NONLINEAR PHYSICS AND MECHANICS

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Vibroimpact Mobile Robot

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A simple model of a capsule robot is studied. The device moves upon a rough horizontal plane and consists of a capsule with an embedded motor and an internal moving mass. The motor generates a harmonic force acting on the bodies. Capsule propulsion is achieved by collisions of the inner body with the right wall of the shell. There is Coulomb friction between the capsule and the support, it prevents a possibility of reversal motion. A periodic motion is constructed such that the robot gains the maximal average velocity.

Keywords: capsule robots, vibroimpact dynamics, Coulomb friction

1. Introduction

Vibroimpact is a well-known instrument to solve various technical tasks ranging from hammering nails to ore mining. The reason is a high efficiency of impacts: they enable us to create large forces without much energy. Since the middle of the last century, the theory of impact oscillations has attracted considerable interest from researchers and has been impressively developed [1–4]. However, to date there have been few applications of this approach to mobile robots. So, a thermally actuated impact-driven locomotive microdevice was considered in [5]. This device is equipped with three legs, which collide with the ground in turn according to a control strategy. The idea of nonlegged locomotion, proposed by Chernous'ko [6], was developed in [7]: a robot-explorer moves like a self-impelling nail beneath Mars' surface. The next steps towards the vibroimpact model were made in [8–10]. In these papers, impact was interpreted as an interaction of a rigid body with a hard spring. In such a setting, analysis becomes rather complicated, thus numerical and experimental methods were implemented. The model of instantaneous collision of two rigid bodies was used in [11], where a sophisticated control pattern was proposed. It was shown that in theory the average velocity of the robot can be made as large as the power of the onboard motor allows.

In this paper, we consider the simplest device of vibroimpact type: it consists of a capsule carrying a motor and an internal moving mass. The motor generates a harmonic force, which causes the motion of the internal mass, accompanied by repeated collisions with the right wall

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of the capsule. It is generally accepted that such a system cannot perform directional motion for symmetry reasons. We show that asymmetry can be gained owing to collisions with just one of two walls. We look for parameters of the capsulbot that allow it to reach the highest average velocity.

2. Problem statement

2.1. System formulation

Fig. 1. The capsulbot model

We consider a device consisting of two rigid bodies, one inside another (Fig. 1). The outer body (capsule) of mass m_1 can move along a certain direction OX upon a rough flat surface. The system includes an onboard motor (not depicted) generating a periodic force $F(t)$ with period $\tau = 2\pi/\omega$, and

$$\Pi(t) = \int_0^t F(t)dt, \quad \Phi(t) = \int_0^t \Pi(t)dt, \quad (2.1)$$

applied to the interior mass m_2 (the opposite force $-F(t)$ acts on the capsule). There is Coulomb friction between the capsule and the ground according to the formula

$$T(v) = -T_0 \text{Sign } v, \quad \text{Sign } v = \begin{cases} \text{sgn } v & \text{if } v \neq 0, \\ [-1, 1] & \text{if } v = 0, \end{cases} \quad T_0 = \mu P, \quad (2.2)$$

where P is the total weight, v is the sliding velocity, and μ is the coefficient of friction. There is no friction between the bodies, but they can collide. We suggest that the parameters are chosen so that no contact is possible between the interior body and the left wall, but so that it can reach the right wall. In this case an instantaneous collision occurs. It will be described by the relation

$$u(t_0 + 0) = -eu(t_0 - 0), \quad (2.3)$$

where t_0 is the instant of the collision, u is the relative velocity, and $e \in [0, 1]$ is the coefficient of restitution.

2.2. Equations of motion

There exist several modes of motion, distinguished by the value of v : sliding ($v \neq 0$), rest ($v = 0$) and impact (v changes by jump). To describe them, we use the coordinates $x \in [0, L]$, equal to the relative displacement of the internal mass (here the boundary values correspond to the contact with a wall) and y being the distance of the capsule from the origin of the inertial frame OX . First note that the reaction N is balanced by the weight of both bodies, while the only external force is friction. Thus, the momentum Q satisfies the equation

$$\begin{aligned} \dot{Q} &= (m_1 + m_2)\dot{v} + m_2\dot{u} = T(v), \\ u &= \dot{x}, \quad v = \dot{y}. \end{aligned} \quad (2.4)$$

It is noteworthy that Eq. (2.4) does not include the motor force $F(t)$. To take it into account, we use Newton's law for m_2 :

$$m_2(\dot{u} + \dot{v}) = F(t). \quad (2.5)$$

Formula (2.3) of elastic collision implies the relations

$$\begin{aligned} v^+ &= v^- + \alpha(1+e)u^-, & u^+ &= -eu^-, \\ \alpha &= \frac{m_2}{M}, & M &= m_1 + m_2, \end{aligned} \quad (2.6)$$

which should be applied each time when $y = 0$ or $y = L$. The superscripts “minus” and “plus” correspond to the pre- and postimpact values.

2.3. Periodic solutions and optimality criterion

We look for solutions to the system (2.4)–(2.5) with period τ such that the average velocity of the capsule is maximal, i. e.,

$$\langle v \rangle = \frac{1}{\tau} \int_0^\tau v(t) dt \rightarrow \max. \quad (2.7)$$

It is assumed that mass m_1 and functions (2.1) are given, while the value m_2 is to be determined from condition (2.7). This means that we have got fixed equipment of the robot with motor and should choose the internal mass.

3. Construction of a periodic solution

3.1. Equations in dimensionless variables

We introduce a new independent variable $\varphi = \omega t$ and dimensionless variables

$$\bar{x} = \frac{x}{L}, \quad \bar{v} = \frac{v}{\omega L}, \quad \bar{F} = \frac{F}{\omega^2 ML}, \quad \bar{T} = \frac{T}{\omega^2 ML}. \quad (3.1)$$

Note that, in view of (2.7), there is no sense to study motions with return running of the capsule. Therefore, we restrict ourselves to considering motions with $v \geq 0$. It is assumed that each period consists of three phases: (i) $v = 0$, the internal mass is moving; (ii) collision with the right wall; (iii) sliding $v > 0$, terminated with a stop. Consider these phases in turn, denoting $x' = d\bar{x}/d\varphi$, etc.

(i) Equation (2.5) takes the form (retaining the previous notation)

$$\alpha x'' = F(\varphi). \quad (3.2)$$

The general solution to (3.2) is

$$x(\varphi) = C_1 + C_2\varphi + \frac{1}{\alpha}\Phi(\varphi) \quad (3.3)$$

with arbitrary constants C_1 and C_2 .

(ii) Since the collision follows the rest phase, formulas (2.6) become

$$v^+ = \alpha(1+e)u, \quad u(+0) = -eu. \quad (3.4)$$

Then the capsule starts moving right.

(iii) When $v > 0$, the system (2.4)–(2.5) becomes

$$v' + \alpha u' = -T_0, \quad \alpha(u' + v') = F(\varphi).$$

Therefore,

$$(1 - \alpha)v' = -T_0 - F(\varphi), \quad \alpha x'' = \frac{\alpha T_0 + F(\varphi)}{1 - \alpha}. \quad (3.5)$$

We set the impact instant as $\varphi = 0$, so $x(0) = 1$, $x'(-0) = u$, $x'(0) = -eu$. Then the solution to the first equation of (3.5) is

$$v = v^+ - \frac{T_0\varphi + \Pi(\varphi)}{1 - \alpha}. \quad (3.6)$$

Relation (3.6) holds as long as $v > 0$, then the capsule stops for $\varphi = \varphi_1$. Meanwhile, the second equation of (3.5) has the solution

$$\begin{aligned} x(\varphi) &= 1 - eu\varphi + T_0 \frac{\varphi^2}{2(1 - \alpha)} + \frac{1}{(1 - \alpha)\alpha} \Phi(\varphi), \\ x'(\varphi) &= -eu + T_0 \frac{\varphi}{1 - \alpha} + \frac{1}{(1 - \alpha)\alpha} \Pi(\varphi). \end{aligned} \quad (3.7)$$

Formula (3.7) is valid for $\varphi \in (0, \varphi_1)$, then it is replaced by relation (3.3).

3.2. Periodicity conditions

To derive these conditions, we should fit the boundary values of the variables at moments of switching phases. There are two such moments: $\varphi = \varphi_1$ and $\varphi = 2\pi$. We define φ_1 as the first positive root to Eq. (3.6). At this moment, the values (3.7) become

$$\begin{aligned} x(\varphi_1) &= 1 - eu\varphi_1 + T_0 \frac{\varphi_1^2}{2(1 - \alpha)} + \frac{1}{(1 - \alpha)\alpha} \Phi(\varphi_1), \\ x'(\varphi_1) &= -eu + T_0 \frac{\varphi_1}{1 - \alpha} + \frac{1}{(1 - \alpha)\alpha} \Pi(\varphi_1). \end{aligned} \quad (3.8)$$

In view of (3.6)

$$T_0\varphi_1 = -\Pi(\varphi_1) + (1 - \alpha)v^+,$$

which, with (3.4) and (3.11) taken into account, implies

$$T_0\varphi_1 = \Pi(\tau) - \Pi(\varphi_1). \quad (3.9)$$

Thus, the value φ_1 can be found as a root of the nonlinear equation (3.9). Integrating Eq. (3.2), we obtain

$$\alpha(x'(\varphi) - x'(\varphi_1)) = \Pi(\varphi) - \Pi(\varphi_1). \quad (3.10)$$

The first periodicity condition $x'(\tau) = u$ implies

$$\Pi(\tau) = \alpha(1 - \alpha)(1 + e)u. \quad (3.11)$$

We thus arrive at the following important conclusion:

To ensure the propulsion of the device, the average force $\langle F \rangle$ must be positive.

For example, a harmonic function is not relevant to generate the directed motion of the type considered.

Note that the left-hand side of (3.9) is the increasing linear function starting from the origin. The right-hand side due to (3.11) is positive for $\varphi = 0$ and vanishes for $\varphi = \tau$. Thus, this equation has at least one root within $(0, \tau)$.

To check the second periodicity condition $x(\tau) = 1$, we integrate Eq. (3.10):

$$x(\varphi) = x(\varphi_1) + (\varphi - \varphi_1)(-eu + v^+) + \frac{\Phi(\varphi) - \Phi(\varphi_1)}{\alpha}. \quad (3.12)$$

Hence,

$$1 - x(\varphi_1) - (\tau - \varphi_1)(-eu + v^+) = \frac{\Phi(\tau) - \Phi(\varphi_1)}{\alpha}. \quad (3.13)$$

Taking (3.8) into account, formula (3.13) transforms to

$$-T_0 \frac{\varphi_1^2}{2(1-\alpha)} + \frac{1}{(1-\alpha)} \Phi(\varphi_1) - (\tau - \varphi_1)v^+ + \tau eu = \frac{\Phi(\tau)}{\alpha}.$$

In view of (3.4) and (3.11), this relation has the form

$$\psi(\alpha) = \alpha \Phi(\varphi_1) - \frac{T_0}{2} \alpha \varphi_1^2 - \frac{((\tau - \varphi_1)\alpha(1+e) - \tau e)}{(1+e)} \Pi(\tau) - (1-\alpha)\Phi(\tau) = 0. \quad (3.14)$$

Equation (3.14) is linear in α , physically $\alpha \in (0, 1)$. To check this condition, we can substitute the values $\alpha = 0$ and $\alpha = 1$ in the left part. The results are

$$\begin{aligned} \psi(0) &= \frac{\tau e}{(1+e)} \Pi(\tau) - \Phi(\tau), \\ \psi(1) &= \Phi(\varphi_1) - \frac{T_0}{2} \varphi_1^2 - \frac{((\tau - \varphi_1)(1+e) - \tau e)}{(1+e)} \Pi(\tau). \end{aligned} \quad (3.15)$$

Then

$$\alpha = \frac{\psi(0)}{\psi(0) - \psi(1)}, \quad \alpha \in (0, 1) \Leftrightarrow \psi(0)\psi(1) < 0. \quad (3.16)$$

3.3. Optimality condition

Since the sliding velocity vanishes without the interval $\varphi \in (0, \varphi_1) \pmod{2\pi}$, the optimality condition (2.7) is

$$I = \int_0^{\varphi_1} v(\varphi) d\varphi \rightarrow \max,$$

where the value φ_1 is the first positive root to Eq. (3.9) and the integrand is defined by formula (3.6). In general, $F(t)$ depends on certain parameters. In such a case, problem (3.14) can be solved by appropriate optimization methods taking into account the geometric condition

$$x(\varphi) \geq 0, \quad \forall \varphi \in (0, \tau). \quad (3.17)$$

In view of (3.6),

$$v(\varphi) = \frac{\Pi(\tau) - T_0\varphi - \Pi(\varphi)}{1-\alpha} \Rightarrow I(\varphi_1) = \frac{1}{1-\alpha} \left(\varphi_1 \Pi(\tau) - \frac{T_0}{2} \varphi_1^2 - \Phi(\varphi_1) \right). \quad (3.18)$$

4. Basic example

In practice, the simplest force to generate by a motor is described by a harmonic law. However, in view of the previous considerations, such a law does not ensure directed motion. So we assume that $F(t)$ belongs to the family

$$F(t) = A \sin(\omega t + \varphi_0) + c, \quad A + c \leq T_0, \quad \varphi_0 \in [0, 2\pi]. \quad (4.1)$$

Here $c > 0$ is a constant that ensures positive displacement for one period of motion in accordance with (3.11). A possible physical interpretation of this value is the weak long spring, which presses the internal moving mass to the capsule. In fact, parameters α and c can be prearranged, but φ_0 cannot. This angle corresponds to a τ -periodic motion with impact at the moments $\varphi = 0 \pmod{2\pi}$. A priori, such a motion may be nonunique or not exist, so a detailed analysis is needed. In formulas (2.1)

$$\begin{aligned} \Pi(\varphi) &= -A(\cos(\varphi + \varphi_0) - \cos \varphi_0) + c\varphi, \quad \Pi(\tau) = c\tau, \\ \Phi(\varphi) &= -A(\sin(\varphi + \varphi_0) - \sin(\varphi_0) - \varphi \cos \varphi_0) + \frac{c}{2}\varphi^2. \end{aligned} \quad (4.2)$$

Now Eq. (3.9) is

$$T_0\varphi_1 = c(\tau - \varphi_1) - 2A \sin\left(\frac{\varphi_1}{2} + \varphi_0\right) \sin \frac{\varphi_1}{2}. \quad (4.3)$$

Formula (4.3) allows us to express φ_0 in terms of φ_1 in elementary functions (two-fold):

$$\varphi_0^1 = \operatorname{Arcsin} \frac{c(\tau - \varphi_1) - T_0\varphi_1}{2A \sin \frac{\varphi_1}{2}} - \frac{\varphi_1}{2}, \quad \varphi_0^2 = \pi - \varphi_1 - \varphi_0^1.$$

A sample graph $\varphi_1(\varphi_0)$ for $c = A = 0.5T_0$ is shown in Fig. 2: this function turns out to be single-valued for any $\varphi_0 \in [0, 2\pi]$. The next step is to find α from the linear equation (3.14). At last, we look for the maximum of $I(\varphi_1)$ in formula (3.15).

Let $e = 1$, then in (3.15)

$$\begin{aligned} \psi(0) &= -A\tau \cos \varphi_0, \\ \psi(1) &= \Phi(\varphi_1) - \frac{T_0}{2}\varphi_1^2 - \frac{\tau - 2\varphi_1}{2}c\tau, \\ \Phi(\varphi_1) &= -A(\sin(\varphi_1 + \varphi_0) - \sin(\varphi_0) - \varphi_1 \cos \varphi_0) + \frac{c}{2}\varphi_1^2 \end{aligned} \quad (4.4)$$

and in (3.18)

$$I(\varphi_1, A, c) = \frac{1}{1 - \alpha} \left(c\varphi_1\tau - \frac{T_0}{2}\varphi_1^2 - \Phi(\varphi_1) \right). \quad (4.5)$$

Numerical calculations show (Fig. 3) that for any admissible values A and c function (2.3) attains maximum close to the point where

$$\varphi_0^2 = \varphi_0^1 + 2\pi = \pi - \varphi_1 - \varphi_0^1 \quad \Rightarrow \quad \varphi_0^1 + \frac{\varphi_1}{2} = -\frac{\pi}{2}.$$

Then (4.3) becomes

$$T_0\varphi_1 = c(\tau - \varphi_1) + 2A \sin \frac{\varphi_1}{2}.$$

For instance, if $c/T_0 = 0.03$, $A/T_0 = 0.2$, then

$$3\varphi_1 = 2\pi + 2 \sin \frac{\varphi_1}{2} \quad \Rightarrow \quad \varphi_1 \simeq 2.748, \quad \varphi_0 \simeq -2.945$$

Fig. 2. Solution to Eq. (4.3)

Fig. 3. A sample graph of the function (4.5)

and $\alpha \simeq 0.5$, while the objective function (4.5) turns out to be

$$I(\varphi_1, A, c) = \frac{5}{6}\bar{T} \left(\varphi_1\tau - \varphi_1^2 - \frac{\Phi(\varphi_1)}{T_0} \right) \simeq 7.65\bar{T},$$

where the dimensionless value $\bar{T}$ is defined by equation (3.1).

To check the geometric condition (3.17), we use formulas (3.7) for $\varphi \in [0, \varphi_1]$ and (3.12) for $\varphi \in [\varphi_1, \tau]$.

5. Discussion

Along with energy concentration, a feature widely used in practice, the phenomenon of impact is characterized by simplicity of description. We have considered a vibroimpact model of a mobile device without external movers, which is based on these advantages. This model has a minimal configuration: a shell, a motor, and a moving internal mass. Due to nonlinearity, inherent in impacts and friction, such a system may exhibit rather complicated dynamics, e.g., grazing bifurcation and chaos. This paper analyzes only periodic motions with one collision per

period. It is shown that the periodicity conditions require an asymmetry of the driving force. This property allows one to determine the unique preimpact velocity of the internal mass, then the mass ratio satisfies a linear equation. Therefore, for a given periodic driving force (depending on the onboard motor) we can construct the motion in question. Note that we suggest that in the absence of impact the device cannot move as the motor is too weak to overcome friction. A basic example of the harmonic force is observed. Here an explicit formula for the average velocity has been derived. This value can be maximized by an appropriate choice of certain parameters.

Note that the numerical results in [8] indicate the existence of unique periodic motion with the largest propulsion, which grows as the excitation amplitude increases. In [9–11], optimization is achieved by an appropriate choice of the excitation (nonharmonic) law. Here only a basic example is presented, which indicates the possibility of optimization for simple harmonic excitation by choosing the moments of collisions.

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