MATHEMATICAL PROBLEMS OF NONLINEARITY

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Optimal Bang-Bang Trajectories in Sub-Finsler Problem on the Cartan Group

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The Cartan group is the free nilpotent Lie group of step 3, with 2 generators. This paper studies the Cartan group endowed with the left-invariant sub-Finsler ℓ_∞ norm. We adopt the viewpoint of time-optimal control theory. By Pontryagin maximum principle, all sub-Finsler length minimizers belong to one of the following types: abnormal, bang-bang, singular, and mixed. Bang-bang controls are piecewise controls with values in the vertices of the set of control parameter.

In a previous work, it was shown that bang-bang trajectories have a finite number of patterns determined by values of the Casimir functions on the dual of the Cartan algebra. In this paper we consider, case by case, all patterns of bang-bang trajectories, and obtain detailed upper bounds on the number of switchings of optimal control.

For bang-bang trajectories with low values of the energy integral, we show optimality for arbitrarily large times.

The bang-bang trajectories with high values of the energy integral are studied via a second order necessary optimality condition due to A. Agrachev and R. Gamkrelidze. This optimality condition provides a quadratic form, whose sign-definiteness is related to optimality of bang-bang trajectories. For each pattern of these trajectories, we compute the maximum number of switchings of optimal control. We show that optimal bang-bang controls may have not more than 11 switchings. For particular patterns of bang-bang controls, we obtain better bounds. In such a way we improve the bounds obtained in previous works.

On the basis of results of this work we can start to study the cut time along bang-bang trajectories, i.e., the time when these trajectories lose their optimality. This question will be considered in subsequent works.

Keywords: sub-Finsler geometry, optimal control, switchings, bang-bang trajectories

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1. Introduction

Sub-Finsler geometry on Lie groups has received considerable attention during last years due to its applications, especially in geometric group theory and in harmonic analysis, see articles [4, 6, 10] and introductions of [16, 18] for a broad explanation of the reasons and for several references of the state-of-the-art. To our knowledge the term sub-Finsler appears for the first time in paper [11].

In the case of step two nilpotent Lie groups and homogeneous spaces there is a good understanding of sub-Finsler structures (Heisenberg group, flat Martinet case, Grushin plane) after work [16]. On the other hand, a detailed study of the left-invariant sub-Finsler structure on the free nilpotent Lie group of step 3 with 2 generators (called the Cartan group) began in works [18, 19]. This paper continues those works.

We adopt the viewpoint of time-optimal control theory. Pontryagin maximum principle [13] implies that sub-Finsler length minimizers are of one of the following types: abnormal, bang-bang, singular, or mixed (concatenations of finite number of bang and singular arcs). In this work we study optimality of bang-bang trajectories. There is a finite number of patterns of these trajectories described in [18, 19], and for each pattern we prove an upper bound on the number of switchings of bang-bang optimal control. The main tool here is a second order necessary optimality condition due to A. Agrachev and R. Gamkrelidze [15].

This work has the following structure. In Section 2 we recall the problem statement and some previously obtained results from [18, 19]. In Section 3 the second order optimality condition by Agrachev–Gamkrelidze [15] is stated. In Section 4 we prove the main results of this paper: we consider all patterns of bang-bang trajectories, and obtain upper bounds on the number of switchings of the optimal control. Results of Section 4 improve Theorem 6 [18] by giving detailed bounds on the number of switchings for all patterns of bang-bang optimal control. Finally, some concluding remarks are given in Section 5.

2. Problem statement and previous results

Consider the 5-dimensional free nilpotent Lie algebra with 2 generators, of step 3. There exists a basis $L = \text{span}(X_1, \dots, X_5)$ in which the product rule in L takes the form

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad \text{ad } X_4 = \text{ad } X_5 = 0.$$

The Lie algebra L is called the Cartan algebra, and the corresponding connected simply connected Lie group M is called the Cartan group. We will use the following model:

$$M = \mathbb{R}_{x,y,z,v,w}^5,$$

with the Lie algebra L modeled by left-invariant vector fields on $\mathbb{R}^5$

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{x^2 + y^2}{2} \frac{\partial}{\partial w}, \\ X_2 &= \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2 + y^2}{2} \frac{\partial}{\partial v}, \\ X_3 &= \frac{\partial}{\partial z} + x \frac{\partial}{\partial v} + y \frac{\partial}{\partial w}, \end{aligned}$$

$$X_4 = \frac{\partial}{\partial v},$$

$$X_5 = \frac{\partial}{\partial w}.$$

The product rule in the Cartan group M in this model is given in [12].

Left-invariant ℓ_∞ sub-Finsler problem on the Cartan group is stated as the following time-optimal problem:

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M, \quad u \in U = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \leq 1\}, \quad (2.1)$$

$$\|u\|_\infty = \max(|u_1|, |u_2|),$$

$$q(0) = q_0 = \text{Id} = (0, \dots, 0), \quad q(T) = q_1, \quad (2.2)$$

$$T \rightarrow \min. \quad (2.3)$$

Problem (2.1)–(2.3) was considered first in papers [18, 19]. We recall some results of those papers.

Existence of optimal controls follows from Rashevsky–Chow and Filippov theorem [13].

Pontryagin Maximum Principle implies that optimal abnormal controls are constant.

Introduce linear-on-fibers Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, $\lambda \in T^*M$, $i = 1, \dots, 5$. A normal extremal arc λ_t , $t \in I = (\alpha, \beta) \subset [0, T]$ is called:

- a bang-bang arc if

$$\text{card}\{t \in I \mid h_1 h_2(\lambda_t) = 0\} < \infty,$$

- a singular arc if one of the condition holds:

$$h_1(\lambda_t) \equiv 0, \quad t \in I \quad (h_1\text{-singular arc}), \text{ or}$$

$$h_2(\lambda_t) \equiv 0, \quad t \in I \quad (h_2\text{-singular arc}),$$

- a mixed arc if it consists of a finite number of bang-bang and singular arcs.

Singular controls have one of components constantly equal to 1 or -1 , thus they are optimal. The fix-time attainable set along singular trajectories was explicitly described and was shown to be semi-algebraic.

Bang-bang extremal trajectories satisfy the Hamiltonian system with the Hamiltonian function $H = |h_1| + |h_2|$:

$$\begin{cases} \dot{h}_1 = -s_2 h_3, \\ \dot{h}_2 = s_1 h_3, \\ \dot{h}_3 = s_1 h_4 + s_2 h_5, \\ \dot{h}_4 = \dot{h}_5 = 0, \\ \dot{q} = s_1 X_1 + s_2 X_2. \end{cases} \quad (2.4)$$

The dual of the Lie algebra $L^* = T_{\text{Id}}^* M$ has Casimir functions $h_4, h_5, E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4$, thus Hamiltonian system (2.4) has integrals h_4, h_5, E , and H .

The mapping $(\lambda, q) \mapsto (k\lambda, q)$, $k > 0$, preserves extremal trajectories, thus we can consider only the reduced case

$$H(\lambda) \equiv 1.$$

With the use of the coordinate $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$:

$$h_1 = \operatorname{sgn}(\cos \theta) \cos^2 \theta, \quad h_2 = \operatorname{sgn}(\sin \theta) \sin^2 \theta,$$

the vertical part of Hamiltonian system (2.4) reduces to the following system:

$$\begin{cases} \dot{\theta} = \frac{h_3}{|\sin 2\theta|}, & \theta \neq \frac{\pi n}{2}, \\ \dot{h}_3 = s_1 h_4 + s_2 h_5, & s_1 = \operatorname{sgn} \cos \theta, \quad s_2 = \operatorname{sgn} \sin \theta. \end{cases} \quad (2.5)$$

Consider the cylinder

$$C = T_{q_0}^* M \cap \{H = 1\}.$$

In work [18] it was shown that bang-bang trajectories can be represented as images of an exponential mapping: $\{q(t)\} = \operatorname{Exp}(\lambda, t)$, $\lambda \in C$, $t > 0$. The exponential mapping is single-valued for generic $\lambda \in C$, and is multi-valued for certain special subsets of C , see [18].

System (2.5) is preserved by the group of symmetries of the square $\{(h_1, h_2) \in \mathbb{R}^2 \mid |h_1| + |h_2| = 1\}$. Thus in the study of system (2.5) we can restrict ourselves by the case $h_4 \geq h_5 \geq 0$. This group of symmetries reduces the cylinder C to the fundamental domain of the group $\{\lambda \in C \mid h_4 \geq h_5 \geq 0\}$. Further, this fundamental domain admits a stratification by invariant subsets of the Hamiltonian system (2.5):

$$\begin{aligned} \{\lambda \in C \mid h_4 \geq h_5 \geq 0\} &= \cup_{i=1}^4 C^i, \\ C^1 &= \{\lambda \in C \mid h_4 > h_5 > 0\}, \\ C^2 &= \{\lambda \in C \mid h_4 > h_5 = 0\}, \\ C^3 &= \{\lambda \in C \mid h_4 = h_5 > 0\}, \\ C^4 &= \{\lambda \in C \mid h_4 = h_5 = 0\}. \end{aligned}$$

Further, we have the following stratifications:

$$\begin{aligned} C^1 &= \cup_{i=1}^8 C_i^1, \\ C_1^1 &= E^{-1}(-h_4), \quad C_2^1 = E^{-1}(-h_4, -h_5), \quad C_3^1 = E^{-1}(-h_5), \quad C_4^1 = E^{-1}(-h_5, h_5), \\ C_5^1 &= E^{-1}(h_5), \quad C_6^1 = E^{-1}(h_5, h_4), \quad C_7^1 = E^{-1}(h_4), \quad C_8^1 = E^{-1}(h_4, +\infty), \\ C^2 &= \cup_{i=1}^6 C_i^2, \\ C_1^2 &= E^{-1}(-h_4), \quad C_2^2 = E^{-1}(-h_4, 0), \quad C_3^2 = E^{-1}(0), \\ C_4^2 &= E^{-1}(0, h_4), \quad C_5^2 = E^{-1}(h_4), \quad C_6^2 = E^{-1}(h_4, +\infty), \\ C^3 &= \cup_{i=1}^4 C_i^3, \\ C_1^3 &= E^{-1}(-h_4), \quad C_2^3 = E^{-1}(-h_4, 0), \quad C_3^3 = E^{-1}(h_4), \quad C_4^3 = E^{-1}(h_4, +\infty), \\ C^4 &= C_1^4 \cup C_2^4, \\ C_1^4 &= E^{-1}(0), \quad C_2^4 = E^{-1}(0, +\infty). \end{aligned}$$

In paper [19] was obtained the following optimality result for bang-bang trajectories with low energy E .

Theorem 1 ([19, Theorem 2]). *If a bang-bang extremal λ_t , $t \in [0, +\infty)$, satisfies the inequality*

$$\min(-|h_4|, -|h_5|) < E \leq \max(-|h_4|, -|h_5|) \quad (2.6)$$

then it is optimal.

3. Theorem by Agrachev – Gamkrelidze

We obtain an upper bound on the number of switchings on optimal bang-bang trajectories via the following theorem due to A. Agrachev and R. Gamkrelidze.

Theorem 2 ([15, 16]). *Let $(q(\cdot), u(\cdot))$ be an extremal pair for problem (2.1)–(2.3) and let $\lambda_\cdot$ be an extremal lift of $q(\cdot)$. Assume that $\lambda_\cdot$ is the unique extremal lift of $q(\cdot)$, up to multiplication by a positive scalar. Assume that there exist $0 = t_0 < t_1 < t_2 < \dots < t_k < \tau_{k+1} = T$ and $u^0, \dots, u^k \in U$ such that $u(\cdot)$ is constantly equal to u^j on (τ_j, τ_{j+1}) for $j = 0, \dots, k$.*

Fix $j = 1, \dots, k$. For $i = 0, \dots, k$ let $Y_i = u_1^i X_1 + u_2^i X_2$ and define recursively the operators

$$\begin{aligned} P_j &= P_{j-1} = \text{Id}_{\text{vec}(M)}, \\ P_i &= P_{i-1} \circ e^{(t_i - t_{i-1}) \text{ad } Y_{i-1}}, \quad i = j+1, \dots, k, \\ P_i &= P_{i+1} \circ e^{-(t_{i+2} - t_{i+1}) \text{ad } Y_{i+1}}, \quad i = 0, \dots, j-2. \end{aligned}$$

Define the vector fields

$$Z_i = P_i(Y_i), \quad i = 0, \dots, k.$$

Let Q be the quadratic form

$$Q(\alpha) = \sum_{0 \leq i < l \leq k} \alpha_i \alpha_l \langle \lambda_{t_j}, [Z_i, Z_l](q(t_j)) \rangle,$$

defined on the space

$$W = \left\{ \alpha = (\alpha_0, \dots, \alpha_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^k \alpha_i = 0, \quad \sum_{i=0}^k \alpha_i Z_i(q(t_j)) = 0 \right\}.$$

If Q is not negative-semidefinite, then $q(\cdot)$ is not optimal.

4. Bounds on the number of switchings

Now we obtain bounds on the number of switchings for bang-bang optimal trajectories $\text{Exp}(\lambda, t)$ with $\lambda \in \cup_{i=1}^4 C^i$, case by case.

4.1. Case $\lambda \in C^1$

In the case $\lambda \in C^1$ system (2.5) has phase portrait given in Fig. 1.

4.1.1. Low values of integral E

Theorem 1 implies the following statement.

Corollary 1. *If $\lambda \in C_1^1 \cup C_2^1 \cup C_3^1$, then the trajectory $\text{Exp}(\lambda, t)$, $t \in [0, +\infty)$, is optimal.*

4.1.2. High values of integral E

We apply Theorem 2 and obtain the following upper bounds on the number of switchings on optimal bang-bang trajectories. An example of detailed computation on the basis of Theorem 2 is given in the proof of Theorem 5 [19].

Fig. 1. Phase portrait of system (2.5) in case $\lambda \in C^1$.

Theorem 3. Let $\lambda \in \cup_{i=4}^8 C_i^1$. Then the bang-bang trajectory $\text{Exp}(\lambda, t)$ with k switchings is not optimal, where k is given by the following tables:

- $\lambda \in C_4^1 \cup C_5^1 \Rightarrow$ Table 1,
- $\lambda \in C_6^1 \Rightarrow$ Table 2,
- $\lambda \in C_7^1 \Rightarrow$ Table 3,
- $\lambda \in C_8^1 \Rightarrow$ Table 4.

REMARK 1. We explain now how Tables 1–4 should be read.

Consider Table 1. The first line — Start — gives the values of $(u_1(0), u_2(0)) = (\text{sgn } h_1(0), \text{sgn } h_2(0))$ and, if necessary, the signs of $h_3(0)$ as a lower index. For example, the first column of Table 1 corresponds to $(u_1(0), u_2(0)) = (\text{sgn } h_1(0), \text{sgn } h_2(0)) = (+1, +1)$. The second column of Table 1 corresponds to the initial values $(u_1(0), u_2(0)) = (\text{sgn } h_1(0), \text{sgn } h_2(0)) = (-1, +1)$ and $\text{sgn } h_3(0) = +1$. The second line of Table 1 gives the number of switchings k for the corresponding $\lambda \in C_4^1 \cup C_5^1$ such that the bang-bang trajectory $\text{Exp}(\lambda, t)$ is not optimal. Similar agreement is applied for Tables 2, 4.

Table 3 should be read as follows. Consider, e.g., entry $N = 10$ of Table 3. The sequence of signs $(+, -) + + -$ has the following meaning:

- the signs $(+, -)$ determine the initial control $(u_1(0), u_2(0)) = (\text{sgn } h_1(0), \text{sgn } h_2(0)) = (+1, -1)$,
- the subsequent signs $+ + -$ determine the signs of $h_3(t)$ between switchings of control, i.e.,
  - $\text{sgn } h_3(t) = +1, t \in [0, t_1]$;
  - $\text{sgn } h_3(t) = +1, t \in [t_1, t_2]$;
  - $\text{sgn } h_3(t) = -1, t \in [t_2, T]$,

where t_1, t_2 are switching times at which $h_3(t)$ vanishes.

The number 9 for entry $N = 10$ of Table 3 gives the number of switchings of a non-optimal bang-bang control.

The same agreement on reading similar tables is used in subsequent subsections.

The below cases $\lambda \in \cup_{i=2}^4 C_i^1$ are considered similarly to the above case $\lambda \in C^1$.

Table 1. $\lambda \in C_4^1 \cup C_5^1$

Start	$(+, +)$	$(-, +)_+$	$(-, -)$	$(-, +)_-$
k	8	9	9	8

 Table 2. $\lambda \in C_6^1$

Start	$(+, +)_+$	$(-, +)_+$	$(-, -)$	$(-, +)_-$	$(+, +)_-$	$(+, -)$
k	9	10	10	10	9	11

 Table 3. $\lambda \in C_7^1$

N	Start	k	N	Start	k	N	Start	k	N	Start	k
1	$(+, +) + ++$	9	9	$(+, -) + ++$	8	17	$(-, +) + ++$	8	25	$(-, -) + ++$	8
2	$(+, +) + +-$	9	10	$(+, -) + +-$	9	18	$(-, +) + +-$	9	26	$(-, -) + +-$	8
3	$(+, +) + --$	8	11	$(+, -) + --$	11	19	$(-, +) + --$	9	27a	$(-, -) + --$	10
4	$(+, +) - ++$	8	12	$(+, -) - ++$	9	20	$(-, +) - ++$	7	27b	$(-, -) - ++$	11
5	$(+, +) - +-$	9	13	$(+, -) - +-$	8	21	$(-, +) - +-$	10	28	$(-, -) - +-$	8
6	$(+, +) - --$	9	14	$(+, -) - --$	12	22a	$(-, +) - --$	12	29	$(-, -) - +-$	7
7	$(+, +) - -+$	10	15a	$(+, -) - -+$	9	22b	$(-, +) - -+$	11	30	$(-, -) - -+$	7
8	$(+, +) - --$	9	15b	$(+, -) - --$	10	23	$(-, +) - --$	10	31	$(-, -) - --$	7
			16	$(+, -) - --$	8	24	$(-, +) - --$	8	32	$(-, -) - --$	7

 Table 4. $\lambda \in C_8^1$

Start	$(+, -)$	$(+, +)$	$(-, +)$	$(-, -)$
k	8	8	6	6

4.2. Case $\lambda \in C^2$

In the case $\lambda \in C^2$ system (2.5) has phase portrait given in Fig. 2.

4.2.1. Low values of integral E

Theorem 1 implies the following statement.

Corollary 2. *If $\lambda \in C_1^2 \cup C_2^2 \cup C_3^2$, then the trajectory $\text{Exp}(\lambda, t)$, $t \in [0, +\infty)$, is optimal.*

4.2.2. High values of integral E

Theorem 4. *Let $\lambda \in \cup_{i=4}^2 C_i^2$. Then the bang-bang trajectory $\text{Exp}(\lambda, t)$ with k switchings is not optimal, where k is given by the following tables:*

- $\lambda \in C_4^2 \Rightarrow$ Table 5,
- $\lambda \in C_6^2 \Rightarrow$ Table 6.

Fig. 2. Phase portrait of system (2.5) in case $\lambda \in C^2$ .Table 5. $\lambda \in C_4^2$

Start	$(+, +)_+$	$(-, +)_+$	$(-, -)$	$(-, +)_-$	$(+, +)_-$	$(+, -)$
k	8	9	8	8	9	8

Table 6. $\lambda \in C_6^2$

Start	$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$
k	7	6	6	7

4.3. Case $\lambda \in C^3$

In the case $\lambda \in C^3$ system (2.5) has phase portrait given in Fig. 3.

Fig. 3. Phase portrait of system (2.5) in case $\lambda \in C^3$.

4.3.1. Low values of integral E

Theorem 1 implies the following statement.

Corollary 3. *If $\lambda \in C_1^3$, then the trajectory $\text{Exp}(\lambda, t)$, $t \in [0, +\infty)$, is optimal.*

4.3.2. High values of integral E

Theorem 5. Let $\lambda \in \cup_{i=2}^4 C_i^3$. Then the bang-bang trajectory $\text{Exp}(\lambda, t)$ with k switchings is not optimal, where k is given by the following tables:

- $\lambda \in C_2^3 \cup C_3^3 \Rightarrow$ Table 7,
- $\lambda \in C_4^3 \Rightarrow$ Table 8.

Table 7. $\lambda \in C_2^3 \cup C_3^3$

Start	$(+, +)$	$(-, +)_+$	$(-, -)$	$(-, +)_-$
k	7	6	7	6

Table 8. $\lambda \in C_3^4$

Start	$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$
k	7	6	7	7

4.4. Case $\lambda \in C^4$

In the case $\lambda \in C^4$ system (2.5) has phase portrait given in Fig. 4.

Fig. 4. Phase portrait of system (2.5) in case $\lambda \in C^4$.

4.4.1. Low values of integral E

Theorem 1 implies the following statement.

Corollary 4. If $\lambda \in C_1^4$, then the trajectory $\text{Exp}(\lambda, t)$, $t \in [0, +\infty)$, is optimal.

4.4.2. High values of integral E

Theorem 6. Let $\lambda \in C_2^4$. Then the bang-bang trajectory $\text{Exp}(\lambda, t)$ with $k = 7$ switchings is not optimal.

5. Conclusion

An obvious next question that arises after the upper bounds on the number of switchings of optimal bang-bang control is the following one: when exactly do the bang-bang trajectories lose their optimality? That is, we would like to describe the cut time along bang-bang trajectories.

We hope that this is possible by (extension of) the symmetry method applied successfully for description of cut time in several sub-Riemannian and Riemannian problems [20–24]. This question will be studied in forthcoming papers.

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