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The Problem of Balancing an Inverted Spherical Pendulum on an Omniwheel Platform

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This paper addresses the problem of balancing an inverted pendulum on an omnidirectional platform in a three-dimensional setting. Equations of motion of the platform – pendulum system in quasi-velocities are constructed. To solve the problem of balancing the pendulum by controlling the motion of the platform, a hybrid genetic algorithm is used. The behavior of the system is investigated under different initial conditions taking into account a necessary stop of the platform or the need for continuation of the motion at the end point of the trajectory. It is shown that the solution of the problem in a two-dimensional setting is a particular case of three-dimensional balancing.

Keywords: balancing of an inverted pendulum, omnidirectional platform, hybrid genetic algorithm, Poincaré equations in quasi-velocities

Introduction

The modeling of complex dynamical systems is, as a rule, the first step towards investigating their behavior and controllability. The problem of controlling a complex dynamical system is a nontrivial problem of dynamics and optimal control. One of the classical problems is that of balancing an inverted pendulum [1]. The center of mass of the inverted pendulum is located above

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the point of its support and attachment. This makes its equilibrium unstable, so that constant balancing is required to hold the pendulum in the vertical position. An inverted pendulum is a mathematical abstraction (model) of many real-world systems [2] such as reusable missile systems [3], Segway, wheeled robots [4] and walking robots [5, 6], and is a classical problem in automatic control.

The model that has received the most attention in the relevant literature to date is the model of a two-dimensional pendulum whose point of support is fastened on a platform capable of moving in a horizontal line along some axis, and the problem on which the focus is placed is that of balancing the pendulum by controlling the motion of the platform. The equations of motion of such a system are presented in [7]. The problem of balancing an inverted pendulum in a two-dimensional setting is often used as a test problem to compare the efficiency of optimization and control algorithms [8]. The condition under which the platform is constrained to move only along one axis simplifies the problem considerably, and in this setting it is solved successfully by both classical and modern methods using fuzzy, neural network and evolutionary approaches.

The classical approach to solving the problem is based on the application of PID [9] and LQR [10] controllers or their combination [11]. The PID controller in [12] and the LQR controller in [13] are applied to solve the problem of stabilizing and controlling the motion of the system with an inverted pendulum along a prescribed trajectory.

Another well-known approach is based on the use of the Bellman equation. For example, S. Kumar [14] considers the balancing of a CartPole system with reinforcement learning by comparing it with the deep Q-neural network algorithm.

Artificial neural networks offer a variety of capabilities to construct complex controllers, therefore they find wide application at present, and various approaches are proposed. The authors of [15] use the convolutional neural network for generation of new pole-balancing networks (which play the role of artificially intelligent agents) without direct access to training data. Another possibility, that of applying the neuroevolutionary approach NEAT for growing the topology of the neural network in developing a controller for stabilization of an inverse pendulum, is shown in [16].

A fuzzy controller based on Mamdani's inference system for controlling a two-wheeled balancing robot is proposed in [4]. A similar fuzzy controller can be constructed immediately in PLD, as shown in [17], where it is able to successfully maintain control over a window of $-10 < \theta < +10$ degrees. In [18] it is shown that fuzzy control is more accurate in reference tracking, faster in response and more robust to disturbance than the PID and LQR controllers.

Much less attention has been given in the literature to the problem of balancing an inverted pendulum in a three-dimensional setting [3, 19], which is by far more complex and closer to real-world systems. The problem of controlling and balancing walking robots [5, 6], where the zero moment point (ZMP) based trajectory and the linear inverted pendulum model (LIPM) reduce the complexity of the dynamics for control purposes, is also a three-dimensional problem. In the case where the pendulum is placed on a cart, the question arises how to organize the motion of the cart in space. One possibility is to use a platform on omniwheels. This method of locomotion has been successfully applied in various systems and holds much promise for development of mobile robots and systems [20, 21]. A key feature is the complete controllability of such a cart: in [22, 23] it is shown that, if the cart is supported by no less than three driving omniwheels so that not all direction vectors of the axes of the wheels' rollers are parallel and the points of contact of the wheels with the plane do not lie on the same straight line, then one can always choose control torques in such a way as to ensure the necessary motion. The complexity of the problem of balancing the pendulum on an omnidirectional cart stems not only from the fact that the motion occurs in space and the dimension of the problem, as well as the number of control parameters, is larger than that in a two-dimensional setting, but also from the fact that the omnidirectional cart in itself is not an abstract platform, but a complex system with a set of real

physical characteristics and with their own laws of motion which must be taken into account in constructing the mathematical model.

In this paper we present a solution to the problem of spatial balancing of the pendulum on a platform with omniwheels using a hybrid genetic algorithm [24, 25], which has proved to be efficient in solving complex problems of global optimization.

1. The mathematical model

1.1. Kinematic relations

Consider the motion of an omnidirectional platform on a horizontal plane with an inverted spherical pendulum placed on it (see Fig. 1). We make the following assumptions concerning the system under consideration:

- 1° The platform moves without slipping of the wheels.
- 2° The pendulum is an axisymmetric rigid rod one end of which is fastened on a spherical hinge located on the platform. In addition, the rod cannot rotate about its symmetry axis.

To describe the motion of the system, we introduce three coordinate systems (see Fig. 1):

- a fixed (inertial) coordinate system $OXYZ$. We assume that the axes OX and OY are parallel to the plane on which the platform moves, and that the axis OZ is directed vertically upwards.
- a moving coordinate system $Cxyz$. Assume that point C is at the center of mass of the platform, the axes Cx and Cy lie in the plane of the platform, and the axis Cz is codirectional with the axis OZ .
- a moving coordinate system $C_px_1x_2x_3$ attached to the pendulum. We assume that the axis C_px_3 is directed along the symmetry axis of the rod, and point C_p coincides with the point of support of the pendulum.

Fig. 1. (a) A schematic representation of the omnidirectional platform, the fixed coordinate system $OXYZ$, and the moving coordinate system $Cxyz$ attached to the platform. The projection of the axis OZ coincides with point O , and the projection of the axis Cz coincides with point C . (b) A schematic representation of the platform, the spherical pendulum, the coordinate system $OXYZ$, the moving coordinate system $Cxyz$ attached to the platform, and the moving coordinate system $C_px_1x_2x_3$ attached to the pendulum

We will specify the position of the platform relative to the fixed coordinate system by the radius vector $\mathbf{R} = (X, Y, 0)$ of point C , and its orientation, by the angle σ between the positive directions of the axes OX and Cx which is measured from the axis OX counterclockwise. For the system considered here the motion of the platform is plane-parallel. Let $\mathbf{v} = (v_1, v_2, 0)$ denote the absolute velocity vector of the center of mass of the platform (point C), where v_1 and v_2 are the projections of this vector onto the moving axes Cx and Cy , and $\mathbf{\Omega} = (0, 0, \Omega)$ is the angular velocity vector of the platform. The following kinematic relations [26] hold:

$$\dot{X} = v_1 \cos \sigma - v_2 \sin \sigma, \quad \dot{Y} = v_1 \sin \sigma + v_2 \cos \sigma, \quad (1.1)$$

$$\dot{\sigma} = \Omega. \quad (1.2)$$

REMARK 1. In the general case, the platform can have any number N of omniwheels. In constructing the mathematical models, we will not fix the quantity N .

As in [29], we specify the position of the k th omniwheel relative to the platform by the radius vector $\mathbf{r}_k$. Let $\boldsymbol{\tau}_k$ and $\mathbf{n}_k$ denote the tangent and the normal unit vectors to the plane of the k th wheel, respectively, and $\mathbf{a}_k = (a_k^1, a_k^2, 0)$ the unit vector directed along the axis of attachment of the rollers (see Fig. 2). Assume that the vector $\boldsymbol{\tau}_k \times \mathbf{n}_k$ is directed vertically upwards. Also, let Ψ_k denote the angle of rotation of the k th wheel, which is measured clockwise when viewed from the top of the vector $\mathbf{n}_k$.

Fig. 2. A schematic representation of the omniwheel and the vectors characterizing it

To construct a nonholonomic constraint describing the no-slip condition of the k th wheel, we calculate the velocity of the motion of its axis

$$\mathbf{v}_a = \mathbf{v} + \mathbf{\Omega} \mathbf{J} \mathbf{r}_k, \quad \mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k = 1, \dots, N \quad (1.3)$$

and the velocity of the contact point

$$\mathbf{v}_q = \mathbf{v}_a + \dot{\Psi}_k h \boldsymbol{\tau}_k, \quad k = 1, \dots, N, \quad (1.4)$$

where $h > 0$ is the radius of the wheel.

The no-slip condition of the k th wheel is defined by the expression

$$(\mathbf{v}_q, \mathbf{a}_k) = 0, \quad k = 1, \dots, N, \quad (1.5)$$

which, in view of (1.3) and (1.4), can be represented in the following form:

$$f_k = \dot{\Psi}_k + \frac{1}{hs_k}(\mathbf{v} + \Omega \mathbf{J} \mathbf{r}_k, \mathbf{a}_k) = 0, \quad k = 1, \dots, N, \quad (1.6)$$

where $s_k = (\boldsymbol{\tau}_k, \mathbf{a}_k) \neq 0$.

Let $\mathbf{d} = (d_1, d_2, 0)$ denote the radius vector of the point of support, C_p , of the spherical pendulum, referred to the axes of the coordinate system $Cxyz$. The spherical pendulum can be considered as a Lagrange top rotating with angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, on which we impose the nonholonomic constraint $\omega_3 = 0$ prohibiting the rotation about the symmetry axis [26]. In what follows, we will use the following notation for this nonholonomic constraint:

$$f_0 = \omega_3 = 0. \quad (1.7)$$

In addition, we assume that the vector $\boldsymbol{\omega}$ is referred to the axes of the moving coordinate system $C_p x_1 x_2 x_3$.

To describe the motion of the pendulum, we let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ denote the vectors directed along the axes of the fixed coordinate system. Their projections onto the axes of the moving coordinate system $C_p x_1 x_2 x_3$ define the orthogonal transfer matrix:

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in \text{SO}(3). \quad (1.8)$$

REMARK 2. We note that the coordinates $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are redundant. However, they turn out to be more convenient for numerical calculations than the spherical coordinates since they have no singular points in contrast to the spherical coordinates and Euler angles.

The evolution of the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ is governed by the Poisson equations

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad (1.9)$$

in which the nonholonomic constraint (1.7) should be taken into account.

1.2. The Lagrangian of the system

The motion of the system under study is described by the Lagrangian

$$L = T - U, \quad (1.10)$$

where T is the kinetic energy and U is the potential energy. In constructing the Lagrangian we will assume that the wheels possess the same mass and the same tensor of inertia.

We write the components of the kinetic energy of the system:

– the kinetic energy of the platform

$$T^c = \frac{1}{2} m_c (\mathbf{v}, \mathbf{v}) + \frac{1}{2} I_c \Omega^2, \quad (1.11)$$

where m_c is the mass of the platform and I_c is the moment of inertia of the platform relative to the axis C_z ;

– the kinetic energy of the k th wheel

$$T^k = \frac{1}{2}m_w(\mathbf{v}, \mathbf{v}) + m_w\Omega(\mathbf{v}, \mathbf{J}\mathbf{r}_k) + \frac{1}{2}\left(\tilde{I}_w + m_w(\mathbf{r}_k, \mathbf{r}_k)\right)\Omega^2 + \frac{1}{2}I_w\dot{\Psi}_k^2,$$

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1.12)$$

where m_w is the mass of the wheel, $\tilde{I}_w$ is the moment of inertia of the wheel relative to the diameter, I_w is the moment of inertia of the wheel relative to the axis, $\mathbf{r}_k = (x_k, y_k, 0)$ is the radius vector of the center of mass of the k th wheel in the coordinate system $Cxyz$, and Ψ_k is the angle of rotation of the k th wheel relative to its axis;

– the kinetic energy of the pendulum

$$T^p = \frac{1}{2}m_p(\mathbf{v}, \mathbf{v}) + m_p\Omega(\mathbf{v}, \mathbf{J}\mathbf{d}) + \frac{1}{2}m_p(\mathbf{d}, \mathbf{d})\Omega^2 +$$

$$+ m_pl((v_1 - d_2\Omega)\cos\sigma - (v_2 + d_1\Omega)\sin\sigma)(\alpha_1\omega_2 - \alpha_2\omega_1) +$$

$$+ m_pl((v_1 - d_2\Omega)\sin\sigma + (v_2 + d_1\Omega)\cos\sigma)(\beta_1\omega_2 - \beta_2\omega_1) +$$

$$+ \frac{1}{2}\frac{4m_pl^2}{3}(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_p\omega_3^2, \quad (1.13)$$

where m_p is the mass of the pendulum, l is the half length of the pendulum, and I_p is the moment of inertia of the pendulum relative to its symmetry axis.

The potential energy of the pendulum has the form

$$U = m_pgl\gamma_3, \quad (1.14)$$

where g is the free-fall acceleration.

Thus, the Lagrangian of the system takes the form

$$L = \frac{1}{2}m(\mathbf{v}, \mathbf{v}) + m\Omega(\mathbf{v}, \mathbf{J}\mathbf{z}) + \frac{1}{2}I\Omega^2 + \frac{1}{2}I_w\sum_{k=1}^N\dot{\Psi}_k^2 +$$

$$+ m_pl((v_1 - d_2\Omega)\cos\sigma - (v_2 + d_1\Omega)\sin\sigma)(\alpha_1\omega_2 - \alpha_2\omega_1) +$$

$$+ m_pl((v_1 - d_2\Omega)\sin\sigma + (v_2 + d_1\Omega)\cos\sigma)(\beta_1\omega_2 - \beta_2\omega_1) +$$

$$+ \frac{1}{2}\frac{4m_pl^2}{3}(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_p\omega_3^2 - m_pgl\gamma_3 \quad (1.15)$$

where

$$m = m_c + Nm_w + m_p, \quad \mathbf{z} = (z_1, z_2, 0) = \frac{m_p\mathbf{d} + m_w\sum_{k=1}^N\mathbf{r}_k}{m},$$

$$I = I_c + N\tilde{I}_w + m_w\sum_{k=1}^N(\mathbf{r}_k, \mathbf{r}_k) + m_p(\mathbf{d}, \mathbf{d}).$$

1.3. Equations of motion

In classical mechanics the motion of nonholonomic systems is described by the Lagrange equations of the second kind with undetermined multipliers, which relate generalized accelerations to generalized coordinates and velocities [27]. As generalized coordinates in the system one can choose the quantities X , Y and σ , which specify the position of the platform, and the Euler angles $\psi \in [0, 2\pi)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, which parameterize the matrix (1.8) as follows [26]:

$$\begin{aligned}\alpha &= (\cos \varphi \cos \psi - \cos \theta \sin \psi \sin \varphi, -\sin \varphi \cos \psi - \cos \theta \sin \psi \cos \varphi, \sin \theta \sin \psi), \\ \beta &= (\cos \varphi \sin \psi + \cos \theta \cos \psi \sin \varphi, -\sin \varphi \sin \psi + \cos \theta \cos \psi \cos \varphi, -\sin \theta \cos \psi), \\ \gamma &= (\sin \varphi \sin \theta, \cos \varphi \sin \theta, \cos \theta).\end{aligned}\quad (1.16)$$

However, instead of the generalized velocities $\dot{\mathbf{q}} = (\dot{X}, \dot{Y}, \dot{\sigma}, \dot{\psi}, \dot{\theta}, \dot{\varphi})$ it is more convenient to use quasi-velocities $\mathbf{w} = (v_1, v_2, \Omega, \omega_1, \omega_2, \omega_3)$.¹ In the problem of interest, the generalized velocities and quasi-velocities are related by the following linear equations:

$$\dot{q}_i = \sum_{s=1}^6 u_i^s w_s, \quad (1.17)$$

where u_i^s are the components of the following vector fields:

$$\begin{aligned}\mathbf{u}^1 &= \cos \sigma \frac{\partial}{\partial X} + \sin \sigma \frac{\partial}{\partial Y}, & \mathbf{u}^2 &= -\sin \sigma \frac{\partial}{\partial X} + \cos \sigma \frac{\partial}{\partial Y}, \\ \mathbf{u}^3 &= \frac{\partial}{\partial \sigma}, & \mathbf{u}^4 &= \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi} + \cos \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ \mathbf{u}^5 &= \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi} - \sin \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, & \mathbf{u}^6 &= \frac{\partial}{\partial \varphi}.\end{aligned}\quad (1.18)$$

REMARK 3. The vector fields $\mathbf{u}^4$ and $\mathbf{u}^5$ have singularities for $\theta = 0$ and $\theta = \pi$. When one transforms to the redundant coordinates α , β and γ , these singularities disappear and do not influence further calculations.

We note that the vector fields (1.18) are left-invariant under the groups $SE(2)$ and $SO(3)$, respectively. A consequence of the left invariance is that there exist constant values of the structural coefficients c_{ij}^s :

$$\begin{aligned}c_{31}^2 &= c_{23}^1 = c_{64}^5 = c_{45}^6 = c_{56}^4 = 1, \\ c_{32}^1 &= c_{13}^2 = c_{54}^6 = c_{65}^4 = c_{46}^5 = -1,\end{aligned}\quad (1.19)$$

which guarantee that the commutation relations (A.3) are satisfied.

¹The quantities $v_1, v_2, \Omega, \omega = (\omega_1, \omega_2, \omega_3)$ are quasi-velocities because they are not the derivatives of any coordinates, in contrast to the generalized velocities $\dot{X}, \dot{Y}, \dot{\sigma}, \dot{\psi}, \dot{\theta}, \dot{\varphi}$.

Thus, taking (1.19) into account, according to Appendix A we obtain the following equations governing the evolution of the quasi-velocities:

$$\begin{aligned}
 \frac{d}{dt} \frac{\partial L}{\partial v_1} &= \frac{\partial L}{\partial v_2} \Omega + \mathbf{u}^1(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial v_1}, & \frac{d}{dt} \frac{\partial L}{\partial v_2} &= -\frac{\partial L}{\partial v_1} \Omega + \mathbf{u}^2(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial v_2}, \\
 \frac{d}{dt} \frac{\partial L}{\partial \Omega} &= \frac{\partial L}{\partial v_1} v_2 - \frac{\partial L}{\partial v_2} v_1 + \mathbf{u}^3(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial \Omega}, \\
 \frac{d}{dt} \frac{\partial L}{\partial \omega_1} &= \frac{\partial L}{\partial \omega_2} \omega_3 - \frac{\partial L}{\partial \omega_3} \omega_2 + \mathbf{u}^4(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial \omega_1}, \\
 \frac{d}{dt} \frac{\partial L}{\partial \omega_2} &= \frac{\partial L}{\partial \omega_3} \omega_1 - \frac{\partial L}{\partial \omega_1} \omega_3 + \mathbf{u}^5(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial \omega_2}, \\
 \frac{d}{dt} \frac{\partial L}{\partial \omega_3} &= \frac{\partial L}{\partial \omega_1} \omega_2 - \frac{\partial L}{\partial \omega_2} \omega_1 + \mathbf{u}^6(L) + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial \omega_3},
 \end{aligned} \tag{1.20}$$

where L is the Lagrangian (1.15).

Equations (1.20) need to be supplemented with equations describing the rotation of the wheels:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\Psi}_k} = \frac{\partial L}{\partial \Psi_k} + \sum_{j=0}^N \lambda_j \frac{\partial f_j}{\partial \dot{\Psi}_k} + M_k, \quad k = 1, \dots, N, \tag{1.21}$$

where M_k is the torque applied to the k th wheel.

Let us write the sixth equation of (1.20) in the explicit form

$$I_p \dot{\omega}_3 = \lambda_0. \tag{1.22}$$

Differentiating the constraint (1.7) yields

$$\lambda_0 = 0. \tag{1.23}$$

Differentiating the nonholonomic constraints (1.6) and substituting the corresponding expressions into equations (1.21), we obtain explicit expressions for the undefined multipliers:

$$\lambda_k = -M_k - \frac{I_w}{h s_k} (\dot{\mathbf{v}} + \dot{\Omega} \mathbf{J} \mathbf{r}_k, \mathbf{a}_k), \quad k = 1, \dots, N. \tag{1.24}$$

Taking into account the expressions (1.23) and (1.24), the first five of equations (1.20) can be represented as

$$\mathbf{S} \dot{\mathbf{w}} = \mathbf{f} + \sum_{k=1}^N M_k \mathbf{g}_k, \tag{1.25}$$

where the nonzero coefficients of the matrix $\mathbf{S}$ are defined by the following expressions:

$$\begin{aligned} s_{11} &= m + \sum_{k=1}^N \frac{I_w(a_k^1)^2}{h^2 s_k^2}, \quad s_{12} = s_{21} = \sum_{k=1}^N \frac{I_w a_k^1 a_k^2}{h^2 s_k^2}, \quad s_{13} = s_{31} = -m z_2 + \sum_{k=1}^N \frac{I_w a_k^1 (\mathbf{J} \mathbf{r}_k, \mathbf{a}_k)}{h^2 s_k^2}, \\ s_{14} &= s_{41} = -m_p l (\alpha_2 \cos \sigma + \beta_2 \sin \sigma), \quad s_{15} = s_{51} = m_p l (\alpha_1 \cos \sigma + \beta_1 \sin \sigma), \\ s_{22} &= m \sum_{k=1}^N \frac{I_w(a_k^2)^2}{h^2 s_k^2}, \quad s_{23} = s_{32} = m z_1 + \sum_{k=1}^N \frac{I_w a_k^2 (\mathbf{J} \mathbf{r}_k, \mathbf{a}_k)}{h^2 s_k^2}, \\ s_{24} &= s_{42} = -m_p l (-\alpha_2 \sin \sigma + \beta_2 \cos \sigma), \quad s_{25} = s_{52} = m_p l (-\alpha_1 \sin \sigma + \beta_1 \cos \sigma), \\ s_{33} &= I + \sum_{k=1}^N \frac{I_w (\mathbf{J} \mathbf{r}_k, \mathbf{a}_k)^2}{h^2 s_k^2}, \\ s_{34} &= s_{43} = m_p l (d_2 (\alpha_2 \cos \sigma + \beta_2 \sin \sigma) - d_1 (-\alpha_2 \sin \sigma + \beta_2 \cos \sigma)), \\ s_{35} &= s_{53} = -m_p l (-d_2 (\alpha_1 \cos \sigma + \beta_1 \sin \sigma) - d_1 (-\alpha_1 \sin \sigma + \beta_1 \cos \sigma)), \\ s_{44} &= s_{55} = \frac{4m_p l^2}{3}, \end{aligned}$$

and the components of the vectors $\mathbf{f}$ and $\mathbf{g}_k$ have the form

$$\begin{aligned} f_1 &= m \Omega (v_2 + z_1 \Omega) + m_p l (\omega_1^2 + \omega_2^2) (\alpha_3 \cos \sigma + \beta_3 \sin \sigma), \\ f_2 &= -m \Omega (v_1 - z_2 \Omega) + m_p l (\omega_1^2 + \omega_2^2) (-\alpha_3 \sin \sigma + \beta_3 \cos \sigma), \\ f_3 &= -m \Omega (z_1 v_1 + z_2 v_2) + m_p l (\omega_1^2 + \omega_2^2) (d_2 (\alpha_3 \cos \sigma - \beta_3 \sin \sigma) + d_1 (\alpha_3 \sin \sigma + \beta_3 \cos \sigma)), \\ f_4 &= m_p l (-\alpha_2 \sin \sigma + \beta_2 \cos \sigma) \Omega (v_1 - d_2 \Omega) - m_p l (\alpha_2 \cos \sigma + \beta_2 \sin \sigma) \Omega (v_2 + d_1 \Omega) + m_p l g \gamma_2, \\ f_5 &= -m_p l (-\alpha_1 \sin \sigma + \beta_1 \cos \sigma) \Omega (v_1 - d_2 \Omega) + m_p l (\alpha_1 \cos \sigma + \beta_1 \sin \sigma) \Omega (v_2 + d_1 \Omega) - m_p l g \gamma_1, \\ \mathbf{g} &= \left(-\frac{a_k^1}{h s_k}, -\frac{a_k^2}{h s_k}, -\frac{(\mathbf{J} \mathbf{r}_k, \mathbf{a}_k)}{h s_k}, 0, 0 \right)^T. \end{aligned}$$

Fig. 3. The configuration of the omnidirectional platform

Next, using Eqs. (1.25), we address the problem of bringing the pendulum to the vertical position. As an example we consider a four-wheeled platform which is characterized by the

symmetric position of the wheels (see Fig. 3):

$$\begin{aligned} \mathbf{r}_1 &= (a, b, 0), \quad \mathbf{r}_2 = (-a, b, 0), \quad \mathbf{r}_3 = (-a, -b, 0), \quad \mathbf{r}_4 = (a, -b, 0), \\ \mathbf{a}_1 &= \mathbf{a}_3 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \quad \mathbf{a}_2 = \mathbf{a}_4 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \\ \boldsymbol{\tau}_k &= (0, 1, 0), \quad \mathbf{n}_k = (-1, 0, 0) \quad k = 1, 2, 3, 4. \end{aligned} \quad (1.26)$$

In addition, we assume that the point of support of the pendulum is at the center of mass of the platform, i. e., $\mathbf{d} = 0$, and hence, in view of (1.26), $\mathbf{z} = 0$. We also fix the following parameter values:

$$m_c = 3.5, \quad m_w = 0.05, \quad m_p = 0.1, \quad a = l = 0.24, \quad b = 0.15. \quad (1.27)$$

2. The problem of balancing the pendulum

Consider the problem of balancing the pendulum in a given time interval T by controlling the value of the torques M_1, M_2, M_3 and M_4 applied to the omniwheels. In addition, we require that the values of velocities $v_1, v_2, \Omega, \omega_1, \omega_2$ and perhaps of the angle of rotation of the platform, σ , are the same at the initial and final times $t = 0$ and $t = T$.

Thus, we have the Mayer problem with the fixed end:

$$\begin{aligned} F &= (\gamma_3(T) - 1)^2 + (v_1(T) - v_1(0))^2 + (v_2(T) - v_2(0))^2 + (\Omega(T) - \Omega(0))^2 + \\ &+ (\omega_1(T) - \omega_1(0))^2 + (\omega_2(T) - \omega_2(0))^2 + p((\sigma(T) - \sigma(0)) \bmod 2\pi)^2 \rightarrow \min \end{aligned} \quad (2.1)$$

and with the restrictions (1.25) and (1.2). In the functional (2.1) we have $p = 1$ if it is required that the orientation of the platform be the same at the initial and final times, and $p = 0$ if it is not required that the above restriction be satisfied.

We will also require that in the process of motion the pendulum not fall on the surface of the platform:

$$\gamma_3(t) > 0, \quad t \in [0, T]. \quad (2.2)$$

To numerically solve the problem (2.1), (1.25), (1.2), we introduce on the interval $[0, T]$ discrete time $t_j, j = 0, \dots, n$ with a constant step τ , and represent the control actions using piecewise linear functions:

$$M_k(t) = M_k(t_j) + (t - t_j) \frac{M_k(t_{j+1}) - M_k(t_j)}{t_{j+1} - t_j}, \quad t \in [t_j, t_{j+1}), \quad t_j = \frac{jT}{n}. \quad (2.3)$$

Since in practice control torques are bounded quantities, we assume that

$$|M_k(t_j)| \leq 1, \quad k = 1, 2, 3, 4, \quad j = 0, \dots, n. \quad (2.4)$$

Next, we define the values of the controls $M_k(t_j), k = 1, \dots, 4, j = 0, \dots, n$ which give a minimum to the functional (2.1) under the restrictions (1.25), (1.2) and (2.4). Thus, the initial problem reduces to that of constrained minimization with $4(n+1)$ unknowns.

To solve the problem described above, we apply a hybrid genetic algorithm [24, 25] based on the genetic algorithm with real encoding with additional training of the leader of the population using the Davidon–Fletcher–Powell variable metric method.

Each individual (chromosome) of the genetic algorithm consists of a set of control torques $M_k(t_j), k = 1, \dots, 4, j = 0, \dots, n$, and accordingly has dimension $4(n+1)$. In the theory

of genetic algorithms the value of the functional (2.1) for a concrete individual is called *Fitness*. In order to take the restriction (2.2) into account, we will assume that

$$Fitness = \begin{cases} F(T), & \gamma_3(t_j) > 0, \forall t_j, j = 0, \dots, n, \\ 1000, & \text{otherwise.} \end{cases} \quad (2.5)$$

To estimate the fitness of individuals and each calculation of the value of the target function, it is necessary to solve the system of differential equations (1.25) and (1.2). To solve it numerically, we have used the Runge–Kutta–Verner method of the sixth order of accuracy with time increment $\Delta t = 0.01$ s.

Next, we consider several cases that differ in the initial conditions and in the motion pattern of the platform.

3. Numerical experiments

3.1. Balancing of the pendulum under deviation in one of the coordinate planes

We consider the case where the platform moves parallel to the axis OY and the pendulum lies in the plane Cyz .² The control can be chosen so as to allow the pendulum to be brought to the vertical position.

1. The motion of the platform in the direction $\pm OY$ can be implemented using the following controls:

$$M_1(t_k) = M_3(t_k) = \mp u(t_k), \quad M_2(t_k) = M_4(t_k) = \pm u(t_k), \quad (3.1)$$

where $u(t_k) > 0$. Thus, in the case of one-dimensional motion the dimension of the optimization problem decreases four-fold.

REMARK 4. The motion in the direction $\pm OX$ can be implemented using the following controls:

$$M_1(t_k) = M_2(t_k) = M_3(t_k) = M_4(t_k) = \mp u(t_k).$$

To show the operation of the numerical algorithm, we choose the following initial conditions³:

$$\begin{aligned} X(0) = Y(0) = 0, \quad \sigma(0) = 0, \quad v_1(0) = v_2(0) = 0, \quad \Omega(0) = 0, \\ \omega_1(0) = \omega_2(0) = 0, \quad \psi(0) = \varphi(0) = 0, \quad \theta(0) = -45^\circ. \end{aligned} \quad (3.2)$$

We will also assume that in the functional (2.1) $p = 0$.

The dependence of the nutation angle $\theta(t)$, which characterizes the deviation of the pendulum from the vertical, is shown in Fig. 4a. During motion the precession angle ψ and the angle of proper rotation φ preserve zero values. The dependences of the translational velocity of the platform v_2 and the angular velocity of the pendulum ω_1 are presented in Figs. 4b and 4c, respectively. The dependence of the control torque $u(t)$ applied to all wheels according to (3.1) is shown in Fig. 4d. The decrease in the absolute values of the velocities of the platform and the pendulum as $t \rightarrow T$ is indicative of their stop at the end of the trajectory. The dependence of the ordinate of the center of the platform $Y(t)$ is shown in Fig. 4e. During motion the component of the translational velocity v_1 and the angular velocity of the platform Ω preserve zero values,

²This case is an analog of the classical problem of balancing an inverted pendulum.

³It is convenient to specify the initial position of the pendulum by using Euler angles and then transforming them to the redundant variables α, β, γ .

Fig. 4. The result of solution to the problem of bringing the pendulum to the vertical position under the initial conditions (3.2) and restrictions on controls (3.1). (a) Dependence of the angle of deviation (nutation) of the pendulum $\theta(t)$. (b) The component v_2 of the translational velocity of the platform. (c) The component ω_1 of the angular velocity of the pendulum. (d) Optimal control $u(t)$ for the platform moving along the axis OY . (e) Dependence of the ordinate of the platform, $Y(t)$, as it moves along the axis OY .

Fig. 5. (a) Time dependence of the nutation angle θ . (b) Time dependence of the precession angle ψ .

and thus the orientation of the platform in space remains unchanged. The component ω_2 of the angular velocity of the pendulum also preserves zero value.

2. To check the adequacy of the model and the algorithm for solving the problem of balancing, we consider a problem similar to the preceding one, but with an independent control of all four omniwheels.

Suppose that at the initial time the state of the system is determined by the same initial conditions (3.2), but the values of the control torques $M_1(t_k)$, $M_2(t_k)$, $M_3(t_k)$ and $M_4(t_k)$ are given independently. In this case, we obtain the solution presented in Figs. 5–8.

Fig. 6. (a) Time dependence of the component v_1 of the translational velocity. (b) Time dependence of the component v_2 of the translational velocity. (c) Time dependence of the angular velocity Ω of the platform. (d) Time dependence of the component ω_1 of the angular velocity of the pendulum. (e) Time dependence of the component ω_2 of the angular velocity of the pendulum

Fig. 7. (a) Time dependence of the control torque M_1 . (b) Deviation of the values of the control torques M_2 , M_3 , M_4 from the torque M_1

A comparison of Fig. 4a and Fig. 5a, Fig. 4b and Fig. 6b, Fig. 4c and Fig. 6d shows that the dependences of the nutation angle θ , the component v_2 of the translational velocity of the point C of the platform and of the component of the angular velocity ω_1 of the pendulum are in qualitative agreement with each other. In addition, the control actions are in qualitative agreement as well (see Figs. 4d and 7). Since the minimization problem is solved numerically with some accuracy,

the control torques applied to different wheels differ only slightly (see Fig. 7b). Mismatch of the control torques gives rise to nonzero values of the component v_1 of the translational velocity of the point C of the platform (see Fig. 6a), the angular velocity Ω of the platform (see Fig. 6c) and the component ω_2 of the angular velocity of the pendulum (see Fig. 6e).

At the end point of the trajectory the deviation of the pendulum from the vertical is about $1\text{--}1.5^\circ$. For this deviation $\gamma_3 = \cos\theta > 0.9996$, which corresponds to the values of the target function (2.1) on the level set $10^{-4} \dots 10^{-3}$ for $p = 0$.

The omnidirectional platform is a fairly mobile device, and in the time of balancing of the pendulum equal to 0.55 sec with given mass-dimensional characteristics [23] and $M_k \in [-1, 1]$ the platform overcomes the distance ≈ 0.6 m (see Fig. 8a). A slight mismatch of the control torques gives rise to small deviations from straight-line motion (see Fig. 8b).

Fig. 8. (a) Time dependence of the ordinate Y of the point C of the platform. (b) Trajectory of the point C of the platform

Thus, despite the fact that in this case the platform was capable of performing an arbitrary motion in the plane, we obtained, as the optimal motion, the motion close to that along the axis OY , which is similar to that shown in Fig. 4e. This confirms the consistency of the two-dimensional and three-dimensional models of the system and the adequacy of the algorithm for solving the control problem.

When it comes to solving the problem of optimization, the problem with independent controls is the most difficult of those considered in this paper. This is due to some special features of the operation of the genetic algorithm. Specifically, since there is the possibility of an arbitrary motion and independently specifying the torques $M_1(t_k)$, $M_2(t_k)$, $M_3(t_k)$ and $M_4(t_k)$, the solution for which the cart moves only along one axis and $M_1(t_k) = M_2(t_k) = M_3(t_k) = M_4(t_k)$ is an optimal solution.

3.2. Spatial balancing of the pendulum

Consider the case where the deviation of the pendulum occurs not in the coordinate planes Cxz or Cyz . As shown in [22, 23], the omnidirectional platform is completely controllable and has the possibility of performing an arbitrary motion in the horizontal plane, and this motion is guaranteed by independently specifying the control torques $M_1(t_k)$, $M_2(t_k)$, $M_3(t_k)$, $M_4(t_k)$.

Let us define the following initial conditions:

$$\begin{aligned} X(0) = Y(0) = 0, \quad \sigma(0) = 0, \quad v_1(0) = v_2(0) = 0, \quad \Omega(0) = 0, \\ \omega_1(0) = \omega_2(0) = 0, \quad \psi(0) = 20^\circ, \quad \theta(0) = 15^\circ, \quad \varphi(0) = 0. \end{aligned} \quad (3.3)$$

At the initial and end points of the trajectory, the components v_1 , v_2 of the translational velocity of the platform, the angular velocity Ω of the platform and the components ω_1 , ω_2 of the

angular velocity of the pendulum are zero (see Fig. 9). This indicates that the platform stops at the end point of the trajectory (see Fig. 10b).

Fig. 9. (a) Time dependences of the components v_1 and v_2 of the translational velocity of the point C of the platform. (b) Dependence of the angular velocity Ω of the platform. (c) Time dependences of the components ω_1 and ω_2 of the angular velocity of the pendulum

One can clearly see from the change in the angular velocity Ω in Fig. 9b that, in addition to performing translational motion, the platform rotates by changing its orientation in space. The components v_1 and v_2 of the translational velocity lie in the interval $[-1.4, 0.6]$, and over certain segments of the trajectory $|\Omega|$ exceeds 8 rad/sec. As a result, to correctly solve the optimization problem, it is necessary to increase the discretization frequency of the control torques (step τ). However, a decrease in the value of τ leads to a growth of time and computational expenditures for the search for an optimal control and balancing of the pendulum. In addition, to ensure a correct numerical solution of the equations of motion, the step of integration with respect to time must not exceed 0.01 s.

Fig. 10. (a) Time dependences of the precession angle ψ and the nutation angle θ . (b) Trajectory of the point C of the platform. (c) Time dependences of control torques

Figure 10c shows the time dependences of the control torques ensuring the motion described above. Figure 10a shows the time dependences of the nutation angle θ and the precession angle ψ . It can be seen from Fig. 10a that the found control ensures that the pendulum is brought to a position close to the vertical one.

3.3. Spatial balancing of the pendulum with preservation of the direction of motion and the velocity of the platform

We now consider the problem of bringing the pendulum to the vertical position under the condition that the orientation of the platform be the same at the initial and final times. For this, we set $p = 1$ in the functional (2.1). Let us define the initial conditions

$$\begin{aligned} X(0) = Y(0) = 0, \quad \sigma(0) = 45^\circ, \quad v_1(0) = 1, \quad v_2(0) = 0, \quad \Omega(0) = 0, \\ \omega_1(0) = \omega_2(0) = 0, \quad \psi(0) = 15^\circ, \quad \theta(0) = 35^\circ, \quad \varphi(0) = 0. \end{aligned} \quad (3.4)$$

The result of the solution of the optimization problem is presented in Figs. 11 and 12. In Fig. 11a it is clearly seen that the components v_1 and v_2 of the translational velocity of the platform at the initial and end points of the trajectory have similar values. In addition, the angular velocities of the platform Ω and the pendulum are close to zero (see Figs. 11b and 11c).

Fig. 11. (a) Time dependences of the components v_1 and v_2 of the translational velocity of the point C of the platform. (b) Dependence of the angular velocity Ω of the platform. (c) Time dependences of the components ω_1 and ω_2 of the angular velocity of the pendulum

Fig. 12. (a) Time dependences of the precession angle ψ , the nutation angle θ and the angle of rotation of the platform σ . (b) Trajectory of the point C of the platform. (c) Time dependences of control torques

Figure 12a shows changes in the nutation angle θ and the precession angle ψ . Figure 12a also shows a change in the rotation angle σ of the platform. We see that the finite value of the angle σ is close to the initial one. The trajectory of the platform which corresponds to the optimal control (see Fig. 12c) is shown in Fig. 12b. It can be seen from Fig. 12b that the initial and final segments of the trajectory have approximately the same angles of inclination of the tangents.

Given that after the balancing of the pendulum the original orientation of the platform σ relative to the fixed coordinate system and the values of the components v_1 and v_2 of the translational velocity and the angular velocity Ω are preserved, it can be concluded that the solution of the problem is correct.

4. Conclusion

In this paper, the problem of balancing an inverted spherical pendulum placed on a platform with omniwheels in a three-dimensional setting is presented. Equations of motion of the system are obtained and the possibility of balancing the pendulum by controlling the torques applied to the omniwheels is shown. To solve the problem of controlling the platform's motion and of balancing the pendulum, a hybrid genetic algorithm is applied. This algorithm uses the Runge–Kutta–Verner method of the sixth order of accuracy to solve the system of differential equations in calculating the target function.

The behavior of the platform–pendulum system is investigated for different initial and final states of the system. The possibility of balancing is shown for the case of the initial deflection of the pendulum by fairly large angles, namely, $\theta_0 = 15^\circ \dots 45^\circ$. As a rule, the results presented in [17] and obtained in a simpler, two-dimensional, setting correspond to the initial values of the deflection angle ranging from 0° to 10° .

In all cases considered above, the hybrid genetic algorithm allows an optimal control to be found. This means that in the future a real-time controller can be constructed by applying an approach similar to that described in [28]. Moreover, the possibility of preserving the prescribed direction and velocity of the motion of the platform during balancing implies that the joint problem of controlling the motion of the omnidirectional platform along the prescribed trajectory and of balancing the inverse pendulum can be considered.

In addition, the approach presented in this paper allows one to pass from the balancing of an abstract spherical pendulum to ensuring the stability of the object with a finite support area on the surface of a platform moving with variable accelerations.

In solving the problems presented it has been shown that it is possible to choose certain combinations and dependences of control torques for the omniwheels in order to ensure the predetermined type of motion of the platform, for example, straight-line motion in the required direction. This leads to another avenue for future research — organizing a combined control which will make it possible to activate, on different trajectory segments, a certain type of control with the corresponding motion pattern: turning of the platform, straight-line motion in the chosen direction, etc.

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Appendix A. Poincaré equations

Assume that the Lagrangian dynamical system is defined by generalized coordinates $\mathbf{q} = (q_1, \dots, q_n)$ and quasi-velocities $\mathbf{w} = (w_1, \dots, w_k)$, which are related to the generalized

velocities by the following linear expressions:

$$\dot{q}_i = \sum_{s=1}^k u_i^s(\mathbf{q}) w_s, \quad i = 1, \dots, n. \quad (\text{A.1})$$

In addition, we assume that the vector fields

$$\mathbf{u}^s = \sum_{i=1}^n u_i^s(\mathbf{q}) \frac{\partial}{\partial q_i} \quad (\text{A.2})$$

form the closed system

$$[\mathbf{u}^i, \mathbf{u}^j] = c_{ij}^s \mathbf{u}^s, \quad i, j, s = 1, \dots, k, \quad (\text{A.3})$$

where c_{ij}^s are constants called *structural coefficients*. Here the Lie bracket is defined as follows:

$$[\mathbf{a}, \mathbf{b}] = \sum_j a_j \frac{\partial b_i}{\partial q_j} - b_j \frac{\partial a_i}{\partial q_j}. \quad (\text{A.4})$$

Then the equations of motion governing the evolution of quasi-velocities have the form

$$\frac{d}{dt} \frac{\partial L}{\partial w_i} = \sum_{r,s} c_{ri}^s w_r \frac{\partial L}{\partial w_s} + \mathbf{u}^i(L), \quad i = 1, \dots, k, \quad (\text{A.5})$$

and are called the *Poincaré equations* [26].

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