilde{u} h_l^0 dx \simeq \int_{-\infty}^{\infty} \frac{-iz^{l-3} |w(z)|^2}{2^{l-2}} dz.$$

On the other hand, in the absence of a discrete spectrum, from the trace formula we have

$$\int_{-\infty}^{\infty} h_l(x) dx = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s^{l-1} ds.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{-iz^{l-3}|w(z)|^2}{2^{l-2}} dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln |a(s)|^2 s^{l-1} ds.$$

This relation is valid for any $l > 0$. Comparing the two expressions on the left and right sides of this equality, we arrive at the final formula for the modulus $|w(z)|$

$$s = \frac{z}{2}, \quad |w(z)|^2 = \frac{z^2}{32\pi} \ln |a(z/2)|^2.$$

REMARK 1. As often happens in constructions of this type, we calculate the asymptotic limits of the integrals of motion using the asymptotic ansatz (4.1). But at the same time, in terms of the function $w(z)$, we obtain the exact values of the integrals of motion, since they are time-independent.

REMARK 2. Now it is clear that, to satisfy condition (4.3), it is enough to require that for the initial given Cauchy problem (1.1) the coefficient $a(k)$ corresponding to it satisfy the condition

$$a(k) = 1, \quad |k| < \delta/2.$$

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