MATHEMATICAL PROBLEMS OF NONLINEARITY

MSC 2010: 37G35, 37G10

Lorenz- and Shilnikov-Shape Attractors in the Model of Two Coupled Parabola Maps

E. Kuryzhov, E. Karatetskaia, D. Mints

We consider the system of two coupled one-dimensional parabola maps. It is well known that the parabola map is the simplest map that can exhibit chaotic dynamics, chaos in this map appears through an infinite cascade of period-doubling bifurcations. For two coupled parabola maps we focus on studying attractors of two types: those which resemble the well-known discrete Lorenz-like attractors and those which are similar to the discrete Shilnikov attractors. We describe and illustrate the scenarios of occurrence of chaotic attractors of both types.

Keywords: strange attractor, discrete Lorenz attractor, hyperchaos, discrete Shilnikov attractor, two-dimensional endomorphism

Received April 19, 2021

Accepted May 21, 2021

This paper was supported by the RSF grant 17-11-01041. Numerical results presented in Section 3 were obtained with the assistance of the Laboratory of Dynamical Systems and Applications NRU HSE, of the Ministry of Science and Higher Education of the RF grant No. 075-15-2019-1931. E. Karatetskaia acknowledges the Russian Foundation for Basic Research, grant No. 19-02-00610 for the support of scientific research.

Evgenii Kuryzhov

eugenkuryzhov@yandex.ru

Efrosiniia Karatetskaia

ekarateczkaya@hse.ru

Dmitrii Mints

dmitriyminc@mail.ru

National Research University Higher School of Economics

ul. Bolshaya Pecherskaya 25/12, Nizhny Novgorod, 603155 Russia

1. Introduction

This paper is devoted to the study of chaotic and hyperchaotic dynamics in a two-dimensional map which describes the dynamics of two coupled identical parabola maps:

$$\begin{cases} \bar{x} = 1 - ax^2 + \varepsilon(x - y), \\ \bar{y} = 1 - ay^2 + \varepsilon(y - x). \end{cases} \quad (1)$$

In this model a and ε are parameters, parameter ε specifies the strength of coupling between two one-dimensional maps; $x, y \in \mathbb{R}$ are variables. Note that map (1) is an endomorphism of a plane.

Some bifurcations of map (1) were studied in [1]. We show that in this model one can observe chaotic attractors whose form and scenario of occurrence resemble the well-known attractors of three-dimensional diffeomorphisms and which can be divided into two types. We call the first type, similar to a discrete Lorenz-like attractor, a *Lorenz-shape attractor*, and the second type, similar to a discrete Shilnikov attractor, a *Shilnikov-shape attractor*.

Discrete Lorenz-like attractors in the case of three-dimensional diffeomorphisms were introduced and studied in [2]. They can appear as a result of quite simple bifurcation scenarios [3–5]. Therefore, they are encountered in various models from applications, for instance, in the non-holonomic model of a Celtic stone [6, 7], in models of diffusion convection [8, 9], etc.

In the first part of the present study we show that in map (1) Lorenz-shape attractors can emerge. Besides, we investigate bifurcation scenarios, which lead to their formation. We pay attention to the fact that these scenarios are in some sense similar to the corresponding scenarios proposed and described for the Lorenz-like attractors of three-dimensional diffeomorphisms [3, 4].

We point out that the Lorenz-shape attractors of two-dimensional endomorphisms can appear near a codimension-two point corresponding to a certain type of fold-flip bifurcation (when the values of multipliers of a fixed point are equal to 1 and -1). In [10], four types of this bifurcation are presented. According to [7] (see also [11]), the presence of fold-flip bifurcation of one of these types in a three-dimensional diffeomorphism can lead to a discrete Lorenz attractor. We show that a similar situation occurs also in the 2D endomorphism (1), i.e., the same type of fold-flip bifurcation leads to a Lorenz-shape attractor in its neighborhood.

In the second part of the paper, we study Shilnikov-shape attractors and discuss their hyperchaoticity. In [12], Shilnikov proposed a universal bifurcation scenario leading to the birth of a spiral attractor containing a saddle-focus equilibrium together with its two-dimensional unstable manifold for one-parameter families of three-dimensional flow systems. In [3, 4] this scenario was extended to the case of one-parameter families of three-dimensional diffeomorphisms. Recall that a discrete Shilnikov attractor was introduced in these works as a homoclinic attractor that contains a saddle-focus fixed (periodic) point with its two-dimensional unstable manifold and homoclinic orbits. We show that, in the case of two-dimensional endomorphisms, Shilnikov-type attractors can also be observed. We note that they contain a fixed (periodic) point, which is an unstable focus.

We show that Shilnikov-shape attractors in two-dimensional endomorphisms can be hyperchaotic [13]. We recall that the hyperchaotic behavior of trajectories is characterized by the presence of at least two directions of hyperbolic instability in an attractor. Such a phenomenon was observed for the first time by Rössler [14] in a four-dimensional dynamical system. In [15], the emergence of hyperchaotic attractors in the system of two weakly coupled oscillators was explained by a transition to chaos via a cascade of period-doubling bifurcations in each oscillator. The scenarios of transition from a stable fixed point to hyperchaos in three-dimensional maps, as well as in four-dimensional flows, were proposed and studied in [13, 16–18]. We show that similar scenarios can lead to hyperchaos in the two-dimensional endomorphism (1).

2. Main bifurcations

The model of two coupled parabolas (1) is symmetric with respect to the following involution:

$$S: \begin{cases} x \rightarrow y, \\ y \rightarrow x. \end{cases} \quad (2)$$

This transformation has a line of fixed points:

$$Fix(S): y = x.$$

Accordingly, here we will call a periodic orbit *symmetric* if one of its components belongs to the line $Fix(S)$. Some bifurcations of map (1) were studied in [1]. This model has four fixed points: two of them are symmetric:

$$\begin{aligned} O(x_1, y_1): \quad x_1 = y_1 &= \frac{-1 + \sqrt{1 + 4a}}{2a}, \\ S(x_2, y_2): \quad x_2 = y_2 &= \frac{-1 - \sqrt{1 + 4a}}{2a}. \end{aligned}$$

We note that only the point O can be stable, its bifurcations can lead to the emergence of other stable regimes, including chaotic and even hyperchaotic ones.

Some bifurcation curves can be found analytically. It is important to note that two types of period-doubling bifurcation with symmetric periodic points (e.g., of period 2^{i-1}) are possible in the system (1). The first one occurs when the bifurcating point belongs to $Fix(S)$, while the point of doubled period, which is born after this bifurcation, does not belong to $Fix(S)$. We denote this bifurcation as PD_i^N . The second type occurs when both initial and doubled points belong to $Fix(S)$. We denote this bifurcation as PD_i^S .

Due to the symmetry S , map (1) admits pitchfork bifurcations. For the fixed point O the subcritical pitchfork bifurcation occurs on a curve:

$$PF_1: \varepsilon = \frac{\sqrt{1 + 4a}}{2}. \quad (3)$$

The saddle-node bifurcation with the fixed point O occurs on the line:

$$SN_1: a = -\frac{1}{4}. \quad (4)$$

The period-doubling bifurcation with the fixed point O can be of two types. After the first type of this bifurcation a newly born stable period-2 orbit is symmetric. We denote the region of its existence as 2^S . After the second type of period-doubling bifurcation the newly born period-2 orbit is asymmetric. We denote its region of stability as 2^N , see Fig. 1. These bifurcations occur on the curves:

$$PD_1^S: a = \frac{3}{4}, \quad PD_1^N: \varepsilon = \frac{\sqrt{1 + 4a} - 2}{2}, \quad (5)$$

respectively. The region of stability of the fixed point O is bounded by the curves (3), (4), (5) on the parameter plane (a, ε) . It is interesting to note that equations for bifurcation curves bounding the region 2^S can be also found analytically. The upper and bottom boundaries are formed by the subcritical bifurcation curve:

$$PF_2: a = \frac{4\varepsilon^2 - 4\varepsilon + 3}{4}.$$

Fig. 1. Bifurcation curves bounding regions of stability 1^S of the fixed point O and 2^S of the period-2 point. PD_i^S — period-doubling bifurcation of a period- 2^{i-1} point, after which a stable point of doubled period 2^i belongs to $Fix(S)$; PD_i^N — period-doubling bifurcation of a period- 2^{i-1} point, after which a stable point of doubled period 2^i does not belong to $Fix(S)$; PF_i — pitchfork bifurcation with a period- 2^{i-1} stable point; SN_1 — saddle-node bifurcation with the fixed point O ; FF — fold-flip bifurcation.

The right boundary of the 2^S -region is formed by the period-doubling bifurcation curves:

$$PD_2^S: a = \frac{5}{4}, \quad PD_2^N: a = \frac{4\epsilon^2 - 4\epsilon + 5}{4}.$$

To the right of the line PD_2^S the stable period-4 point is symmetric, while to the right of the curve PD_2^N it is asymmetric.

3. Fold-flip bifurcation and the Lorenz-shape attractor

3.1. Fold-flip bifurcation

The point FF in the (a, ϵ) -bifurcation diagram (Fig. 1) corresponds to the codimension-two bifurcation fold-flip, when the fixed point O has a pair of multipliers $(-1, 1)$. In [10] it was shown that, depending on coefficients of the corresponding normal form, this bifurcation can be of one of four types. One type of this bifurcation, as was shown in [7], can lead to the appearance of a discrete Lorenz attractor in three-dimensional diffeomorphisms. Let us study bifurcations in the neighborhood of the FF -point in map (1), where this bifurcation occurs when $(a, \epsilon) = (-0.25, -1)$.

Fig. 2. Bifurcation diagram in the neighborhood of the codimension-two bifurcation fold-flip of map (1). The red curve corresponds to the saddle-node bifurcation, the blue curve to a period-doubling bifurcation, the black curve to an invariant curve bifurcation, and the dashed region indicates a series of heteroclinic bifurcations.

The saddle-node bifurcation occurs along the vertical line SN : $a = -0.25$, to the left of it, in the region ①, the system does not contain any fixed points. The fold-flip point divides the curve SN into two segments SN^- and SN^+ . When passing to the region ② through SN^+ , a completely unstable point and a saddle fixed point are born. Otherwise, when passing to ⑥ through SN^- , a stable point and a saddle fixed point occur. The period-doubling bifurcation curve PD also consists of two segments PD^- and PD^+ . When passing from ② to ③ through PD^+ , the completely unstable fixed point becomes a saddle, and a period-2 completely unstable point is born in its neighborhood. The transition between the regions ③ and ④ is accompanied by a series of heteroclinic bifurcations, the wedge-shaped area, where they occur, is marked in the diagram as J . The boundaries of this area are curves of heteroclinic tangencies, between which transversal heteroclinic structures exist. Here, after bifurcations associated with heteroclinic tangencies, stable invariant curves are born. With the transition from J to region ⑤ these curves collide with the period-2 unstable point and then disappear, leaving these points stable. When passing from region ⑤ to ⑥ the stable period-2 point merges with a saddle fixed point, and the fixed point gets stability. Finally, the diagram is closed with the transition from ⑥ to ① when the saddle and stable fixed points merge and disappear through the saddle-node bifurcation.

As was shown in [2], the discrete Lorenz attractor of 3D diffeomorphisms can be born as a result of the codimension-three bifurcation, when the fixed point has a triplet of multipliers $(-1, -1, 1)$. Despite this fact, the fold-flip bifurcation, whose unfolding is the same as in Fig. 2, gives important necessary conditions supporting its appearance near region J quite far from this

point [7]. Further, we illustrate a similar phenomenon near the same fold-flip bifurcation of the two-dimensional endomorphism (1). Namely, we show that the Lorenz-shape attractor can appear as a result of the simple bifurcation scenario.

3.2. Lorenz-shape attractors

Figure 3 shows a Lyapunov diagram superimposed with some bifurcation curves near the fold-flip bifurcation of map (1). For this diagram we use the following color code: blue and red — periodic and quasi-periodic regimes, yellow and gray — chaotic and hyperchaotic attractors, see the legend, inserted in the top left corner. Let us fix $a = 0.05$ and consider a bifurcation scenario leading from the stable fixed point O to the chaotic attractor along a pathway AB : $\varepsilon \in (-0.4, -0.781)$. Some phase portraits along this pathway are shown in Fig. 4.

Fig. 3. The chart of Lyapunov exponents superimposed with bifurcation curves. The blue curve PD_1^N corresponds to the period-doubling bifurcation of the stable fixed point O , after which a period-2 stable asymmetric orbit (p_1, p_2) is born; the black curve IC_2 corresponds to a bifurcation of the birth of a 2-component stable invariant curve (S_1, S_2) ; on the curve $\sigma = 1$ the saddle value of O is equal to one; the point FF corresponds to a fold-flip bifurcation.

The symmetric stable fixed point O undergoes the supercritical period-doubling bifurcation on the curve PD_1^N . It becomes a saddle, and an asymmetric stable period-2 orbit (p_1, p_2) is born in its neighborhood (Fig. 4a). Note that the unstable separatrices W_1^U and W_2^U , forming together with O the unstable manifold of this point, tend to p_1 and p_2 , respectively. When

Fig. 4. Towards the birth of the Lorenz-shape attractor along the pathway AB .

crossing the curve IC_2 along this route, the orbit (p_1, p_2) undergoes a bifurcation of the birth of an invariant curve: it becomes completely unstable (an unstable focus) and a two-component stable curve (S_1, S_2) surrounds it below this curve (Fig. 4b). From this figure one can see that the unstable separatrix $W_1^U(W_2^U)$ still tends to the attractor formed from the point $p_1(p_2)$. As the parameter ε decreases further, this invariant curve loses its smoothness and soon becomes chaotic, see Fig. 4c. For a certain $\varepsilon \in (-0.7476, -0.75)$ the unstable separatrices return to the neighborhood of O , creating something similar to a homoclinic “butterfly bifurcation” (Fig. 4d). At this instant, a Lorenz-shape attractor emerges. Figures 4e and 4f show other examples of Lorenz-shape attractors along the pathway AB .

Recall that, by definition (see, e.g., [19]), discrete Lorenz-like attractors contain a saddle fixed (periodic) point whose saddle value is greater than one. Along the pathway AB considered above, the resulting attractor contains a fixed point whose saddle value is less than one. This attractor is geometrically similar to the Lorenz-like attractors of the three-dimensional maps, but the area-expanding condition is violated for it at least at the point O . However, in map (1) there exists a region with chaotic attractors where the saddle value of the fixed point O is greater than one. This region exists to the right of a curve $\sigma = 1$:

$$\varepsilon = -\frac{-1 - 4a + 2\sqrt{1 + 4a}}{2(-1 + \sqrt{1 + 4a})}.$$

Finally, in this section, we study bifurcations along the pathway CD : $a = 0.5$, $\varepsilon \in (-0.1, -0.55)$, where the resulting attractor contains an area-expanding saddle (with the saddle value greater than one). The first part of the bifurcation scenario is the same as that along the pathway AB . It leads to the two-component torus-chaos attractor presented in Figure 5a. As ε decreases further, a “homoclinic butterfly”-like bifurcation occurs. In this case, since $\sigma > 1$, a pair of saddle invariant curves together with a nontrivial hyperbolic set are born after this bifurcation. After it two components of the attractor merge to form a single attractor, and a Lorenz-shape attractor is born, see Fig. 5b for $\varepsilon = -0.5$ and Fig. 5c for $\varepsilon = -0.544$.

Fig. 5. Phase portraits of Lorenz-shape attractor in map (1) for $a = 0.5$ with $\sigma > 1$.

4. Shilnikov-shape attractor and hyperchaos

Map (1) consists of two one-dimensional identical parabola maps coupled via parameter ε . Thus, in the case of weak coupling (when $\varepsilon \ll 1$) the emergence of hyperchaotic attractors in this system can be explained by the transition to chaos via a cascade of period-doubling bifurcations in each parabola map. Indeed, when $\varepsilon = 0$, each one-dimensional parabola map with increasing parameter a demonstrates the classical transition from a stable fixed point to a chaotic attractor through the Feigenbaum cascade of period-doubling bifurcations. We have found out that in map (1) hyperchaotic regimes can also appear when the value of the coupling parameter ε is not very small. More precisely, we show that Shilnikov-shape attractors containing an unstable periodic point can lead to hyperchaos.

Recall that discrete Shilnikov attractors of three-dimensional diffeomorphisms are defined as homoclinic chaotic attractors containing a saddle-focus fixed (periodic) point with its two-dimensional unstable manifold and homoclinic points. In the case of two-dimensional endomorphisms we call a chaotic attractor a *Shilnikov-shape attractor* if its trajectories come arbitrarily close to a fixed (periodic) point which is an unstable focus.

In map (1) we observe a scenario of the emergence of a hyperchaotic Shilnikov-shape attractor, similar to scenarios described in [13, 16, 17]. Let us demonstrate it, moving along the pathway EF ($\varepsilon = 0.11$, $a \in [0.6, 1.32]$).

At the beginning of this pathway the stable fixed point O is a global attractor in map (1) (Fig. 6a). This point undergoes a period-doubling bifurcation, after which a stable point of period 2, belonging to $Fix(S)$, is born (Fig. 6b). Then another period-doubling bifurcation occurs, and the newly born stable point of period 4 does not lie on the line $Fix(S)$ (Fig. 6c). As the parameter a increases further, this stable orbit undergoes an invariant curve bifurcation. As a result, the period-4 point becomes completely unstable and a stable curve of period 4 is born in its neighborhood (Fig. 6d). Further, this curve starts losing smoothness (Fig. 6e), becomes resonant, and, as a result, a stable point of high period appears on it. In turn, this point undergoes one more bifurcation of the birth of an invariant curve (a similar scenario in the case of four-dimensional flows was studied in detail in [17, 20]), and a multicomponent curve is born (Figs. 6f and 6g). As the governing parameter a increases further, this curve breaks down and, as a result, a chaotic attractor emerges (Fig. 6h). Then, as a consequence of the absorption of the completely unstable period-4 point, a 4-component Shilnikov-shape attractor appears. After that, as the parameter a grows further, this attractor absorbs the set of completely unstable periodic points, which appear as a result of period-doubling bifurcations with periodic saddle points and invariant curve bifurcations with stable ones, and the Shilnikov-shape attractor becomes hyperchaotic (Fig. 6i).

Fig. 6. Evolution of attractors in map (1) along the pathway EF ($\varepsilon = 0.11$, $a \in [0.6, 1.32]$): from the stable fixed point O to a hyperchaotic four-component Shilnikov-shape attractor.

Finally, we would like to note that in this study we have not focused on some interesting aspects related to cascades of period-doubling bifurcations with periodic saddle points and bifurcations of the birth of an invariant curve. A more detailed analysis of these and other aspects (the transition from Shilnikov-shape attractors to Lorenz-shape attractors through absorption of symmetric saddle periodic points, as well as mechanisms of hyperchaos emergence, including the appearance of multicomponent hyperchaotic Shilnikov-shape attractors, in map (1)) will be carried out in future research.

The authors thank A. Kazakov for the problem statement and fruitful discussions.

References

- [1] Biragov, V. S., Ovsyannikov, I. M., and Turaev, D. V., A Study of One Endomorphism of a Plane, in *Methods of Qualitative Theory and Theory of Bifurcations*, E. A. Leontovich-Andronova (Ed.), Gorky: Gorky Gos. Univ., 1988, pp. 72–86 (Russian).
- [2] Gonchenko, S. V., Ovsyannikov, I. I., Simó, C., and Turaev, D., Three-Dimensional Hénon-Like Maps and Wild Lorenz-Like Attractors, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2005, vol. 15, no. 11, pp. 3493–3508.
- [3] Gonchenko, A. S., Gonchenko, S. V., and Shilnikov, L. P., Towards Scenarios of Chaos Appearance in Three-Dimensional Maps, *Nelin. Dinam.*, 2012, vol. 8, no. 1, pp. 3–28 (Russian).
- [4] Gonchenko, A. S., Gonchenko, S. V., Kazakov, A. O., and Turaev, D. V., Simple Scenarios of Onset of Chaos in Three-Dimensional Maps, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2014, vol. 24, no. 8, 1440005, 25 pp.
- [5] Gonchenko, A. S. and Gonchenko, S. V., Variety of Strange Pseudohyperbolic Attractors in Three-Dimensional Generalized Hénon Maps, *Phys. D*, 2016, vol. 337, pp. 43–57.
- [6] Gonchenko, A. S., Gonchenko, S. V., and Kazakov, A. O., Richness of Chaotic Dynamics in the Nonholonomic Model of Celtic Stone, *Regul. Chaotic Dyn.*, 2013, vol. 18, no. 5, pp. 521–538.
- [7] Gonchenko, A. S. and Samylna, E. A., On the Region of Existence of a Discrete Lorenz Attractor in the Nonholonomic Model of a Celtic Stone, *Radiophys. Quantum El.*, 2019, vol. 62, no. 5, pp. 369–384; see also: *Izv. Vyssh. Uchebn. Zaved. Radiofizika*, 2019, vol. 62, no. 5, pp. 412–428.
- [8] Eilertsen, J. S. and Magnan, J. F., On the Chaotic Dynamics Associated with the Center Manifold Equations of Double-Diffusive Convection near a Codimension-Four Bifurcation Point at Moderate Thermal Rayleigh Number, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2018, vol. 28, no. 8, 1850094, 24 pp.
- [9] Eilertsen, J. S. and Magnan, J. F., Asymptotically Exact Codimension-Four Dynamics and Bifurcations in Two-Dimensional Thermosolutal Convection at High Thermal Rayleigh Number: Chaos from a Quasi-Periodic Homoclinic Explosion and Quasi-Periodic Intermittency, *Phys. D*, 2018, vol. 382/383, pp. 1–21.
- [10] Kuznetsov, Yu. A., Meijer, H. G. E., and van Veen, L., The Fold-Flip Bifurcation, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 2004, vol. 14, no. 7, pp. 2253–2282.
- [11] Ovsyannikov, I., Global and Local Bifurcations, Three-Dimensional Hénon Maps and Discrete Lorenz Attractors, arXiv:2104.01262 (2021).
- [12] Shilnikov, L. P., Bifurcation Theory and Turbulence, *Methods of Qualitative Theory of Differential Equations*, E. A. Leontovich (Ed.), Gorky: Gorky Gos. Univ., 1986, pp. 150–165, 215 (Russian). See also: *Selecta Math. Soviet.*, 1991, vol. 10, no. 1, 43–53.
- [13] Garashchuk, I. R., Sinelshchikov, D. I., Kazakov, A. O., and Kudryashov, N. A., Hyperchaos and Multistability in the Model of Two Interacting Microbubble Contrast Agents, *Chaos*, 2019, vol. 29, no. 6, 063131, 16 pp.
- [14] Rössler, O. E., An Equation for Hyperchaos, *Phys. Lett. A*, 1979, vol. 71, nos. 2–3, pp. 155–157.
- [15] Harrison, M. A. and Lai, Y.-Ch., Route to High-Dimensional Chaos, *Phys. Rev. E*, 1999, vol. 59, no. 4, R3799–R3802.
- [16] Shykhmamedov, A., Karatetskaia, E., Kazakov, A., and Stankevich, N., Hyperchaotic Attractors of Three-Dimensional Maps and Scenarios of Their Appearance, arXiv:2012.05099 (2020).
- [17] Stankevich, N., Kazakov, A., and Gonchenko, S., Scenarios of Hyperchaos Occurrence in 4D Rössler System, *Chaos*, 2020, vol. 30, no. 12, 123129, 16 pp.
- [18] Karatetskaia, E., Shykhmamedov, A., and Kazakov, A., Shilnikov Attractors in Three-Dimensional Orientation-Reversing Maps, *Chaos*, 2021, vol. 31, no. 1, 011102, 11 pp.
- [19] Gonchenko, S., Gonchenko, A., Kazakov, A., and Samylna, E., On Discrete Lorenz-Like Attractors, *Chaos*, 2021, vol. 31, no. 2, 023117, 20 pp.
- [20] Stankevich, N., Kuznetsov, A., Popova, E., and Seleznev, E., Chaos and Hyperchaos via Secondary Neimark – Sacker Bifurcation in a Model of Radiophysical Generator, *Nonlinear Dynam.*, 2019, vol. 97, no. 4, pp. 2355–2370.