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Bifurcation Analysis of Periodic Motions Originating from Regular Precessions of a Dynamically Symmetric Satellite

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We deal with motions of a dynamically symmetric rigid-body satellite about its center of mass in a central Newtonian gravitational field. In this case the equations of motion possess particular solutions representing the so-called regular precessions: cylindrical, conical and hyperboloidal precession. If a regular precession is stable there exist two types of periodic motions in its neighborhood: short-periodic motions with a period close to $2\pi/\omega_2$ and long-periodic motions with a period close to $2\pi/\omega_1$ where ω_2 and ω_1 are the frequencies of the linearized system ($\omega_2 > \omega_1$).

In this work we obtain analytically and numerically families of short-periodic motions arising from regular precessions of a symmetric satellite in a nonresonant case and long-periodic motions arising from hyperboloidal precession in cases of third- and fourth-order resonances. We investigate the bifurcation problem for these families of periodic motions and present the results in the form of bifurcation diagrams and Poincaré maps.

Keywords: Hamiltonian mechanics, satellite dynamics, bifurcations, periodic motions, orbital stability

1. Introduction

The motion of a dynamically symmetric satellite in a circular orbit was studied in many papers [1–4, 6–16]. One of the important subjects of study in satellite dynamics is periodic motions. Since equations of motion in satellite dynamics usually cannot be integrated analytically, a relevant problem is to develop asymptotical and numerical methods for computing and studying periodic motions. The problem of numerical computation of periodic motions was

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addressed in [17] where an approach that allows a reduction of the boundary-value problem to a simpler Cauchy problem was proposed. Numerical methods based on this approach were developed in [18–22] where motions arising from the satellite's hyperboloidal precession were computed analytically and numerically and their linear orbital stability was studied.

In this paper we compute analytically and numerically families of short-periodic motions originating from regular precessions of a symmetric satellite in a nonresonant case and families of long-periodic motions originating from hyperboloidal precession in the case of third- and fourth-order resonances. We consider the bifurcation problem of these motions.

2. Formulation of the problem

We consider the motion of a satellite around its center of mass O which moves in a circular orbit in a central Newtonian gravitational field at angular velocity ω_0 . The satellite is considered to be a dynamically symmetric rigid body with principal moments of inertia J_1 , J_2 and J_3 ($J_1 = J_2$). To describe the satellite's motion around its center of mass, we introduce an orbital reference frame $OXYZ$ and a mobile reference frame $Oxyz$. Axes OX , OY and OZ are aligned with transversal and normal vectors to the orbit and with the radius vector of the satellite's center of mass, respectively. Axes Ox , Oy and Oz are aligned with the satellite's principal axes of inertia. The relative position of these reference frames is defined by Euler's angles ψ , θ , φ . In this case the system possesses a cyclical coordinate φ and its respective impulse p_φ retains constant value. Following [3, 7], the equations of motion of a dynamically symmetric satellite can be written in a canonical form with the following Hamiltonian:

$$H = \frac{p_\psi^2}{2\sin^2\theta} + \frac{p_\theta^2}{2} - \left(\frac{\gamma \cos\theta}{\sin^2\theta} + \cos\psi \cot\theta \right) p_\psi - \sin\psi p_\psi + \frac{1}{2}\gamma^2 \cot^2\theta + \gamma \frac{\cos\psi}{\sin\theta} + \frac{1}{2}\delta \cos^2\theta, \quad (2.1)$$

where p_ψ and p_θ are dimensionless impulses corresponding to variables ψ and θ , $\delta = 3(J_3/J_1 - 1)$ and $\gamma = \frac{J_3 r_0}{J_1 \omega_0}$ are dimensionless parameters and r_0 is the projection of the satellite's absolute angular velocity along its principal axis Oz . The independent variable is a true anomaly $\nu = \omega_0 t$. The equations of motions with Hamiltonian (2.1) possess particular solutions

$$\theta_0 = \frac{\pi}{2}, \quad \psi_0 = \pi, \quad p_{\theta_0} = 0, \quad p_{\psi_0} = 0, \quad (2.2)$$

$$\sin\theta_0 = \frac{\gamma}{\delta - 1}, \quad \psi_0 = 0, \quad p_{\theta_0} = 0, \quad p_{\psi_0} = \delta \sin\theta_0 \cos\theta_0. \quad (2.3)$$

$$\theta_0 = \frac{\pi}{2}, \quad \cos\psi_0 = -\gamma, \quad p_{\theta_0} = \sin\psi_0, \quad p_{\psi_0} = 0, \quad (2.4)$$

known as regular precessions [2, 3].

Solution (2.2) exists for all values of γ , δ and is known as cylindrical precession. It defines a relative equilibrium of the satellite in the orbital reference frame when its principal axis Oz is aligned with the normal vector to the satellite's orbit and describes a cylinder in absolute space (Fig. 1a). Figure 1b presents a stability diagram for cylindrical precession where gray areas represent domains of instability and white areas represent domains of stability.

Solution (2.3) exists for $|\gamma| \leq |\delta - 1|$ (subdomains I–III in Fig. 1d) and is known as conical precession. In this case the satellite's principal axis Oz remains orthogonal to the transversal

Fig. 1. Regular precessions of a symmetric satellite (a, c, d) and their existence and stability domains in the plane of problem parameters γ , δ (b, e, f). Here G_0 is the gravitating center, Y is aligned with the normal vector to the satellite's orbit and X is aligned with the transversal vector.

vector of the orbital reference and describes a cone in absolute space with an angle $\pi - 2\theta_0$ at its tip (Fig. 1c). Figure 1d presents a stability diagram for conical precession where gray areas represent domains of instability and white areas represent domains of stability.

Solution (2.4) is known as hyperboloidal precession and defines a relative equilibrium of the satellite in the orbital reference frame when its principal axis Oz lies in a plane perpendicular to the radius vector of the satellite's center of mass and remains at a constant angle $\pi - \psi_0$ to the normal vector (Fig. 1d). The satellite's principal axis Oz describes a hyperboloid in absolute space. Hyperboloidal precession exists for $|\gamma| \leq 1$ and is Lyapunov stable if $\delta > 0$ [4] (Fig. 1f).

Fig. 2. Periodic motion in the neighborhood of hyperboloidal precession. Here G_0 is the gravitating center, z_0 is the position of the satellite's figure axis Oz in the case of hyperboloidal precession and s is the trace of the satellite's principle axis Oz on the unit sphere.

Two types of periodic motions exist in the neighborhood of a stable regular precession: short-periodic motions with period close to $2\pi/\omega_2$ and long-periodic motions with period close to $2\pi/\omega_1$, respectively. These periodic motions represent oscillations of the satellite's principle axis Oz in the neighborhood of regular precession (Fig. 2) and can be obtained as small parameter power series of the oscillation amplitude $c = c(\Delta h)$ where Δh is a deviation of the energy constant h from its value for a corresponding regular precession. Here $\omega_{1,2}$ are the frequencies of the linearized system

$$\omega_{1,2} = \sqrt{F_1^j \mp \sqrt{F_2^j}}, \quad (2.5)$$

where coefficients F_1^j and F_2^j ($j = \Gamma, Z, K$) in (2.5) corresponding to cylindrical (Z), conical (K) and hyperboloidal (Γ) precession are given by the following formulae:

$$\begin{aligned} F_1^\Gamma &= \frac{\delta + 1}{2}, \quad F_2^\Gamma = \frac{(\delta - 1)^2 + 4\gamma^2\delta}{4}, \\ F_1^Z &= \frac{1}{2}\gamma^2 - \gamma + \frac{1}{2}\delta + 1, \quad F_2^Z = \frac{1}{4}\gamma^4 - \gamma^3 + \gamma^2 - 2\gamma\delta + \delta + \frac{1}{2}\gamma^2\delta + \frac{1}{4}\delta^2, \\ F_1^K &= \frac{1}{2(\delta - 1)^2} \left[1 + 5\delta^2 + \delta^2\gamma^2 + 3\delta\gamma^2 - 2\delta^3 - 4\delta \right], \\ F_2^K &= \frac{1}{4(\delta - 1)^4} \left[1 - 4\gamma^2\delta^5 + (\gamma^2 + 1)^2\delta^4 + (-4 + 10\gamma^2 + 6\gamma^4)\delta^3 + \right. \\ &\quad \left. + (9\gamma^4 + 6 - 10\gamma^2)\delta^2 + (-4 + 2\gamma^2)\delta \right]. \end{aligned}$$

3. Analytical computation of periodic motions originating from hyperboloidal precession

In this paragraph, following [14–16, 23], we obtain families of periodic motions originating from regular precessions of a dynamically symmetric satellite in the form of small parameter power series of amplitude c .

3.1. Nonresonant case

To obtain the periodic motions originating from regular precessions of a satellite in a non-resonant case, we follow the method developed by Lyapunov [5] and introduce local coordinates

$$\psi = \psi_0 + \xi_1, \quad \theta = \theta_0 + \xi_2, \quad p_\psi = p_{\psi_0} + \eta_1, \quad p_\theta = p_{\theta_0} + \eta_2, \quad (3.1)$$

in the neighborhood of a regular precession and expand the Hamiltonian (2.1) into series of ξ_i, η_i ($i = 1, 2$)

$$H = H_2^j + H_3^j + H_4^j + \dots, \quad (3.2)$$

where H_k^j ($j = Z, K, \Gamma$) are polynomials of degree k in variables ξ_i, η_i . We then apply a canonical transformation

$$\begin{aligned} \xi_1 &= \kappa_1^j A_{11}^j x_1 + \kappa_2^j A_{12}^j x_2, & \xi_2 &= \kappa_1^j A_{21}^j y_1 + \kappa_2^j A_{22}^j y_2, \\ \eta_1 &= \kappa_1^j A_{31}^j x_1 + \kappa_2^j A_{32}^j x_2, & \eta_2 &= \kappa_1^j A_{41}^j y_1 + \kappa_2^j A_{42}^j y_2, \end{aligned} \quad (3.3)$$

which transforms (3.2) to the following normal form:

$$\begin{aligned} H &= \frac{1}{2} \omega_1 (x_1^2 + y_1^2) + \frac{1}{2} \omega_2 (x_2^2 + y_2^2) + \sum_{n_1+n_2+m_1+m_2=3} h_{n_1 n_2 m_1 m_2} x_1^{n_1} x_2^{n_2} y_1^{m_1} y_2^{m_2} + \\ &+ \sum_{n_1+n_2+m_1+m_2=4} h_{n_1 n_2 m_1 m_2} x_1^{n_1} x_2^{n_2} y_1^{m_1} y_2^{m_2} + \dots, \end{aligned} \quad (3.4)$$

where $h_{n_1 n_2 m_1 m_2}$ are constant coefficients dependent upon γ and δ . The coefficients $\kappa_1^j, \kappa_2^j, A_{i1}^j, A_{i2}^j$ ($i = 1, \dots, 4, k = Z, K, \Gamma$) of the transformation (3.3) are given by the following formulae:

$$\begin{aligned} A_{11}^Z &= \frac{1}{\omega_1} (1 - \gamma - \delta + \omega_1^2) \omega_1, & A_{12}^Z &= \frac{1}{\omega_2} (1 - \gamma - \delta + \omega_2^2), \\ A_{21}^Z &= A_{22}^Z = \gamma - 2, \\ A_{31}^Z &= (2\gamma - 1 - \gamma^2 - \delta + \omega_1^2), & A_{32}^Z &= (2\gamma - 1 - \gamma^2 - \delta + \omega_2^2), \\ A_{41}^Z &= \frac{1}{\omega_1} (\gamma + \delta - \gamma \omega_1^2 - 1 + \omega_1^2), & A_{42}^Z &= \frac{1}{\omega_2} (\gamma + \delta - \gamma \omega_2^2 - 1 + \omega_2^2), \\ \kappa_i^Z &= A^Z(\omega_i), \quad i = 1, 2 \end{aligned} \quad (3.5)$$

$$\begin{aligned} A^Z(\omega) &= \frac{1}{\omega} \gamma^3 + (\delta - 4) \gamma^2 + (6 - 2\omega^2 - 2\delta) \gamma - 3 + \\ &+ 2\delta + \omega^4 + \delta^2 + 2\omega^2 - 2\delta \omega^2, \\ A_{11}^K &= \frac{\delta - 1}{\omega_1 \omega_1 \gamma^2} (\omega_2 \delta^2 - 2\omega_2 \delta + \omega_2 + \omega_2 \delta \omega_1^2 - \omega_2 \gamma^2 - \omega_2 \omega_1^2), \\ A_{12}^K &= \frac{\delta - 1}{\omega_1 \omega_1 \gamma^2} (\omega_1 \delta^2 - 2\omega_1 \delta + \omega_1 \delta \omega_2^2 - \omega_1 \gamma^2 + \omega_1 - \omega_1 \omega_2^2), \\ A_{21}^K &= A_{22}^K = \delta + 1, \\ A_{31}^K &= \frac{1}{(\delta - 1)^2} (\omega_1^2 - 4\delta^2 - \delta^2 \gamma^2 - 3\delta \gamma^2 + 2\delta^3 + 2\delta - 2\omega_1^2 \delta + \omega_1^2 \delta^2), \\ A_{32}^K &= \frac{1}{(\delta - 1)^2} (\omega_2^2 + \omega_2^2 \delta^2 - 2\omega_2^2 \delta - \delta^2 \gamma^2 - 3\delta \gamma^2 + 2\delta^3 + 2\delta - 4\delta^2), \end{aligned} \quad (3.6)$$

$$\begin{aligned}
A_{41}^K &= \frac{1}{\omega_1 \omega_1 \gamma^2} (\omega_2 \omega_1^2 - \omega_2 - 2\omega_2 \delta \omega_1^2 + \omega_2 \omega_1^2 \delta^2 - \\
&\quad - \omega_2 \gamma^2 \omega_1^2 - 3\omega_2 \delta^2 - \omega_2 \delta \gamma^2 + \omega_2 \delta^3 + 3\omega_2 \delta + \omega_2 \gamma^2 - \omega_2 \gamma^2 \omega_1^2 \delta), \\
A_{42}^K &= \frac{1}{\omega_1 \omega_1 \gamma^2} (\omega_1 \delta^3 + 3\omega_1 \delta + \omega_1 \gamma^2 + \omega_1 \omega_2^2 + \omega_1 \omega_2^2 \delta^2 - \\
&\quad - 2\omega_1 \delta \omega_2^2 - \omega_1 - \omega_1 \gamma^2 \omega_2^2 - \omega_1 \gamma^2 \omega_2^2 \delta - 3\omega_1 \delta^2 - \omega_1 \delta \gamma^2), \\
\kappa_i^K &= \frac{1}{(\delta - 1) \gamma^2 \omega_i} A(\omega_i), \quad i = 1, 2 \\
A^K(\omega) &= 2\omega^2 - \omega^4 - \gamma^2 \delta^4 - 2\gamma^2 \delta^3 + (8 - 2\omega^2) \gamma^2 \delta^2 + \gamma^4 \delta^2 + 3\gamma^4 \delta + \\
&\quad + (5 + 3\omega^4 - 8\omega^2) \delta + (1 - 2\omega^2) \gamma^2 + (12\omega^2 - 3\omega^4 - 10) \delta^2 + \delta^5 + \\
&\quad + (-6 + 4\omega^2) \gamma^2 \delta + (-5 + 2\omega^2) \delta^4 + (\omega^4 + 10 - 8\omega^2) \delta^3 - 1. \\
A_{11}^\Gamma &= \frac{\delta - \omega_1^2}{\omega_1}, \quad A_{12}^\Gamma = \frac{\delta - \omega_2^2}{\omega_2}, \\
A_{21}^\Gamma &= A_{22}^\Gamma = \gamma, \\
A_{31}^\Gamma &= \delta - \omega_1^2, \quad A_\Gamma^Z = \delta - \omega_2^2, \\
A_{41}^\Gamma &= -\frac{\gamma \delta}{\omega_1}, \quad A_{42}^Z = -\frac{\gamma \delta}{\omega_2}, \\
\kappa_i^\Gamma &= \sqrt{\frac{\omega_i}{\omega_i^4 + \gamma^2 \delta - 2 \delta \omega_i^2 + \delta^2}}, \quad i = 1, 2.
\end{aligned} \tag{3.7}$$

The expressions (3.5), (3.6) and (3.7) correspond to the cases of cylindrical (Z), conical (K) and hyperboloidal (Γ) precession, respectively.

We then apply another canonical transformation $x_i, y_i \rightarrow q_i, p_i$ ($i = 1, 2$) which transforms (3.4) to the form

$$\begin{aligned}
K &= \frac{1}{2} \omega_1 (q_1^2 + p_1^2) + \frac{1}{2} \omega_2 (q_2^2 + p_2^2) + \\
&\quad + a_{20} (q_1^2 + p_1^2)^2 + a_{11} (q_1^2 + p_1^2) (q_2^2 + p_2^2) + a_{02} (q_2^2 + p_2^2)^2 + K^{(5)},
\end{aligned} \tag{3.8}$$

where a_{20}, a_{11}, a_{02} are constant coefficients dependent upon γ and δ and K_5 are terms of order 5 and higher. The canonical system with Hamiltonian (3.8) admits a family of short-periodic motions

$$q_2 = c \sin \Omega_2 (\nu - \nu_0), \quad p_2 = c \cos \Omega_2 (\nu - \nu_0), \quad q_1 = p_1 = 0, \tag{3.9}$$

with period $T_2 = 2\pi/\Omega_2$ where

$$\Omega_2 = \omega_2 + 4c^2 a_{02} + O(c^4).$$

The parameter c in (3.9) represents the small amplitude of oscillations of the satellite's axis Oz in the neighborhood of a regular precession. Returning to the initial variables $\psi, \theta, p_\psi, p_\theta$, we obtain the following asymptotic expressions for the families of short-periodic motions originating from regular precessions:

$$\begin{aligned}
\psi &= \psi_0 + c \kappa_2 A_{12}^j \sin \Omega_2 (\nu - \nu_0) + O(c^2), \\
p_\psi &= p_{\psi_0} + c \kappa_2 A_{32}^j \cos \Omega_2 (\nu - \nu_0) + O(c^2), \\
\theta &= \theta_0 + c \kappa_2 A_{22}^j \cos \Omega_2 (\nu - \nu_0) + O(c^2), \\
p_\theta &= p_{\theta_0} + c \kappa_2 A_{42}^j \sin \Omega_2 (\nu - \nu_0) + O(c^2),
\end{aligned} \tag{3.10}$$

where coefficients $\kappa_1^j, \kappa_2^j, A_{i1}^j, A_{i2}^j$ ($i = 1, \dots, 4, k = Z, K, \Gamma$) are given by (3.5)–(3.7). In presenting further results, we will refer to the families (3.10) of short-periodic motions originating from cylindrical, conical and hyperboloidal precessions as Z_s, K_s and Γ_s , respectively.

The canonical system with Hamiltonian (3.8) also admits a family of long-periodic motions

$$q_2 = p_2 = 0, \quad q_1 = c \sin \Omega_1(\nu - \nu_0), \quad p_1 = c \cos \Omega_1(\nu - \nu_0) \quad (3.11)$$

with period $T_1 = 2\pi/\Omega_1$ ($T_1 > T_2$) where $\Omega_1 = \omega_1 + 4c^2 a_{20} + O(c^4)$. Upon returning to the initial variables $\psi, \theta, p_\psi, p_\theta$ expressions (3.11) give the following series of amplitude c :

$$\begin{aligned} \psi &= \psi_0 + c \kappa_1 A_{11}^j \sin \Omega_1(\nu - \nu_0) + O(c^2), \\ p_\psi &= p_{\psi_0} + c \kappa_1 A_{31}^j \cos \Omega_1(\nu - \nu_0) + O(c^2), \\ \theta &= \theta_0 + c \kappa_1 A_{21}^j \cos \Omega_1(\nu - \nu_0) + O(c^2), \\ p_\theta &= p_{\theta_0} + c \kappa_1 A_{41}^j \sin \Omega_1(\nu - \nu_0) + O(c^2), \end{aligned} \quad (3.12)$$

which represent the long-periodic motions originating from regular precessions in a nonresonant case.

3.2. Third-order resonance

To analytically obtain the families of periodic motions originating from hyperboloidal precession in the neighborhood of third-order resonance ($\omega_2 = 2\omega_1$), we apply a canonical transformation $\psi, \theta, p_\psi, p_\theta \rightarrow \xi_1, \xi_2, \eta_1, \eta_2$ which transforms the Hamiltonian (2.1) into the following normal form:

$$K^I = \frac{1}{2}\omega_1(\xi_1^2 + \eta_1^2) + \frac{1}{2}\omega_2(\xi_2^2 + \eta_2^2) + A[\xi_2(\eta_1^2 - \xi_1^2) - 2\eta_1\xi_1\eta_1] + O_4, \quad (3.13)$$

where A is a resonance coefficient. In the case of precise resonance $\omega_2 = 2\omega_1$ this coefficient is given by the expression

$$A = \sqrt{\frac{(3\omega_1^2 - 1)^2 (4\omega_1^2 - 1)}{3\omega_1^5}}.$$

Following [10, 16], we introduce new time $\tau = \omega_1\nu$, apply a scaling canonical transformation $\xi_i = \varepsilon ax_i, \eta_i = \varepsilon ay_i$, ($i = 1, 2$) with valence $n = 1/\varepsilon^2 a^2$ and then canonical transformations $x_i = \sqrt{2r_i} \sin \phi_i, y_i = \sqrt{2r_i} \cos \phi_i$, ($i = 1, 2$) and $\phi_1 = Q_1, \phi_2 = Q_2 + 2Q_1, r_1 = P_1 - 2P_2, r_2 = P_2$ which transform (3.13) to the form [10, 15]

$$K^{II} = P_1 + \mu P_2 + \varepsilon(P_1 - 2P_2)\sqrt{P_2} \sin Q_2 + O(\varepsilon^2), \quad (3.14)$$

where $\mu = \omega_2/\omega_1 - 2$ is a small parameter, ε is a scaling coefficient and $P_1 \geq 2P_2$. The system with Hamiltonian (3.14) possesses a first integral $P_1 = C$ using which and omitting the terms $O(\varepsilon^2)$ we arrive at a truncated system

$$K^{III} = \mu P_2 + \varepsilon(C - 2P_2)\sqrt{P_2} \sin Q_2, \quad (3.15)$$

which describes long-periodic motions in the neighborhood of hyperboloidal precession in the case of the third-order resonance. The system (3.15) possesses equilibria

$$Q_2^* = \frac{\pi}{2}, \quad P_2^* = \frac{\mu + \sqrt{\mu^2 + 6\varepsilon^2 C}}{6\varepsilon}, \quad (3.16)$$

and

$$Q_2^* = -\frac{\pi}{2}, \quad P_2^* = \frac{-\mu + \sqrt{\mu^2 + 6\varepsilon^2 C}}{6\varepsilon}. \quad (3.17)$$

In the initial system with Hamiltonian (2.1) the equilibria (3.16) and (3.17) correspond to families of long-periodic motions, which we will refer to as Γ_1 and Γ_2 , respectively.

Depending upon the values of the problem's parameters γ , δ and h there may exist one or both of these families in the neighborhood of the third-order resonance curve $\gamma_0 = \pm \frac{\sqrt{-4\delta^3 + 17\delta^2 - 4\delta}}{5\delta}$. Above the surface given by the following series of h

$$\gamma(\delta, h) = \gamma_0 \pm \frac{\sqrt{6}|3\delta - 2|(5 + 5\delta)^{1/4}}{25\sqrt{\delta(4 - \delta)}}\sqrt{h} + O(h), \quad (3.18)$$

which emanates from the resonant curve γ_0 , there exist both Γ_1 and Γ_2 . Below the surface (3.18) there exist one family Γ_1 in the case $\gamma < \gamma_0$ and one family Γ_2 in the case $\gamma > \gamma_0$. Due to their size we do not present the expressions for the families Γ_1 and Γ_2 in the initial variables in this work.

3.3. Fourth-order resonance

To obtain the long-periodic motions in the case of precise fourth-order resonance ($\omega_2 = 3\omega_1$), we apply a canonical transformation $\psi, \theta, p_\psi, p_\theta \rightarrow q_1, q_2, p_1, p_2$ that transforms the Hamiltonian (2.1) into normal form [11, 23, 24]

$$K^I = \frac{1}{2}\omega_1(q_1^2 + p_1^2) + \frac{1}{2}\omega_2(q_2^2 + p_2^2) + \frac{1}{4}c_{20}(q_1^2 + p_1^2)^2 + \frac{1}{4}c_{11}(q_1^2 + p_1^2)(q_2^2 + p_2^2) + \frac{1}{4}c_{02}(q_2^2 + p_2^2)^2 - \frac{b}{4}[q_1^2(q_1q_2 + 3p_1p_2) - p_1^2(p_1p_2 + 3q_1q_2)] + O_6, \quad (3.19)$$

where c_{11} , c_{20} , c_{02} are constant coefficients dependent upon the frequencies (2.5). In the case of precise fourth-order resonance they take on the following form:

$$\begin{aligned} c_{11} &= \frac{1}{560\omega_1^6}(678\omega_1^6 + 13\omega_1^4 - 20\omega_1^2 + 1), \\ c_{20} &= \frac{1}{320\omega_1^6}(118\omega_1^6 - 319\omega_1^4 + 60\omega_1^2 - 3), \\ c_{02} &= -\frac{1}{6720\omega_1^6}(906\omega_1^6 - 983\omega_1^4 - 220\omega_1^2 + 11). \end{aligned}$$

In (3.19) b is a resonant coefficient given by the following expression:

$$b = \frac{1}{120} \frac{(1 - 15\omega_1^2 + 56\omega_1^4) \sqrt{-3 + 30\omega_1^2 - 27\omega_1^4}}{\omega_1^6}.$$

We introduce new time $\tau = \omega_1\nu$, apply a scaling canonical transformation $q_i = \sqrt{\varepsilon\omega_1/b}x_i$, $p_i = \sqrt{\varepsilon\omega_1/b}y_i$, ($i = 1, 2$) with valence $n = b/(\varepsilon\omega_1)$, a canonical transformation $x_i = \sqrt{2r_i}\sin\phi_i$, $y_i = \sqrt{2r_i}\cos\phi_i$, ($i = 1, 2$) and omit the terms of order ε^2 and higher, which brings us to a truncated canonical system with a Hamiltonian [11, 23]

$$K^{II} = r_1 + 3r_2 + \varepsilon \left[c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + r_1^{3/2}\sqrt{r_2}\cos\phi_1 - 3\phi_2 \right] \quad (3.20)$$

and a first integral $3r_1 + r_2 = \text{const}$ where ε is a scaling coefficient. We apply a canonical transformation $\Psi = \phi_1$, $\Theta = \phi_2 - 3\phi_1$, $J = 3r_2 + r_1$, $R = r_2$, which transforms a system with the Hamiltonian (3.20) to the following form:

$$\frac{d\Theta}{d\tau} = \varepsilon \frac{1}{2} \frac{\sqrt{J-3R}(J-12R)\cos\Theta}{\sqrt{R}} + \sigma J + \mu R, \quad \frac{dR}{d\tau} = \varepsilon \sqrt{R}(J-12R)^{3/2} \sin\Theta \quad (3.21)$$

with a first integral $J = J_0 = \text{const}$ where $\sigma = c_{11} - 6c_{20}$ and $\mu = c_{02} + 9c_{20} - 3c_{11}$ are constant coefficients. The system (3.21) possesses equilibria

$$\Theta = \frac{\pi}{2} 1 - \text{sign}[(12x_* - 1)(\sigma + 2\mu x_*)], \quad R = x_* J_0, \quad (3.22)$$

where x_* is a real-value root of a cubic equation

$$(432 + 16\mu^2)x^3 + (16\mu\sigma - 216)x^2 + (4\sigma^2 + 27)x - 1 = 0. \quad (3.23)$$

For the values of the problem's parameters γ and δ considered in this work, Eq. (3.23) possesses three real-valued roots x_* which upon substituting into (3.22) and returning to the initial variables ψ , θ , p_ψ , p_θ give three families of long-periodic motions that originate from the hyperboloidal precession in the case of fourth-order resonance. We refer to these families as Γ_2 , Γ_3 and Γ_4 . Numerical analysis carried out in [23] shows that the family Γ_2 obtained here coincides with the family Γ_2 which was obtained in the previous subsection.

4. Bifurcation analysis of periodic motions originating from regular precessions

Analytical representations of the families Z_s , K_s , Γ_s , Γ_i ($i = 1, \dots, 4$) obtained in the previous section are only valid for small values of the amplitude c and small deviations Δh of the energy constant h from its value on a corresponding regular precession. To obtain the existence domains of the said families for nonsmall values of c , a numerical method has to be used. In Refs. [14–16, 23], using a method described in [18, 22], the families Γ_s , Γ_i ($i = 1, \dots, 4$) were continued numerically to the borders of their existence domains. Linear orbital stability of these periodic motions was studied in [15, 22, 23]. In this section we construct existence domains of the families of short-periodic motions Z_s , K_s , Γ_s and consider the bifurcation problem of the families Z_s , K_s , Γ_s and Γ_i ($i = 1, \dots, 4$).

4.1. Existence domains of periodic motions

Using the method described in [18, 22] the families Z_s , K_s , Γ_s of short-periodic motions arising from regular precessions of a symmetric satellite were numerically continued to the borders of their existence domains in the problem's three-dimensional parameter space. In Fig. 3 we present existence domains of the aforementioned families for fixed values of $\delta = 0.5$, $\delta = 1.0$, $\delta = 2.8$ and assuming $\gamma \geq 0$.

The family Z_s of short-periodic motions originating from cylindrical precession exists in a domain between the curves S_0^Z and S_1^Z . In Fig. 3 this domain is colored gray. The family Z_s arises from the cylindrical precession on the curve S_0^Z and terminates on curve S_1^Z . In the case $\delta \leq 1.0$ the existence domain takes on the form shown in Figs. 3a, 3b. The form of the aforementioned domain in the case $\delta > 1.0$ is shown in Fig. 3c. Numerical computation has

Fig. 3. Existence domains for the families of short-periodic motions Z_s (gray areas), K_s (horizontally-hatched areas), Γ_s (cross-hatched areas) arising from cylindrical, conical and hyperboloidal precession of a symmetric satellite.

shown that in this case the family Z_s cannot be continued using the method described in [18, 22] into two subdomains enclosed by the curves $S_{3,4,5}^Z$.

The horizontally-hatched areas in Fig. 3 correspond to the existence domain of the family K_s of short-periodic motions arising from conical precession. For $\delta < 1.0$ (Fig. 3a) the aforementioned domain is enclosed by curves S_0^K , S_1^K and S_2^K . Here the curve S_0^K corresponds to conical precession. The family K_s terminates on the border S_2^K . On the curve S_1^K a bifurcation occurs and the family K_s coincides with the family Z_s of short-periodic motions arising from cylindrical precession. With increasing δ point P_{01} of the curve S_1^K approaches point P_{02} and at the value $\delta = 1.0$ the existence domain becomes degenerate. For $\delta > 1.0$ the existence domain of the family K_s takes on the form shown in Fig. 3d.

The cross-hatched areas in Fig. 3 correspond to the existence domain of the family Γ_s of short-periodic motions arising from hyperboloidal precession. The existence domain of the family Γ_s is enclosed by curves S_0^Γ , S_1^Γ and S_2^Γ where the curve S_0^Γ corresponds to hyperboloidal precession. The family Γ_s arises from hyperboloidal precession on the curve S_0^Γ and terminates on the border S_2^Γ . The curve S_1^Γ corresponds to the bifurcation values of the problem's param-

Fig. 4. Existence domains for the families of periodic motions Γ_s and Γ_i ($i = 1, \dots, 4$) emanating from hyperboloidal precession. Gray corresponds to the existence domain of the family Γ_s of short-periodic motions.

eters. On S_1^Γ the family Γ_s coincides with the family Z_s of short-periodic motions arising from cylindrical precession.

Figure 4 shows existence domains for the families of long-periodic motions Γ_i ($i = 1, \dots, 4$) arising from hyperboloidal precession for fixed value of $\delta = 1.0$. This pattern of the existence domains remains valid for $0.115 < \delta < 3.0$. Numerical computation shows that these domains are symmetric with respect to the axis h , so further results are presented assuming $\gamma > 0$. In this figure the solid lines divide the parameter space into subdomains in which there may exist different numbers of long-periodic motions. On the borders of these subdomains, a bifurcation of families Γ_s, Γ_i ($i = 1, \dots, 4$) occurs.

4.2. Bifurcation analysis of short-periodic motions

It has been noted above that a bifurcation of short-periodic motions originating from regular precessions may occur on the borders of their existence domains.

Figure 5 shows existence domains and a diagram of periods T_K, T_Z, T_Γ of the families Z_s, K_s, Γ_s for fixed values of $\delta = 0.5$ and $h = 0.35$. In Fig. 5b the solid lines represent intervals of linear orbital stability and the dashed lines represent intervals of orbital instability of the aforementioned families of periodic motions. The method of investigating the linear orbital stability of the said periodic motions is described in Refs. [19, 22].

For $\gamma < 0.16$ there exist only one family of linear orbitally stable short-periodic motions – Γ_s . At the value $\gamma = 0.16$ a family of linear orbitally stable short-periodic motions Z_s arises from cylindrical precession and at the value $\gamma = 0.42$ a family of orbitally unstable short-periodic motions K_s arises from conical precession. With increasing γ the periods T_K and T_Z of the families K_s and Z_s converge and coincide at the point B_1 ($\gamma = 0.78$). At this point a bifurcation occurs: the family K_s coincides with Z_s and periodic motions of the family Z_s become orbitally unstable. With further growth of parameter γ the period T_Γ of the family Γ_s converges with the period T_Z of the family Z_s and these two families coincide at the point B_2 ($\Gamma = 1.11$). To the right of the point B_2 only one family of short-periodic motions remains — the family Z_s of short-periodic motions arising from cylindrical precession. Periodic motions of the family Z_s are linear orbitally stable for $\gamma > 1.11$. The family Z_s coincides with cylindrical precession at point P_3 ($\gamma = 1.64$). This bifurcation pattern remains valid for $0 < \delta < 1.0$.

Fig. 5. Existence domains and a bifurcation diagram of the families Z_s , K_s , Γ_s of short-periodic motions emanating from cylindrical, conical and hyperboloidal precession for fixed values of $\delta = 0.5$, $h = 0.35$. Here T_Z , T_K , T_Γ are periods of the periodic motions belonging to the families Z_s , K_s , Γ_s , respectively.

Fig. 6. Poincaré maps computed in the neighborhood of bifurcation points B_1 (a) and B_2 (b) for fixed values of $\gamma = 0.77$ and $\gamma = 1.09$, respectively.

To verify the results described above, Poincaré maps were constructed in the neighborhood of the bifurcation points B_1 and B_2 . Figure 6a shows a map constructed near the point B_1 for the values $\gamma = 0.77$, $\delta = 0.5$, $h = 0.35$. On this Poincaré map a point marked Z_s corresponds to a linear orbitally stable short-periodic motion of the family Z_s and the point marked K_s corresponds to an orbitally unstable short-periodic motion of the family K_s . With γ approaching the bifurcation value the points corresponding to the families K_s and Z_s converge and coincide forming a point corresponding to an orbitally unstable periodic motion of the family Z_s (Fig. 6b).

Figure 6b shows a Poincaré map constructed near the point B_2 for the values $\gamma = 1.09$, $\delta = 0.5$, $h = 0.35$. On this map the point marked Z_s corresponds to an orbitally unstable short-periodic motion of the family Z_s and the point marked Γ_s corresponds to a linear orbitally stable short-periodic motion of the family Γ_s . With γ approaching the bifurcation value at point B_2 the points corresponding to the families Z_s and Γ_s converge and coincide forming a point corresponding to a linear orbitally stable periodic motion of the family Z_s (Fig. 7).

Fig. 7. Poincaré map computed in the neighborhood of the bifurcation point B_2 for fixed value of $\gamma = 1.15$.

In Fig. 6 and Fig. 7 there also exist points marked Z_i ($i = 1, \dots, 5$), K_1 , $\Gamma_{3,4}$ corresponding to long-periodic motions arising from regular precessions. The families $\Gamma_{3,4}$ of long-periodic motions arising from hyperboloidal precession have been obtained analytically and numerically in [22]. Numerical computation shows that the families Z_i ($i = 1, \dots, 5$) and K_1 arise from cylindrical and conical precessions, respectively.

4.3. Bifurcation analysis of long-periodic motions originating from hyperboloidal precession

Figures 8 and 10 show bifurcation diagrams of periodic motions originating from the hyperboloidal precession of a satellite for $\delta = 1.0$ and different values of the energy constant h . In these diagrams the periods T_s , T_i ($i = 1, \dots, 4$) of the families Γ_s , Γ_i ($i = 1, \dots, 4$) are mapped against the parameter γ for fixed values of h . The curves NT_s represent period T_s of the family Γ_s multiplied N times. The solid lines represent intervals of linear orbital stability, while gray lines represent intervals of orbital instability.

Fig. 8. Bifurcation diagrams for families of periodic motions originating from the hyperboloidal precession of a symmetric satellite obtained for fixed values of $h = 0.001$, $\delta = 1.0$ and $h = 0.1$, $\delta = 1.0$.

Figures 8 show a diagram for $h = 0.001$ which represents relatively small deviations of the energy constant from its value on hyperboloidal precession and corresponds to the results presented in the analytical studies [10, 13]. In this case, for small values of γ there exist a family of short-periodic motions Γ_s and a family of long-periodic motions Γ_1 . At point B_1 a family of long-periodic motions Γ_2 branches off from the family Γ_s , so between the points B_1 and B_2 there exist two families of long-periodic motions which arise from hyperboloidal precession. At point B_2 the period T_1 of the family Γ_1 becomes twice as large as the period T_s of the family Γ_s and the families Γ_1 and Γ_s coincide. The family Γ_2 can be numerically continued to the point B_4 where it coincides with Γ_s . At this point the period T_2 becomes equal to $3T_s$. At point B_3 two families arise: Γ_3 and Γ_4 . The family Γ_3 can be continued to the point B_4 where it coincides with Γ_s . Upon approaching point B_3 the period T_3 gets closer to $3T_s$. The family Γ_4 coincides with Γ_s at point B_5 with its period T_4 becoming equal to $4T_s$. The family Γ_s can be continued along the γ axis to the value $\gamma \approx 1.01$ where it coincides with a family of periodic motions that arises from cylindrical precession.

Figure 9 shows Poincaré maps computed for fixed values of $h = 0.001$ and $\delta = 1.0$ in the neighborhood of the bifurcation point B_1 . The left map shows families Γ_s and Γ_1 before the bifurcation. The right map shows the family Γ_2 branching off from Γ_s with Γ_s becoming orbitally unstable.

Fig. 9. Poincaré maps computed for fixed values of $h = 0.001$ and $\delta = 1.0$ in the neighborhood of the bifurcation point B_1 shown in Fig. 3a.

Figure 10a shows a diagram for $h = 1.0$. In this case, for small values of γ there exist a family Γ_s and a family Γ_s^* . At point B_1^* families Γ_1^* and Γ_2 appear. The family Γ_1^* exists for $h > 0.05$ and has been obtained numerically using Poincaré maps. With increasing γ the period T_1 becomes closer to $2T_s$ and at point B_2 the families Γ_1 and Γ_s coincide. Γ_2 coincides with Γ_s at point B_3 with its period T_2 becoming equal to $2T_s$. Approaching the point B_1 , period T_1^* of the family Γ_1^* gets closer to $2T_s$ and at point B_1 the family Γ_1^* coincides with Γ_s .

At point B_3 two families of long-periodic motions arise: Γ_3 and Γ_4 . The family Γ_3 can be continued to the point B_4 where it coincides with Γ_s and its period T_3 becomes equal to $3T_s$. The family Γ_4 coincides with Γ_s at point B_5 where its period T_4 equals T_s . The family Γ_s can be continued along the γ axis to the value $\gamma \approx 1.04$ where it coincides with a family of periodic motions that arises from cylindrical precession.

Fig. 10. Bifurcation diagrams for families of periodic motions originating from the hyperboloidal precession of a symmetric satellite obtained for fixed values of $h = 0.205$, $\delta = 1.0$ and $h = 0.3$, $\delta = 1.0$.

The diagram shown in Fig. 10b represents the case of $h = 0.205$. In the interval $0.204 < h < 0.206$ there exists a value h^* for which the curves T_1^* and T_2 form a sharp angle at the point B_1^* which coincides with the curve T_1 . For $h > h^*$ the family Γ_1^* takes the place of Γ_1 and coincides with Γ_s at the point B_2 , while the family Γ_1 coincides with Γ_s at the point B_1 . The families Γ_2 , Γ_3 and Γ_4 retain similar behavior as in the previous case. A similar change in the bifurcation diagram occurs at the value $h \approx 0.25$ where the curves T_2 , T_3 and T_4 converge at the point B_3 and the families Γ_2 and Γ_3 switch places. Figure 10c shows a diagram computed for $h = 0.3$ which demonstrates the aforementioned change.

Conclusion

In this work we have constructed families of periodic motions emanating from the regular precession of a dynamically symmetric satellite for all admissible values of the problem parameters. A bifurcation problem for the said families of periodic motions has been addressed. The results have been presented in the form of bifurcation diagrams and Poincaré maps.

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