NONLINEAR PHYSICS AND MECHANICS

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On Integrability of the FitzHugh – Rinzel Model

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The integrability of the FitzHugh – Rinzel model is considered. This model is an example of the system of equations having the expansion of the general solution in the Puiseux series with three arbitrary constants. It is shown that the FitzHugh – Rinzel model is not integrable in the general case, but there are two formal first integrals of the system of equations for its description. Exact solutions of the FitzHugh – Rinzel system of equations are given.

Keywords: FitzHugh – Rinzel model, Painlevé test, first integral, general solution, exact solution

1. Introduction

One of the most important problems now is understanding how impulses are propagated from one neuron to another. The solution of this problem would not only provide a better understanding of complex processes, but also allow significant progress in the creation of artificial intelligence.

It is well known that the classical standard model in neuroscience is the FitzHugh – Nagumo model. This model was presented in [1–3] as a simplification of the famous Hodgkin – Huxley model [4] for description of the impulse propagation from one neuron to another.

The FitzHugh – Nagumo model has been studied many times in the periodical literature. Some recent results can be found, for example, in [5–12].

There is an important question of the existence of exact solutions for many models in neuroscience. Firstly, we need to obtain analytical dependencies on time for activation functions in constructing neuron networks. Secondly, it is necessary to test different applications of models in neuroscience, and the best way is to check the mathematical model taking into account the exact solutions of nonlinear differential equations. To investigate the question of integrability and to find exact solutions of the nonlinear mathematical model, we can use the Painlevé test.

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In the recent paper [15], the authors considered the analytical properties of the so-called FitzHugh–Rinzel model and built some simple solutions of the system of equations used to describe the process of neuron propagation. They showed that the general solution of a third-order equation is represented by a Puiseux series with three arbitrary constants.

This raises the question of the integrability of a third-order equation that corresponds to this system of equations.

The FitzHugh–Rinzel model is described by the following system of equations [13–15]:

$$\begin{aligned} v_t &= v - \frac{1}{3}v^3 - w + y + I_{ext}, \\ w_t &= \delta(v - \beta w), \\ y_t &= \mu(\sigma - v - \gamma y). \end{aligned} \quad (1.1)$$

The system of equations (1.1) can be written in the form of third-order nonlinear ordinary equations at $\beta\delta \neq \mu\gamma$ as follows:

$$v_{ttt} + (v^2 + a - 1)v_{tt} + 2vv_t^2 + (av^2 + c)v_t + \frac{1}{3}bv^3 + ev - (I_{ext} + \sigma)\beta\mu\delta = 0, \quad (1.2)$$

where

$$a = \beta\delta + \gamma\mu, \quad b = \beta\delta\gamma\mu, \quad c = b - a + \delta + \mu, \quad e = \mu\delta(\beta + \gamma) - b. \quad (1.3)$$

The origin of the expansion for the general solution of the system of equations (1.1) with three arbitrary constants in the Puiseux series is explained by the fact that two of the three equations of the system are linear and therefore by definition do not have critical movable singular points. The first-order equation of the system (1.1) is nonlinear, but it always has a critical pole at $t = t_0$ with the exception of quadratic nonlinearity.

The aim of this note is to consider the first integrals of the third-order differential equation (1.2) and to find the general solution of this equation taking into account the values of the Fuchs indices at the expansion of the general solution in the Puiseux series.

2. First integrals of the FitzHugh–Rinzel model

The first term of the expansion of the general solution in the Puiseux series can be found from the equation with the leading terms

$$v_{ttt} + v^2v_{tt} + 2vv_t^2 = \frac{d^2}{dt^2} \left(v_t + \frac{1}{3}v^3 \right) = 0. \quad (2.1)$$

We can note that the equation with leading terms is determined by calculating the second derivative of two terms of the first equation (1.1). We can see that Eq. (1.2) does not pass the Painlevé test because there is the critical movable point in the expansion [16–21]. However, there is the expansion of the general solution of Eq. (1.2) in the Puiseux series with three arbitrary constants t_0 , a_3 and a_5 in the form [15]

$$\begin{aligned} v(t) &= \pm \frac{\sqrt{6}}{2\sqrt{t-t_0}} \pm \frac{\sqrt{6}}{4}\sqrt{t-t_0} \mp \\ &\mp a_3(t-t_0) \pm \frac{\sqrt{6}}{48}(1+16b-16a-16c)(t-t_0)^{3/2} \mp a_5(t-t_0)^2 \dots \end{aligned} \quad (2.2)$$

Let us note that we have a critical movable pole of the general solution and the three arbitrary constants t_0 , a_3 and a_5 for the expansion of the general solution in the Puiseux series. These arbitrary constants may correspond to first integrals with the leading terms of $v_t + m v^3$ and $v_{tt} + m v^2 v_t$ [22, 23].

With these observations in mind, we can look for the first integrals for Eq. (1.2) in the form

$$v_{tt} = S(t) - v^2 v_t - m v^3 - n v_t - p v, \quad (2.3)$$

where the coefficients m , n , p and the function $S(t)$ should be determined from the compatibility conditions with Eq. (1.2).

Differentiating Eq. (2.3) with respect to t and substituting V_{ttt} and v_{tt} into Eq. (1.2), we can find parameters m , n and p in the form

$$\begin{aligned} m_1 = \beta \delta - 1, \quad n_1 = \frac{1}{3} \beta \delta, \quad p_1 = \delta + \mu - \beta \delta, \\ m_2 = \gamma \mu - 1, \quad n_2 = \frac{1}{3} \gamma \mu, \quad p_2 = \delta + \mu - \gamma \mu. \end{aligned} \quad (2.4)$$

To obtain the equation for the function $S(t)$, we have to take into account the additional condition for parameters β , δ , γ and μ of the third-order differential equation (1.2) in the form

$$\mu (\beta \delta - \gamma \mu) = 0. \quad (2.5)$$

In this case we get the equation for the function $S(t)$ in the form

$$S_t + \beta \delta S = 0. \quad (2.6)$$

At $\mu = 0$ and $\beta \delta \neq \gamma \mu$ we obtain a first integral in the form

$$v_{tt} + v^2 v_t + (\beta \delta - 1) u_t + \frac{1}{3} u^3 + \delta(1 - \beta) u = C_1. \quad (2.7)$$

This first integral corresponds to the partial case of Eq. (1.2) at $\mu = 0$

$$v_{ttt} + (v^2 + \beta \delta - 1) v_{tt} + 2 v v_t^2 + (\beta \delta u^2 - \beta \delta + \delta) u_t = 0. \quad (2.8)$$

Let us forget how Eq. (1.2) was obtained, then the first integral of Eq. (1.2) at $\mu = \frac{\beta \delta}{\gamma}$ takes the form

$$v_{tt} + v^2 v_t + 2 v v_t^2 + (\beta \delta - 1) v_t + \frac{\beta \delta}{3} v^3 + \delta \left(1 + \frac{\beta}{\gamma} - \beta \right) v = C_1 e^{-\beta \delta t}. \quad (2.9)$$

This integral is also a special case of Eq. (1.2)

$$\begin{aligned} v_{ttt} + (v^2 + \beta \delta - 1) v_{tt} + 2 v v_t^2 + \beta \delta \left(2 v^2 + 2 \beta \delta - 2 + \frac{1}{\beta} + \frac{1}{\gamma} \right) v_t + \\ + \frac{\beta^2 \delta^2}{3} v^3 + \beta^2 \delta^2 \left(\frac{1}{\beta} + \frac{1}{\gamma} - 1 \right) v = 0. \end{aligned} \quad (2.10)$$

In the case $\delta = 0$ the system of equations is transformed into the system of two equations. At $\delta \neq 0$ we have $\gamma = -\beta$ and the first integral of Eq. (2.9) can be written as

$$v_t + \frac{1}{3} v^3 - v = (C_2 + C_1 t) e^{-\beta \delta t}. \quad (2.11)$$

Equation (1.2) at $\gamma = -\beta$ and $\mu = -\delta$ takes the form

$$v_{ttt} + (v^2 + \beta\delta - 1)v_t + 2v v_t^2 + 2\beta\delta(v^2 + \beta\delta - 1)v_t + \frac{\beta^2\delta^2}{3}v^3 - \beta^2\delta^2v = 0. \quad (2.12)$$

The Cauchy problem for this equation can be solved using Eq. (2.11) with two arbitrary constants C_1 and C_2 .

Let us consider the case $\mu = \frac{\beta\delta}{\gamma}$ using the system of equations (1.1). Substituting this condition into the system of equations (1.1), we have a system in the form

$$\begin{aligned} v_t &= v - \frac{1}{3}v^3 - w + y + I_{ext}, \\ w_t + \delta\beta w &= \delta v, \\ y_t + \beta\delta y &= \frac{\beta\delta\sigma}{\gamma}. \end{aligned} \quad (2.13)$$

Substituting $y - w$ from the first equation of the system of equations (2.13) into the difference of the two last Eqs. (2.13), we have a second-order differential equation in the form

$$v_{tt} + v^2 v_t + \beta\delta v_t - v_t - \beta\delta v - \beta\delta I_{ext} + \frac{1}{3}\beta\delta v^3 + \frac{\beta\delta}{\gamma}v_t - \frac{\beta\delta\sigma}{\gamma} = 0. \quad (2.14)$$

Taking into account the Fuchs index equal to $\frac{3}{2}$, we obtain a first integral in the form

$$v_t + \frac{1}{3}v^3 - v = I_{ext} + \frac{\sigma}{\beta} + C_2 e^{-\beta\delta t}. \quad (2.15)$$

Let us note that the first integral (2.15) coincides with the first integral of the third-order differential equation (1.2) at $C_1 = 0$. These first integrals can be used for construction of exact solutions for the system of equations (1.1).

3. Exact solutions of the FitzHugh – Rinzel model

It seems to us that the general solution of the nonautonomous first-order differential equation (2.11) cannot be found. We could not find a general solution of Eq. (2.11) in the handbooks [24–26]. Naturally, the general solution of the equation at $C_1 \neq 0$ and $C_2 \neq 0$ cannot be found by the conventional methods presented, for the example, in [27–31].

At $C_1 = C_2 = 0$ we have the Bernoulli equation from (2.11) in the form

$$v_t + \frac{1}{3}v^3 - v = 0, \quad (3.1)$$

which is linearized by means of transformations

$$v(t) = \frac{\sqrt{3}}{2\omega(t)^{1/2}}. \quad (3.2)$$

As a result, we have the equation

$$\omega_t + 2\omega = 1 \quad (3.3)$$

with the solution

$$\omega(t) = 1 + C_1 e^{-2t}. \quad (3.4)$$

Taking into account formula (3.2), we get a solution for $u(t)$ in the form

$$v(t) = \frac{\sqrt{3}}{\sqrt{1 + 3C_1 e^{-2t}}}. \quad (3.5)$$

Assuming $C_1 = 0$ and $C_2 \neq 0$, we get an equation from (2.11) in the form

$$v_t + \frac{1}{3} v^3 - v = C_2 e^{-\beta \delta t}. \quad (3.6)$$

Let us look for a solution of Eq. (3.6) in the form

$$v(t) = w(t) e^t. \quad (3.7)$$

Substituting expression (3.7) into (3.6), we obtain

$$w_t e^t + \frac{1}{3} w^3 e^{3t} = C_2 e^{-\beta \delta t}. \quad (3.8)$$

The last equation is transformed at $\beta \delta = -3$ to the following integral:

$$w(t) = (3C_2)^{1/3} F(t), \quad \int \frac{dF}{1 - F^3} = C_3 + \frac{C_2}{2(3C_2)^{1/3}} e^{2t}. \quad (3.9)$$

Calculating integral (3.9) with respect to F , we have a solution in the form

$$v(t) = e^t w(t) = e^t (3C_2)^{1/3} F(t), \quad (3.10)$$

where the function $F(t)$ is determined from the equation

$$\ln \left[\frac{F^2 + F + 1}{F^2 - 2F + 1} \right] + 2\sqrt{3} \arctan \left[\frac{\sqrt{3}}{3} (2F + 1) \right] = 6C_3 + \frac{6C_2}{2(3C_2)^{1/3}} e^{2t}. \quad (3.11)$$

The solution of the FitzHugh–Rinzel system of equations (3.9) has two arbitrary constants.

The authors of [15] considered the same dependence of t for functions $y(t)$ and $w(t)$ in the form

$$y = w - \frac{\sigma}{\beta}. \quad (3.12)$$

In this case there are two equations in the form

$$\begin{aligned} v_t + \frac{1}{3} v^3 - v &= 0, \\ w_t + \beta \gamma &= \delta v. \end{aligned} \quad (3.13)$$

The first equation of this system is linearized as we have shown before, and the second equation of the system is the first-order linear equation. However, this integrable case without the Painlevé property corresponds to the second-order differential equation of the FitzHugh–Rinzel system of equations.

4. Conclusion

In this note we have considered the FitzHugh–Rinzel model for description of the propagation from one neuron to another. This topic was studied in the paper [15], and this note has been written under the influence of this article.

The main interest in this system of equations is due to the fact that the general solution of this system, due to the specificity of the system, has an expansion of the general solution in the Puiseux series with three arbitrary constants. In fact, there is a local general (though nonmeromorphic) solution to this system of equations. This raises the question of the global solution and the existence of first integrals for the third-order equation obtained from the FitzHugh–Rinzel system of equations.

In this note we have shown that there are two nonautonomous first integrals for the third-order differential equation corresponding to the FitzHugh–Rinzel system of equations which allow us to obtain some exact solutions of this system.

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