NONLINEAR PHYSICS AND MECHANICS

MSC 2010: 70E18

On the Motion of a Body with a Moving Internal Mass on a Rough Horizontal Plane

B. S. Bardin, A. S. Panev

We consider a vibration-driven system which consists of a rigid body and an internal mass. The internal mass is a particle moving in a circle inside the body. The center of the circle is located at the mass center of the body and the absolute value of particle velocity is a constant. The body performs rectilinear motion on a horizontal plane, whereas the particle moves in a vertical plane. We suppose that dry friction acts between the plane and the body.

We have investigated the dynamics of the above system in detail and given a full description of the body's motion for any values of its initial velocity. In particular, it is shown that there always exists a periodic mode of motion. Depending on parameter values, one of three types of this periodic mode takes place. At any initial velocity the body either enters a periodic mode during a finite time interval or it asymptotically approaches the periodic mode.

Keywords: periodic motion, dry friction, rigid body, vibration-driven system

Received August 16, 2018
Accepted September 15, 2018

This research was supported by a grant of the Russian Science Foundation (project No 14-21-00068).

Boris S. Bardin
bsbardin@yandex.ru
Moscow Aviation Institute (National Research University)
Volokolamskoe sh. 4, Moscow, 125993 Russia
Mechanical Engineering Research Institute of the Russian Academy of Sciences (IMASH RAN)
M. Kharitonyevskiy per. 4, Moscow, 101990 Russia

Alexander S. Panev
a.s.panev@gmail.com
Moscow Aviation Institute (National Research University)
Volokolamskoe sh. 4, Moscow, 125993 Russia

1. Introduction

Consider a mechanical system consisting of a rigid body of mass M , located on a flat horizontal rough surface, and an internal mass – a material point of mass m moving inside the rigid body in a circle of radius R with its center at the center of mass of the body. The internal mass does not interact with the environment, and the angular velocity ω of the radius vector tracing out its circular motion is constant.

Fig. 1. Mechanical system.

We shall consider the motion of the system in a vertical plane, in which we introduce an absolute coordinate system OXY (Fig. 1). Forces of dry Coulomb friction act between the rigid body and the surface, and the system parameters are chosen so that the relative motion of the internal mass can bring the body from rest, and at and after the start of motion the body performs translational motion without loss of contact with the horizontal surface. The position of the rigid body will be given by the coordinates X and Y of its center of mass O_1 .

The investigation of the dynamics of such mechanical systems is not only of theoretical interest, but can also be of interest for applications, in particular, for development of vibration robots moving by changing the position of their internal masses. An important advantage of such devices is that they do not require special propulsion devices (wheels, caterpillar tracks etc.) and can be built in the form of soldered capsules. This makes them resistant to environmental impacts, and so there are good prospects for vibration robots to be used in aggressive media, both on solid surfaces and in fluids. In particular, they can be of interest for space projects, in investigating the surface of celestial bodies.

There is a lot of published research [1–12] on applied problems of dynamics, mathematical modeling of motion and problems of designing mobile robots capable of moving on the surface by changing the position of internal masses.

A rigorous theoretical investigation of problems concerned with the dynamics and optimal motion control of mechanical systems consisting of a rigid body and internal moving masses was started in [13] and [14]. The papers [15, 16] are devoted to the search for optimal control to maximize the average velocity of the rigid body. Stationary periodic modes of motion are studied in [15] by averaging methods under the assumption of smallness of the coefficient of sliding friction.

An analysis was also made of motions of the rigid body in the presence of viscous friction [17] and on an inclined plane [18, 19]. In [20–22], the results of investigation of two-dimensional motion are presented. In particular, methods of maneuvering (turns) and motion of the body

from the initial position to the desired position were studied. An analysis of shock-free jumps of a body containing two moving masses on a horizontal plane was carried out in [23].

A number of papers study the dynamics and construction of optimal control of a two-mass system consisting of a rigid body and a material point moving inside the body along some trajectory. In [24], the dynamics of a rigid body is investigated under the assumption that the internal mass moves inside it in the vertical plane rectilinearly, and the coordinates of its relative motion change according to a harmonic law. In [25], a study was made of motion modes in which the body comes to a stop, remains at rest for a finite time interval, and then continues sliding on the horizontal plane, and an analysis of bifurcations of the above-mentioned modes was carried out. Problems of constructing the optimal control of the motion of a rigid body with the internal mass performing rectilinear relative motion are dealt with in [26–28]. Problems of the optimal control of motion of a rigid body in the case of circular relative motion of the internal mass under some restrictions imposed on its acceleration were considered in [29, 30]. Trajectories of relative motion of the internal mass which ensure the optimal control of a rigid body are constructed in [31].

The goal of this paper is to carry out a complete qualitative investigation of the dynamics of the above-mentioned mechanical system for all admissible parameter values and initial conditions. For the zero initial velocity of a rigid body, an analysis of the dynamics of this system was made in [32]. Some modes of rigid body motion at nonzero initial velocity were studied in [33]. Using the methods of the above-mentioned papers, an analogous investigation was carried out in [34, 35] in the case where the rigid body is acted upon both by dry friction and by the force of viscous friction.

2. Equations of motion

Following [32], we introduce the following dimensionless quantities: a coordinate x , time t' and parameter μ :

$$X = \frac{Rmx}{M+m}, \quad t = \frac{t'}{\omega}, \quad \mu = \frac{(M+m)g}{Rm\omega^2}, \quad (2.1)$$

where g is the free-fall acceleration. In what follows, we will drop the prime which indicates dimensionless time. Then the equation of motion of the rigid body on a horizontal surface takes the form [32] (see also [15])

$$\dot{u} = \sin t + f_c, \quad (2.2)$$

where $u = \dot{x}$ is the dimensionless velocity of the rigid body and the function f_c is given by

$$f_c = \begin{cases} -k(\mu + \cos t) \operatorname{sign} u, & \text{if } u \neq 0, \\ -\sin t, & \text{if } u = 0 \text{ and } |\sin t| \leq k(\mu + \cos t), \\ -k(\mu + \cos t) \operatorname{sign}(\sin t), & \text{if } u = 0 \text{ and } |\sin t| > k(\mu + \cos t). \end{cases} \quad (2.3)$$

Here and in what follows, k is the coefficient of dry friction. According to (2.2) and (2.3), the motion of the rigid body in the positive direction is described by

$$\dot{u} = \sin t - k(\mu + \cos t), \quad (2.4)$$

and the motion in the negative direction is described by

$$\dot{u} = \sin t + k(\mu + \cos t). \quad (2.5)$$

We note that dimensionless time has been introduced so that the period of relative motion of the internal mass is 2π . Without loss of generality we assume that at time $t = 0$ the internal mass moving counterclockwise passes the lower point of the circle. Then dimensionless time t can be interpreted as an angle defining the position of the internal mass in the coordinate system $O_1\xi\eta$ attached to the rigid body (see Fig. 1).

3. On the pattern of motion of a rigid body with zero initial velocity. Periodic modes of motion

In what follows, we will assume that the system parameters satisfy the following inequalities:

$$k^2 < \frac{1}{\mu^2 - 1}, \quad \mu \geq 1. \quad (3.1)$$

The first inequality guarantees that there exists such an instant of time at which the horizontal component of the inertial force applied to the internal mass exceeds in absolute value the force of dry friction, i.e., the relative motion of the internal mass can bring the rigid body from rest. The second inequality implies that throughout the motion the vertical component of the inertial force does not exceed in absolute value the gravity force applied at the center of mass of the system, i.e., the body moves without loss of contact with the surface. We will also assume that the geometry of the body is such that it does not overturn during the motion.

We start the analysis of the body dynamics by defining the instants of time at which the acceleration of the body is zero. These instants are found from the equation

$$\sin t = \pm k(\mu + \cos t). \quad (3.2)$$

On the interval $t \in (0, 2\pi)$ this equation has the following roots:

$$\begin{aligned} t_1 &= 2 \operatorname{arctg} \left(\frac{1 - \sqrt{1 - k^2(\mu^2 - 1)}}{k(\mu - 1)} \right), \\ t_2 &= 2 \operatorname{arctg} \left(\frac{1 + \sqrt{1 - k^2(\mu^2 - 1)}}{k(\mu - 1)} \right), \\ t_3 &= 2\pi - t_2, \quad t_4 = 2\pi - t_1. \end{aligned} \quad (3.3)$$

The time instants t_i ($i = 1, 2, 3, 4$) define the intervals (t_2, t_3) and $(t_4, t_1 + 2\pi)$ on which the acceleration of the rigid body is opposite to the direction of its velocity or is zero. When the first of the above-mentioned intervals is passed, the internal mass is in the upper part of the trajectory, while when the second interval is passed, it is in the lower part. We call these intervals, respectively, the upper and lower deceleration zones (Fig. 2). They play an important role in the analysis of the pattern of motion of the body. In particular, if the body stops at the instant when the internal mass passes a deceleration zone, it will remain at rest until the internal mass leaves the deceleration zone. This phenomenon is called sticking of the body [26].

We also note that the following relations hold:

$$t_1 + t_2 > \pi, \quad (3.4)$$

and

$$\cos t_1 + k \sin t_1 = -(\cos t_2 - k \sin t_2). \quad (3.5)$$

We will use them below.

Fig. 2. Deceleration zones.

Further, in this section we assume that the velocity of the rigid body at $t = t_1$ is zero, i.e., t_1 is the instant of the start of motion. In [32] it is shown that in this case, depending on the values of parameters k and μ , the motion of the system will be qualitatively different. In particular, on the plane of parameters k, μ there exist three regions (see Fig. 3). In regions I and II the body, starting its motion at time t_1 , will move with 2π -periodically changing velocity, and in region III, under the same initial condition, the motion of the body is no longer periodic, but asymptotic, approaching some periodic mode of motion. Outside the above-mentioned regions, conditions (3.1) are not satisfied.

 Fig. 3. Regions of values of parameters k and μ which correspond to different types of periodic motions of a rigid body.

Region I is defined by the system of inequalities [32]:

$$\begin{aligned} k^2 &< \frac{1}{\mu^2 - 1}, \\ -\operatorname{arctg} k + \frac{\sqrt{1 - k^2(\mu^2 - 1)} + \mu k^2}{(k^2 + 1)\mu k} &< 0. \end{aligned} \quad (3.6)$$

The last inequality defines the condition for the rigid body to stop in the upper deceleration zone. For parameter values from region I, the body performs 2π -periodic back-and-forth motion

with stops at times $t_*^I + 2\pi n$ and $t_{**}^I + 2\pi n$ ($n \in \mathbf{N}$) in the upper and the lower deceleration zone, respectively. In the deceleration zones, the body undergoes sticking, i.e., after a stop at time $t_*^I + 2\pi n$ the body remains at rest until time $t_3 + 2\pi n$, and then it starts moving in the negative direction, and after a stop at time $t_{**}^I + 2\pi n$ the body remains at rest until time $t_1 + 2\pi(n+1)$. Thus, the body performs periodic motion with period 2π corresponding to a complete revolution of the internal mass in a circle. In this case, not only the velocity of the body, but also its coordinate changes periodically, and hence the displacement of the body within a period is zero [32].

The parameter region II is defined by the system of inequalities [32]:

$$\begin{aligned} \mu &\geq 1, \\ -\operatorname{arctg} k + \frac{\sqrt{1 - k^2(\mu^2 - 1)} + \mu k^2}{(k^2 + 1)\mu k} &\geq 0, \\ \cos t_1 + \cos \left[t_1 + \frac{\sin t_1 + \sqrt{1 - (\cos t_1 - \pi k \mu)^2}}{\mu} \right] &< \pi k \mu. \end{aligned} \quad (3.7)$$

In this case, the body passes the upper deceleration zone without a stop, comes to a stop at time $t_*^{II} + 2\pi n$ (where $t_3 < t_*^{II} < t_4$), and then starts moving immediately in the negative direction. Moving in the negative direction, the body comes to a stop in the lower deceleration zone at time $t_{**}^{II} + 2\pi n$ and remains at rest until time $t_1 + 2\pi(n+1)$. Thus, in region II the body performs motion with sticking only in the lower deceleration zone. In this case, only the velocity of the body changes periodically, and the displacement of the body within a period is positive [32].

The pattern of periodic motion in region III given by the inequalities

$$\begin{aligned} \mu &\geq 1, \\ \cos t_1 + \cos \left[t_1 + \frac{\sin t_1 + \sqrt{1 - (\cos t_1 - \pi k \mu)^2}}{\mu} \right] &\geq \pi k \mu \end{aligned} \quad (3.8)$$

will be described in the next section.

4. Preliminary remarks on the properties of solutions of the equations of motion of the rigid body

Equation (2.2), which describes the motion of the rigid body, has at $u = 0$ a discontinuity on the right-hand side and does not satisfy the Lifshitz condition, and so the classical theorems from the theory of ordinary differential equations cannot be applied to it. General conclusions on the existence, uniqueness and continuity of the solution to Eq. (2.2) can be obtained using a theory developed for systems with a discontinuous right-hand side [36, 37]. In particular, it can be shown that for Eq. (2.2) right-handed uniqueness of the solution takes place, i.e., for $t > t_0$ there always exists a unique solution satisfying the initial condition $u(t_0) = u_0$. It is also easy to show that, when $t > t_0$, the solution of Eq. (2.2) continuously depends on t , the initial condition and the system parameters k and μ . In this section we establish some general properties of the solutions to Eq. (2.2), which will be used below in analyzing the motion of the rigid body with an arbitrary initial velocity.

We first show that if the function $u(t)$ is a solution to Eq. (2.2), then the sequence of its values $u(t + 2\pi n)$, $n \in \mathbf{N}$, possesses the monotonicity property. For this purpose, we consider two solutions $u^{(1)}(t)$ and $u^{(2)}(t)$ to Eq. (2.2). These solutions are continuous functions of t and, when $t \geq t_1$, are uniquely defined by its initial conditions $u^{(1)}(t_1) = u_0^{(1)}$ and $u^{(2)}(t_1) = u_0^{(2)}$.

Let $u_0^{(1)} < u_0^{(2)}$, then two cases are possible: either $u^{(1)}(t) \neq u^{(2)}(t)$ for all $t \geq t_1$ and the continuity of function $u^{(1)}(t)$ and $u^{(2)}(t)$ implies immediately that $u^{(1)}(t) < u^{(2)}(t)$; or there exists such an instant of time t^* that $u^{(1)}(t^*) = u^{(2)}(t^*) = 0$ and $u^{(1)}(t) \neq u^{(2)}(t)$ at $t_1 \leq t < t^*$. In the latter case, the continuity of $u^{(1)}(t)$ and $u^{(2)}(t)$ implies that $u^{(1)}(t) < u^{(2)}(t)$ at $t_1 \leq t < t^*$, and the right-handed uniqueness of the solution to Eq. (2.2) with the initial condition $u(t^*) = 0$ implies that, when $t \geq t^*$, the identical equality $u^{(1)}(t) \equiv u^{(2)}(t)$ holds. Thus, if $u_0^{(1)} < u_0^{(2)}$, then for all $t \geq t_1$ the inequality $u^{(1)}(t) \leq u^{(2)}(t)$ holds and, in particular,

$$u^{(1)}(t_1 + 2\pi) \leq u^{(2)}(t_1 + 2\pi). \quad (4.1)$$

Now assume that $u^{(1)}(t_1) < u^{(1)}(t_1 + 2\pi)$. In this case, choosing the initial condition defining the solution $u^{(2)}(t)$ so that $u^{(2)}(t_1) = u^{(1)}(t_1 + 2\pi)$, we arrive by virtue of 2π -periodicity of the right-hand side of Eq. (2.2) at the identical equality $u^{(2)}(t) \equiv u^{(1)}(t + 2\pi)$, which allows us to rewrite inequality (4.1) as

$$u^{(1)}(t_1 + 2\pi) \leq u^{(1)}(t_1 + 4\pi). \quad (4.2)$$

Continuing further by induction, we have

$$u^{(1)}(t_1 + 2\pi n) \leq u^{(1)}(t_1 + 2\pi(n + 1)), \quad n \in \mathbf{N}. \quad (4.3)$$

Thus, the sequence of values $u^{(1)}(t_1 + 2\pi n)$ is nondecreasing.

In a similar way, it can be shown that in the case $u^{(1)}(t_1) > u^{(1)}(t_1 + 2\pi)$ the inequality $u^{(1)}(t_1 + 2\pi n) \geq u^{(1)}(t_1 + 2\pi(n + 1))$, $n \in \mathbf{N}$, holds, i.e., the sequence of values $u^{(1)}(t_1 + 2\pi n)$ is nonincreasing.

In the limiting case $u^{(1)}(t_1) = u^{(1)}(t_1 + 2\pi)$, the initial conditions of the solutions $u^{(1)}(t)$ and $u^{(2)}(t)$ coincide, and hence, by virtue of uniqueness of the solution to Eq. (2.2) at $t > t_1$, these solutions are also identically equal: $u^{(1)}(t) \equiv u^{(2)}(t)$. On the other hand, as was shown above, $u^{(2)}(t) \equiv u^{(1)}(t + 2\pi)$, therefore, for all $t > t_1$ the following equality holds:

$$u^{(1)}(t) = u^{(1)}(t + 2\pi), \quad (4.4)$$

which means that $u^{(1)}(t)$ is a periodic function of t .

The above reasoning remains valid if we choose not t_1 , but any arbitrary value $t^* \geq t_1$ as the initial instant of time. Thus, we arrive at the following statement.

Lemma 1. *Let the function $u(t)$ given on the interval $[t_1; +\infty)$ be a solution to Eq. (2.2). Then its values $u(t^* + 2\pi n)$, where $n \in \mathbf{N}$, form a monotonic sequence for any $t^* \geq t_1$.*

We now consider in more detail the limiting case where the solution to Eq. (2.2) is 2π -periodic. As stated above, in the parameter regions I and II there exist periodic solutions which are defined by zero initial conditions and describe the motion of the rigid body with sticking in the deceleration zones. Let us examine the question of existence of 2π -periodic solutions describing the motion of the body without sticking.

Assume that Eq. (2.2) admits a 2π -periodic solution $u_*(t)$ describing the motion of the body without sticking. Then on the interval $[t_1; t_1 + 2\pi]$ the function $u_*(t)$ vanishes and changes sign

twice. Indeed, reasoning by contradiction, we assume that $u_*(t) \neq 0$ on the interval $[t_1; t_1 + 2\pi]$ and for definiteness we set $u_*(t) > 0$. Then, integrating Eq. (2.4), we have $u_*(t_1 + 2\pi) = u_*(t_1) + 2\pi k\mu$, which contradicts the condition of periodicity of $u_*(t)$. Therefore, the function $u_*(t)$ necessarily vanishes and changes sign on the interval $[t_1; t_1 + 2\pi]$. Moreover, the continuity and 2π -periodicity of $u_*(t)$ imply that on the above-mentioned interval $u_*(t)$ changes sign an even number of times. We now note that, since the function $u_*(t)$ describes the motion of the rigid body without sticking, it can vanish only on the intervals $[t_1; t_2]$ and $[t_3; t_4]$. The right-hand side of Eq. (2.2) is positive on the interval $[t_1; t_2]$ and negative on the interval $[t_3; t_4]$, therefore the function $u_*(t)$ increases on the interval $[t_1; t_2]$ and decreases on the interval $[t_3; t_4]$. This means that on each of the above-mentioned intervals $u_*(t)$ can vanish only once. Thus, the function $u_*(t)$ behaves as follows: it vanishes at times $t_* \in [t_1; t_2)$ and $t_{**} \in [t_3; t_4]$; on the interval $(t_*; t_{**})$ it is positive and at first increases to its maximal value $u_*(t_2)$ and then decreases to zero; on the interval $(t_{**}; t_* + 2\pi)$ it is negative and at first decreases to its minimal value $u_*(t_4)$ and then increases to zero.

We now determine at what values of the parameters k and μ Eq. (2.2) admits a periodic solution that possesses the above properties. For this purpose, we integrate Eqs. (2.4) and (2.5) with the initial conditions $u(t_*) = 0$ and $u(t_{**}) = 0$ on the intervals $[t_*; t_{**}]$ and $[t_{**}; t_* + 2\pi]$, respectively. As a result, we arrive at the following system of equations in t_* and t_{**} :

$$\begin{aligned} \cos t_* - \cos t_{**} - k\mu(t_{**} - t_*) - k(\sin t_{**} - \sin t_*) &= 0, \\ \cos t_{**} - \cos t_* + k\mu(2\pi + t_* - t_{**}) + k(\sin t_* - \sin t_{**}) &= 0, \end{aligned} \quad (4.5)$$

which needs to be solved taking into account the restrictions

$$t_1 \leq t_* < t_2, \quad t_3 \leq t_{**} < t_4. \quad (4.6)$$

Adding and subtracting Eqs. (4.5) and then applying the well-known trigonometric formulae, we obtain the following equivalent system of equations:

$$\begin{aligned} 2 \sin \frac{t_* - t_{**}}{2} \cos \frac{t_{**} + t_*}{2} &= \mu(t_{**} - t_* - \pi), \\ 2 \sin \frac{t_{**} - t_*}{2} \sin \frac{t_{**} + t_*}{2} &= \pi k\mu. \end{aligned} \quad (4.7)$$

We introduce the notation

$$x = t_{**} - t_* - \pi, \quad y = t_{**} + t_*, \quad (4.8)$$

in which the system of equations (4.7) takes the form

$$\begin{aligned} -2 \cos \frac{x}{2} \cos \frac{y}{2} &= \mu x, \\ 2 \cos \frac{x}{2} \sin \frac{y}{2} &= \pi k\mu. \end{aligned} \quad (4.9)$$

By virtue of (4.6), taking (3.3) into account, the solutions to the system (4.9) must satisfy the restrictions

$$\pi - 2t_2 < x < \pi - 2t_1, \quad 2\pi + t_1 - t_2 < y < 2\pi + t_2 - t_1. \quad (4.10)$$

Since $0 < t_1 < t_2 < \pi$, it follows that, when inequalities (4.10) hold, the inequalities

$$-\pi < x < \pi, \quad \pi < y < 3\pi \quad (4.11)$$

hold also, which implies that $\cos \frac{x}{2} > 0$, $\cos \frac{y}{2} < 0$. Therefore, on the solutions satisfying the restrictions (4.10), the left-hand side of the first equation of (4.9) is positive, and hence its right-hand side is also positive. The latter means that the restrictions (4.10) will be satisfied only by such solutions (x_0, y_0) of system (4.9) for which $x_0 > 0$.

Let us square both sides of the equations of system (4.9) and add the resulting equalities. After elementary transformations we have the following equation for quantity x :

$$2(1 + \cos x) = \mu^2 x^2 + \pi^2 k^2 \mu^2. \quad (4.12)$$

A straightforward analysis shows that Eq. (4.12) can have real roots only when $\pi k \mu \leq 2$. The first of conditions (3.1) guarantees that the inequality $\pi k \mu < 2$ is always satisfied. In this case, Eq. (4.12) has exactly two real roots, one of which is negative and the other is positive. Dividing term by term the first equation of system (4.9) into its second equation, we obtain an equation for finding the quantity y

$$\operatorname{ctg} \frac{y}{2} = -\frac{x_0}{\pi k}, \quad (4.13)$$

where x_0 is the positive root of Eq. (4.12).

Equation (4.13) has the unique solution

$$y_0 = 2\pi - \operatorname{arccctg} \frac{x_0}{\pi k}, \quad (4.14)$$

which satisfies the restriction (4.10).

Now, substituting (4.14) into (4.8), we obtain the following expressions for t_* and t_{**} :

$$\begin{aligned} t_* &= \frac{\pi}{2} - \frac{x_0}{2} - \frac{1}{2} \operatorname{arccctg} \frac{x_0}{\pi k}, \\ t_{**} &= \frac{3\pi}{2} + \frac{x_0}{2} - \frac{1}{2} \operatorname{arccctg} \frac{x_0}{\pi k}. \end{aligned} \quad (4.15)$$

Thus, among all possible solutions of (4.5) the restrictions (4.6) can be satisfied only by the solution (4.15). However, even this solution satisfies the restrictions (4.6) not for all values of k and μ . Let us examine this question in more detail.

We first note that the inequality $t_* < t_2$ on the solution (4.15) is always satisfied since $t_* < \frac{\pi}{2}$ and $t_2 > \frac{\pi}{2}$. We further note that if the inequality $t_1 \leq t_*$ is satisfied on the solution (4.15), then, in view of (3.4), the inequalities $t_3 < t_{**} < t_4$ are also satisfied. The latter follows immediately from the following two chains of inequalities:

$$t_3 < t_1 + \pi \leq t_* + \pi < t_* + \pi + x_0 = t_{**} \quad (4.16)$$

and

$$t_4 = 2\pi - t_1 \geq 2\pi - t_* = \frac{3\pi}{2} + \frac{x_0}{2} + \frac{1}{2} \operatorname{arccctg} \frac{x_0}{\pi k} > \frac{3\pi}{2} + \frac{x_0}{2} - \frac{1}{2} \operatorname{arccctg} \frac{x_0}{\pi k} = t_{**}. \quad (4.17)$$

Thus, if the inequality $t_1 \leq t_*$ is satisfied, then the solution (4.15) satisfies all restrictions (4.6) and Eq. (2.2) has a periodic solution describing the motion of the rigid body without sticking.

Quantities t_1 and t_* , which are given by (3.3) and (4.15), are continuous functions of k and μ , therefore the inequality $t_1 \leq t_*$ defines in the parameter plane some region on whose

boundary the equality $t_* = t_1$ is satisfied. Assuming $t_* = t_1$ in Eqs. (4.5) and then eliminating t_{**} from them, we arrive at the following equality:

$$\cos t_1 + \cos \left[t_1 + \frac{\sin t_1 + \sqrt{1 - (\cos t_1 - \pi k \mu)^2}}{\mu} \right] = \pi k \mu, \quad (4.18)$$

which defines the boundary of the region under consideration.

On the other hand, comparing (4.18) with (3.7) and (3.8), one can notice that Eq. (4.18) defines the boundary dividing regions II and III (see Fig. 3). By virtue of continuous dependence of t_* on k and μ , the inequality $t_* \geq t_1$ is either satisfied at all points of region II or is not satisfied at any point of this region. The same is true of region III. Therefore, it suffices to check the validity of the inequality $t_* \geq t_1$ at one (arbitrary) internal point of each of the regions II and III. Numerical verification has shown that this inequality is satisfied in region III and is not satisfied in region II and, as a consequence, in region I. Thus, the following statement is proved.

Lemma 2. *In region III given by inequalities (3.8), Eq. (2.2) has a unique periodic solution $u_*(t)$ describing the motion of the rigid body without sticking. This solution is uniquely defined by the initial condition $u_*(t_*) = 0$, where the time instant t_* is calculated by the formula (4.15). Outside region III, there exist no periodic solutions that describe the motion of the body without sticking.*

We also note the monotonicity property of solutions to Eq. (2.2) on the interval $[t_1; t_1 + 2\pi]$. A straightforward analysis shows that the right-hand side of Eq. (2.4) is positive on the interval $(t_1; t_2)$ and negative on the interval $(t_2; t_1 + 2\pi)$, and the right-hand side of Eq. (2.5) is positive on the intervals $[t_1; t_3]$, $(t_4; t_1 + 2\pi)$ and negative on the interval $(t_3; t_4)$. Hence, the following statement holds, which is important for the investigation of the general pattern of motion of the rigid body.

Lemma 3. *Let the function $u(t)$ given on the interval $[t_1; t_1 + 2\pi]$ be a solution to Eq. (2.2). Then the values of t_i ($i = 2, 3, 4$) divide the interval $[t_1; t_1 + 2\pi]$ into monotonicity intervals so that:*

1. *The function $u(t)$ increases on the interval $[t_1; t_2]$ and decreases on the interval $[t_3; t_4]$.*
2. *If $u(t_2) > 0$, then the function $u(t)$ either decreases on the entire interval $[t_2; t_3]$ to the value $u(t_3) \geq 0$ or it decreases and vanishes at some internal point of the interval $[t_2; t_3]$ and then retains the zero value until the end of this interval.*
3. *If $u(t_2) < 0$, then the function $u(t)$ either increases on the entire interval $[t_2; t_3]$ to the value $u(t_3) \leq 0$ or it increases to a zero value, which it takes at some internal point of the interval $[t_2; t_3]$ and then remains identically equal to zero until the end of this interval.*
4. *If $u(t_4) > 0$, then the function $u(t)$ either decreases on the entire interval $[t_4; t_1 + 2\pi]$ to the value $u(t_1 + \pi) \geq 0$ or it decreases and vanishes at some internal point of the interval $[t_4; t_1 + 2\pi]$ and then retains the zero value until the end of this interval.*
5. *If $u(t_4) < 0$, then the function $u(t)$ either increases on the entire interval $[t_4; t_1 + 2\pi]$ to the value $u(t_1 + \pi) \leq 0$ or it increases to a zero value, which it takes at some internal point of the interval $[t_4; t_1 + 2\pi]$ and then remains identically equal to zero until the end of this interval.*

REMARK. It follows from Lemma 3 that the solutions to Eq. (2.2) do not change sign on the intervals $[t_2; t_3]$ and $[t_4; t_1 + 2\pi]$.

5. Analysis of the motion of the rigid body in region I

Assuming that the parameters k and μ take values from region I, we explore the pattern of motion of the rigid body at nonzero initial velocity. We first assume that at time t_1 the body has a positive velocity $u(t_1) > 0$. Then until the first stop its motion will be described by Eq. (2.4). Integrating this equation yields

$$\begin{aligned} u(t) &= u(t_1) + \int_{t_1}^t [\sin \tau - k(\mu + \cos \tau)] d\tau = \\ &= u(t_1) + \cos t_1 - \cos t - k\mu(t - t_1) - k(\sin t - \sin t_1). \end{aligned} \quad (5.1)$$

We note that the function $u(t)$ on the interval $(t_1, t_1 + 2\pi]$ has two extremal points $t = t_2$ and $t = t_1 + 2\pi$. The function $u(t)$ attains a maximum at point $t = t_2$ and a minimum at the point $t = t_1 + 2\pi$. Thus, $u(t)$ increases on the interval (t_1, t_2) , takes the largest value $u(t_2)$, and then monotonically decreases on the interval $(t_2, t_1 + 2\pi)$. Since it follows from (5.1) that

$$u(t_1 + 2\pi) = u(t_1) - 2\pi k\mu, \quad (5.2)$$

the velocity of the rigid body, retaining a positive value, decreases by $2\pi k\mu$ when $u(t_1) > 2\pi k\mu$, on the interval $(t_1, t_1 + 2\pi]$.

We now consider the case of negative initial velocity, i.e., we assume that $u(t_1) < 0$. Then, integrating Eq. (2.5), we have

$$\begin{aligned} u(t) &= u(t_1) + \int_{t_1}^t [\sin \tau + k(\mu + \cos \tau)] d\tau = \\ &= u(t_1) + \cos t_1 - \cos t + k\mu(t - t_1) + k(\sin t - \sin t_1). \end{aligned} \quad (5.3)$$

In this case, the function $u(t)$ increases on the intervals (t_1, t_3) , $(t_4, t_1 + 2\pi)$ and decreases on the interval (t_3, t_4) . Thus, on the interval $(t_1, t_1 + 2\pi]$ it has two extremal points: the maximum point $t = t_3$ and the minimum point $t = t_4$. The function $u(t)$ can take its largest value either at the maximum point $t = t_3$ or on the boundary of the interval under consideration at the point $t = t_1 + 2\pi$. We show that $u(t_1 + 2\pi)$ is the maximal value of the function $u(t)$ on the interval $(t_1, t_1 + 2\pi]$. Indeed, in region I there exists periodic motion of the rigid body with sticking both in the upper and in the lower deceleration zone, which is described by the solution $u_0(t)$ of Eq. (2.2) with the initial condition $u_0(t_1) = 0$. On this solution, the equality $u_0(t_3) = u_0(\tilde{t}) = 0$ is satisfied, where $\tilde{t}$ is the instant of stop of the rigid body in the lower deceleration zone, i.e., $\tilde{t} \in (t_4, t_1 + 2\pi)$. Now, taking into account the fact that on the interval $[t_3, \tilde{t}]$ the function $u_0(t)$ is a solution to Eq. (2.5) with the initial condition $u_0(t_3) = 0$, the equality $u_0(\tilde{t}) = 0$ can be rewritten as

$$\int_{t_3}^{\tilde{t}} [\sin \tau + k(\mu + \cos \tau)] d\tau = 0. \quad (5.4)$$

Now, integrating Eq. (2.5) on the interval $[t_3, t_1 + 2\pi]$ and taking (5.4) into account, we have

$$u(t_1 + 2\pi) = u(t_3) + \int_{t_3}^{t_1 + 2\pi} [\sin \tau + k(\mu + \cos \tau)] d\tau = u(t_3) + \int_{\tilde{t}}^{t_1 + 2\pi} [\sin \tau + k(\mu + \cos \tau)] d\tau. \quad (5.5)$$

Since $\sin \tau + k(\mu + \cos \tau) > 0$ with $\tau \in (t_4, t_1 + 2\pi)$, the integral on the right-hand side of the last equality is positive, and hence the inequality $u(t_1 + 2\pi) > u(t_3)$ is satisfied, which obviously means that $u(t_1 + 2\pi)$ is the maximal value of the function $u(t)$ on the interval $(t_1, t_1 + 2\pi]$.

On the other hand, if on the interval $(t_1, t_1 + 2\pi]$ the velocity of the body is negative, then, as (5.3) implies, $u(t_1 + 2\pi) = u(t_1) + 2\pi k\mu$. Therefore, when $u(t_1) < -2\pi k\mu$ on the interval $(t_1, t_1 + 2\pi]$, the velocity of the body, retaining a negative value, increases by $2\pi k\mu$.

Thus, starting motion at time $t = t_1$ with velocity $|u(t_1)| > 2\pi k\mu$ and moving without a stop, the body will have at time $t_1 + 2\pi n$ a velocity satisfying the inequality $|u(t_1 + 2\pi n)| < 2\pi k\mu$, where $n = \left\lceil \frac{|u(t_1)|}{2\pi k\mu} \right\rceil$, and the square brackets denote the operation of taking the integer part of the number. Therefore, without loss of generality, we will assume in what follows that $|u(t_1)| < 2\pi k\mu$.

To draw conclusions on the motion of the body in region I, we investigate the behavior of the integral curves of Eq. (2.2) on the time interval $(t_1, t_1 + 2\pi]$. An important role is played here by the integral curves corresponding to the motion of the body with a stop on one of the boundaries of the upper or lower deceleration zone. They divide the space of solutions to Eq. (2.2) into regions in which the integral curves and hence the corresponding motions of the body have a qualitatively different form.

Let $u^{(2)}(t)$ denote a solution that describes the motion of the body with a stop at time $t = t_2$. We immediately note that, as Lemma 3 implies, $u^{(2)}(t)$ increases on the interval $[t_1, t_2]$, therefore, a stop at time $t = t_2$ is possible only in the case of motion of the body in the negative direction. Consequently, inside the interval $(t_1, t_2]$ the function $u^{(2)}(t)$ takes only negative values and vanishes on its boundary. On the interval $(t_2, t_3]$ the function $u^{(2)}(t)$ vanishes identically. Thus, $u^{(2)}(t_3) = u_0(t_3) = 0$ and hence, by virtue of the right-handed uniqueness of solution (2.2), when $t \geq t_3$, the identical equality $u^{(2)}(t) \equiv u_0(t)$ is satisfied.

Integrating Eq. (2.5) on the interval $(t_1, t_2]$ and taking the formulae (3.3) into account, we can show that the solution $u^{(2)}(t)$ satisfies the following initial condition:

$$u^{(2)}(t_1) = -2\sqrt{1 - k^2(\mu^2 - 1)} + 2k\mu \operatorname{arctg} \left(\frac{\sqrt{1 - k^2(\mu^2 - 1)}}{k\mu} \right). \quad (5.6)$$

Let $u_+^{(3)}(t)$ and $u_-^{(3)}(t)$ denote the solutions to Eq. (2.2) which describe the motion during which the body, moving in the positive and negative direction, respectively, comes to a stop at time t_3 . The initial conditions of these solutions must satisfy the obvious inequalities $u_+^{(3)}(t_1) > u_0(t_1) = 0$ and $u_-^{(3)}(t_1) < u^{(2)}(t_1)$. Therefore, by virtue of Lemma 3 the function $u_+^{(3)}(t)$ increases on the interval $(t_1, t_2]$ and decreases on the interval $(t_2, t_3]$, vanishing on its right boundary, and the function $u_-^{(3)}(t)$ increases on the entire interval $(t_1, t_3]$ and takes a zero value on its right boundary. For all $t \geq t_3$ the identical equalities $u_+^{(3)}(t) \equiv u_0(t)$ and $u_-^{(3)}(t) \equiv u_0(t)$ are satisfied. It can be shown that the solutions $u_+^{(3)}(t)$ and $u_-^{(3)}(t)$ are defined by the initial conditions

$$u_+^{(3)}(t_1) = 2k\mu \operatorname{arctg} k - 2 \frac{\sqrt{1 - k^2(\mu^2 - 1)} + k^2\mu}{k^2 + 1}, \quad (5.7)$$

$$u_-^{(3)}(t_1) = - \frac{2\sqrt{1 - k^2(\mu^2 - 1)} + 2\mu k^2}{k^2 + 1} - 2k\mu \operatorname{arctg} k. \quad (5.8)$$

Now let $u_-(t)$ and $u_+(t)$ be the solutions to Eq. (2.2) with initial conditions satisfying the inequalities $u_-^{(3)}(t_1) < u_-(t_1) < u^{(2)}(t_1)$ and $u^{(2)}(t_1) < u_+(t_1) < u_+^{(3)}(t_1)$, respectively. Then by Lemma 3 the function $u_-(t)$ increases on the interval $(t_1, t_2]$ and takes on its right boundary the negative value $u_-(t_2) < u^{(2)}(t_2) = 0$, it also increases on the interval $(t_2, t_3]$, vanishes at some internal point of this interval and retains the zero value to its right boundary. Analogously, by Lemma 3 the function $u_+(t)$ increases on the interval $(t_1, t_2]$ and takes on its right boundary the positive value $u_+(t_2) > u^{(2)}(t_2) = 0$, decreases on the interval $(t_2, t_3]$, vanishes at some internal point of this interval, and then retains the zero value to its right boundary. Since $u_-(t_3) = u_+(t_3) = u_0(t_3) = 0$, it follows that, when $t \geq t_3$, the identical equalities $u_-(t) \equiv u_0(t)$ and $u_+(t) \equiv u_0(t)$ are satisfied.

In the space of solutions to Eq. (2.2), the integral curves $u_-(t)$ lie in the region bounded by the curves $u^{(2)}(t)$ and $u_-^{(3)}(t)$, and the integral curves $u_+(t)$ lie in the region bounded by the curves $u^{(2)}(t)$ and $u_+^{(3)}(t)$. If $t \geq t_3$, these regions degenerate to the curve $u_0(t)$. Thus, if at time $t = t_1$ the velocity of the rigid body ranges from $u_-^{(3)}(t_1)$ to $u_+^{(3)}(t_1)$, then the body comes to a stop in the upper deceleration zone $[t_2, t_3]$ and remains at rest until time $t = t_3$, and then begins to perform periodic motion which is described by the solution $u_0(t)$, i.e., when $t \geq t_3$, the body performs back-and-forth motion with sticking in the upper and the lower deceleration zone.

Reasoning in a similar way, one can show the following. Solutions with initial conditions satisfying the inequality $-2\pi k\mu < u(t_1) < u_-^{(3)}(t_1)$ at first increase on the interval $(t_1, t_3]$ and then decrease on the interval $(t_3, t_4]$, and then increase again on the interval $(t_4, t_1 + 2\pi]$, vanish at some internal point of this interval and retain the zero value to its right boundary. Solutions with initial conditions satisfying the inequality $u_+^{(3)}(t_1) < u(t_1) < u^{(4)}(t_1)$, where

$$u^{(4)}(t_1) = 4k\mu \operatorname{arccotg} \left(\frac{1 - \sqrt{1 - k^2(\mu^2 - 1)}}{k(\mu - 1)} \right) - \frac{2k^2(\mu - 1) \left(1 - \sqrt{1 - k^2(\mu^2 - 1)} \right)}{k^2 - k^2\mu + 1 - \sqrt{1 - k^2(\mu^2 - 1)}}, \quad (5.9)$$

increase on the interval $(t_1, t_2]$, then decrease on the interval $(t_2, t_4]$, and then increase again on the interval $(t_4, t_1 + 2\pi]$, vanish at some internal point of this interval and retain the zero value to its right boundary. Solutions with initial conditions satisfying the inequality $u^{(4)}(t_1) \leq u(t_1) < 2\pi k\mu$ increase on the interval $(t_1, t_2]$, then decrease and vanish at some internal point of the interval $(t_4, t_1 + 2\pi]$, and then retain the zero value to its right boundary. We note that the solution $u^{(4)}(t)$ given by the initial condition (5.9) describes the motion during which the body, moving at time t_1 with positive velocity, comes to a stop at time t_4 and reaches the periodic mode of motion.

Thus, if the inequality $-2\pi k\mu \leq u(t_1) < u_-^{(3)}(t_1)$ or the inequality $u_+^{(3)}(t_1) < u(t_1) \leq 2\pi k\mu$ is satisfied at the initial instant, then the rigid body comes to a stop in the lower deceleration zone and remains at rest to its right boundary, and then begins to perform periodic back-and-forth motions with sticking in the upper and the lower deceleration zone.

Figure 4a presents the space of solutions to Eq. (2.2), which is based on the above analysis and allows one to obtain a complete qualitative picture of the body motion for parameter values from region I. The solid lines indicate the integral curves dividing the space of solutions into regions for which the motion of the body is qualitatively different. In particular, the integral curves $u_-^{(1)}(t)$ and $u_+^{(1)}(t)$ given by the initial conditions $u_-^{(1)}(t_1) = -2\pi k\mu$ and $u_+^{(1)}(t_1) = 2\pi k\mu$ bound the region corresponding to motion during which the body comes to a stop on the time interval $[t_1; t_1 + 2\pi]$ in the deceleration zone and reaches the periodic mode. Figure 4b shows an enlarged part of Fig. 4a which is bounded by horizontal intermittent straight lines.

Fig. 4. The space of solutions to Eq. (2.2) in region I ($k = 0.7, \mu = 1.5$).

Summing up the results of this section, one can draw the following conclusion. *If the parameters of the problem take values from region I, then at any initial velocity the body reaches in a finite time interval the periodic mode, i.e., it performs back-and-forth motion with sticking in the upper and the lower deceleration zone.*

6. On the qualitative pattern of motion of the rigid body in region II

The analysis of motion of the rigid body on the interval $[t_1, t_1 + 2\pi]$ for parameter values from region II can be carried out in the same way as it was done in the previous section for parameter values from region I. Without going into details of this analysis, we describe its main results. We first note that in region II, in contrast to region I, the solution to Eq. (2.5) on the interval $[t_1, t_1 + 2\pi]$ can take its largest value both at $t = t_1 + 2\pi$ and at $t = t_3$. The pattern of the body motion on this interval will depend on which of these cases will take place.

If the inequality $u_-^{(3)}(t_1) > -2\pi k\mu$ is satisfied, where the quantity $u_-^{(3)}(t_1)$ is calculated by the formula (5.8), then the first case takes place, i.e., the solution to Eq. (2.5) on the interval $[t_1, t_1 + 2\pi]$ takes its largest value at $t = t_1 + 2\pi$. In this case, the motion of the rigid body has the same qualitative pattern as for the parameter values from region I. In particular, if the initial velocity is such that $|u(t_1)| > 2\pi k\mu$, then on the entire interval $[t_1, t_1 + 2\pi]$ the body moves without a stop and its velocity on this interval decreases in absolute value by $2\pi k\mu$. If $|u(t_1)| \leq 2\pi k\mu$, then on the interval $[t_1, t_1 + 2\pi]$ the body comes to a stop in the deceleration zone and remains

at rest until the end of this interval, after which it begins to move 2π -periodically, stopping and sticking only in the lower deceleration zone.

Region II is divided by the curve $u_-^{(3)}(t_1) = -2\pi k\mu$ into two subregions IIa and IIb (see Fig. 5). Figure 6a shows the space of solutions to Eq. (2.2), which gives a complete characteristic of the body motion on the interval $[t_1, t_1 + 2\pi]$ in subregion IIa, where the inequality $u_-^{(3)}(t_1) > -2\pi k\mu$ is satisfied. Figure 6b shows an enlarged rectangular region bounded in Fig. 6a by intermittent lines.

Fig. 5. Subregions IIa and IIb.

Fig. 6. The space of solutions to Eq. (2.2) in subregion IIa ($k = 0.35, \mu = 1.5$).

In subregion IIb, the inequality $u_-^{(3)}(t_1) < -2\pi k\mu$ is satisfied. In this case, the solution to Eq. (2.5) takes on the interval $[t_1, t_1 + 2\pi]$ its largest value at $t = t_3$. This implies that $u_-^{(3)}(t_1)(t_1 + 2\pi) = u_-^{(3)}(t_1) + 2\pi k\mu < 0$, i.e., on the interval $[t_1, t_1 + 2\pi]$ the solution $u_-^{(3)}(t)$ describes the body motion in the negative direction with a stop at time $t = t_3$ and without a stop in the lower deceleration zone. Therefore, at the initial velocity $u(t_1) < u_-^{(3)}(t_1)$ the body moves on the entire interval $(t_1, t_1 + 2\pi]$ in the negative direction, without stops and during the motion its velocity decreases in absolute value by $2\pi k\mu$. In the space of solutions presented in Fig. 7a, such a motion corresponds to integral curves lying below the curve $u_-^{(3)}(t)$.

Fig. 7. The space of solutions to Eq. (2.2) in subregion IIb ($k = 0.225, \mu = 1.5$).

When $u(t_1) > 2\pi k\mu$, the rigid body moves on the interval $(t_1, t_1 + 2\pi]$ in the positive direction without stops, and its velocity decreases by $2\pi k\mu$. This motion corresponds to a region of the space of solutions that is situated above the integral curve $u_+^{(1)}(t)$ (see Fig. 7a), which is given by the initial condition $u_+^{(1)}(t_1) = 2\pi k\mu$. Now assume that at the initial instant the inequality $u_-^{(3)}(t_1) \leq u(t_1) \leq 2\pi k\mu$ is satisfied. In this case, the integral curve $u_-^{(1)}(t)$ plays an important role in describing the pattern of motion of the body. On the interval $(t_1, t_1 + 2\pi]$ this curve corresponds to motion of the body with three stops: at times $t_*^{(1)}$ and $t_{**}^{(1)}$ on the intervals $[t_1, t_2]$ and $[t_3, t_4]$, respectively, as well as at time $t = t_1 + 2\pi$ on the right boundary of the lower deceleration zone. After each stop the body starts moving in the opposite direction. It can be shown that at parameter values from subregion IIb such an integral curve exists, is unique, and is given by the initial condition

$$u_-^{(1)}(t_1) = \cos t_*^{(1)} - \cos t_1 - k\mu(t_*^{(1)} - t_1) - k(\sin t_*^{(1)} - \sin t_1), \quad (6.1)$$

where $t_*^{(1)}$ is defined by solving the system of equations

$$\begin{aligned}\cos t_*^{(1)} - \cos t_{**}^{(1)} - k \left[\mu(t_{**}^{(1)} - t_*^{(1)}) + \sin t_{**}^{(1)} - \sin t_*^{(1)} \right] &= 0, \\ \cos t_{**}^{(1)} - \cos t_1 + k \left[\mu(t_1 + 2\pi - t_{**}^{(1)}) + \sin t_1 - \sin t_{**}^{(1)} \right] &= 0,\end{aligned}\tag{6.2}$$

for $t_*^{(1)}$ and $t_{**}^{(1)}$.

We note that at $t \geq t_1 + 2\pi$ the integral curve $u_-^{(1)}(t)$ describes periodic motion with sticking in the lower deceleration zone, i.e., $u_-^{(1)}(t) \equiv u_0(t)$, where, as before, $u_0(t)$ denotes a periodic solution to Eq. (2.2), which is given by the initial condition $u_0(t_1) = 0$.

Based on the analysis of the behavior of the integral curves (see Fig. 7), we arrive at the following conclusions on the motion of the body in region IIb.

1. If at the initial instant the inequality $u_-^{(1)}(t_1) < u(t_1) < 2\pi k\mu$ is satisfied, then the rigid body does not stop in the upper deceleration zone, but comes to a stop in the lower deceleration zone and remains at rest until time $t = t_1 + 2\pi$, after which it starts moving 2π -periodically with sticking only in the lower deceleration zone. In Fig. 7a, such a motion corresponds to a region bounded by the integral curves $u_-^{(1)}(t)$ and $u_+^{(1)}(t)$, and at $t \geq t_1 + 2\pi$ it degenerates to the curve $u_0(t)$.
2. If at the initial instant the inequality $u_-^{(3)}(t_1) \leq u(t_1) < u_+^{(3)}(t_1)$ is satisfied, then the rigid body comes to a stop in the upper deceleration zone and remains at rest until time $t = t_3$, then it starts moving in the negative direction and moves with velocity $u_-^{(3)}(t)$. In Fig. 7a, such a motion corresponds to a region bounded by the integral curves $u_-^{(3)}(t)$ and $u_+^{(3)}(t)$, for $t \geq t_3$ these integral curves coincide ($u_+^{(3)}(t) \equiv u_-^{(3)}(t)$) and this region degenerates to one integral curve.
3. If at the initial instant the inequality $u_+^{(3)}(t_1) \leq u(t_1) < u_-^{(1)}(t_1)$ is satisfied, then on the interval $(t_1, t_1 + 2\pi]$ the body moves without stops in the deceleration zones and at time $t = t_1 + 2\pi$ it has nonzero (negative) velocity. In Fig. 7a, such a motion corresponds to a region bounded by the integral curves $u_-^{(1)}(t)$ and $u_+^{(3)}(t)$; this region is shown in grey.

We now prove that for parameter values from subregion IIb, irrespective of the value of the initial velocity, the rigid body will reach after a finite time interval a periodic mode of motion corresponding to the integral curve $u_0(t)$. To do so, it suffices to show that if at $t = t_1$ the velocity of the body satisfies the inequality $u_-^{(3)}(t_1) \leq u(t_1) < u_-^{(1)}(t_1)$, then the body will come to a stop after a finite time interval in the lower deceleration zone.

We first show that for parameter values from subregion IIb there always exists such a natural number N that $u_+^{(3)}(t) \equiv u_0(t)$ at $t > t_1 + 2\pi N$. In region II the integral curve $u_+^{(3)}(t)$ corresponds to motion during which the body, moving at time $t = t_1$ in the negative direction, comes to a stop on the interval (t_1, t_2) at time $t = t_*^{(3)}$. The integral curve $u_+^{(3)}(t)$ in region II (in contrast to region I) is given by the initial condition

$$u_+^{(3)}(t_1) = \cos t_*^{(3)} - \cos t_1 - k\mu(t_*^{(3)} - t_1) - k(\sin t_*^{(3)} - \sin t_1),\tag{6.3}$$

where $t_*^{(3)}$ is defined by solving the equation

$$\cos t_3 - \cos t_*^{(3)} + k \left[\mu(t_3 - t_*^{(3)}) + \sin t_3 - \sin t_*^{(3)} \right] = 0.\tag{6.4}$$

A straightforward integration of Eq. (2.2) yields

$$u_+^{(3)}(t_1 + 2\pi) = \cos t_3 - \cos t_1 + k\mu(t_1 + 2\pi - t_3) + k(\sin t_1 - \sin t_3). \quad (6.5)$$

Now, taking into account (6.3), (6.4) and (6.5), we calculate the difference

$$u_+^{(3)}(t_1 + 2\pi) - u_+^{(3)}(t_1) = 2k\mu(\pi + t_*^{(3)} - t_3) + 2k(\sin t_*^{(3)} - \sin t_3). \quad (6.6)$$

By virtue of the obvious chain of inequalities $\pi + t_*^{(3)} > \pi + t_1 > t_3$ the first term on the right-hand side (6.6) is positive. The second term on the right-hand side (6.6) is also positive, since $\pi < t_3 < 2\pi$ and $0 < t_*^{(3)} < \pi$. Thus, the difference (6.6) is positive and hence the inequality $u_+^{(3)}(t_1 + 2\pi) > u_+^{(3)}(t_1)$ is satisfied.

By virtue of Lemma 1 the values $u_+^{(3)}(t_1 + 2\pi n)$, $n \in \mathbb{N}$, form a monotonic sequence, and by virtue of the last inequality this sequence is nondecreasing. Reasoning by contradiction, we show that this sequence is bounded, namely, $u_+^{(3)}(t_1 + 2\pi n) \leq 0$. Indeed, let $u_+^{(3)}(t_1 + 2\pi n_*) > 0$, where n_* is some natural number. Then, taking into account the fact that the above conclusions on the motion of the body on the interval $[t_1, t_1 + 2\pi]$ hold true also on the interval $[t_1 + 2\pi n_*, t_1 + 2\pi(n_* + 1)]$, we arrive at the following conclusion: either $u_+^{(3)}(t_1 + 2\pi(n_* + 1)) = 0$ or $u_+^{(3)}(t_1 + 2\pi(n_* + 1)) = u_+^{(3)}(t_1 + 2\pi n_*) - 2\pi k\mu$. In both cases, $u_+^{(3)}(t_1 + 2\pi(n_* + 1)) < u_+^{(3)}(t_1 + 2\pi n_*)$, which contradicts the monotonicity property.

By virtue of monotonicity and boundedness the sequence $u_+^{(3)}(t_1 + 2\pi n)$ has a limit. We show that this limit is zero. Indeed, if this is not the case, then there exists a limiting mode of motion of the body with 2π -periodically changing velocity during which the body stops neither in the lower nor in the upper deceleration zone, i.e., it moves without sticking. The latter contradicts Lemma 2, which implies that the above-mentioned mode of body motion does not exist in region II, therefore $\lim_{n \rightarrow \infty} u_+^{(3)}(t_1 + 2\pi n) = 0$. Moreover, starting from some $n = N_*$, all terms of the sequence $u_+^{(3)}(t_1 + 2\pi n)$ vanish, since otherwise there exists a limiting 2π -periodic mode of motion with a stop of the body on the right boundary of the lower deceleration zone, which is impossible by virtue of existence of a periodic mode of motion with a stop inside the lower deceleration zone. Consequently, there exists such a natural number N_* that $u_+^{(3)}(t) \equiv u_0(t)$ with $t > t_1 + 2\pi N_*$.

We now recall that if the initial velocity of the body satisfies the inequality $u_-^{(3)}(t_1) \leq u(t_1) < u_+^{(3)}(t_1)$, then at $t > t_3$ the body moves with velocity $u_+^{(3)}(t)$. This means that in this case, with $t > t_1 + 2\pi N_*$, the body reaches the mode of motion with 2π -periodically changing velocity $u_0(t)$.

If the initial velocity of the body lies in the interval of values $u_+^{(3)}(t_1) < u(t_1) < u_-^{(1)}(t_1)$, then, at $t > t_1$, the inequality $u_+^{(3)}(t) \leq u(t) \leq u_-^{(1)}(t)$ is satisfied. Therefore, $u(t) \equiv u_0(t)$ for $t > t_1 + 2\pi N$, where N is some natural number ($1 \leq N \leq N_*$). Consequently, in this case also the body reaches in finite time the mode of motion with 2π -periodically changing velocity $u_0(t)$.

Based on the results of this section, we arrive at the following conclusion. *If the system parameters take values from region II, then at any initial velocity the body reaches in finite time the 2π -periodic mode, i.e., it moves with 2π -periodically changing velocity, sticking only in the lower deceleration zone and moving during the period in the positive direction.*

7. Investigation of the motion of the rigid body in region III

The pattern of motion of the rigid body for parameter values from region III can be investigated in the same way as it was done for regions I and II. In particular, at the initial velocity $u(t_1) < u_-^{(3)}(t_1)$ (where $u_-^{(3)}(t_1)$ is calculated by the formula (5.8)), the body moves on the interval $(t_1, t_1 + 2\pi]$ in the negative direction, and during the motion its velocity decreases in absolute value by $2\pi k\mu$, and when $u(t_1) > 2\pi k\mu$, the body moves on this interval in the positive direction and its velocity decreases by $2\pi k\mu$. If the initial velocity of the body lies in the range $u_-^{(3)}(t_1) \leq u(t_1) \leq 2\pi k\mu$, then on the interval $(t_1, t_1 + 2\pi]$ it comes to stops, reversing the direction of motion or sticking in the deceleration zones. We consider this case in greater detail.

The space of solutions to Eq. (2.2), constructed for parameter values from region III, is presented in Fig. 8. As in region II, a crucial role is played here by the integral curves $u_-^{(1)}(t)$, $u_+^{(3)}(t)$ and $u_-^{(3)}(t)$, which are indicated by solid lines in Fig. 8a. We only note that, in contrast to region II, in region III the integral curve $u_-^{(1)}(t)$ corresponds to motion of the rigid body with two stops: at time $t_*^{(1)}$ on the interval $[t_3, t_4]$ and at time $t = t_1 + 2\pi$ on the right boundary of the lower deceleration zone. In region III, this integral curve is given by the initial condition

$$u_-^{(1)}(t_1) = \cos t_*^{(1)} - \cos t_1 + k\mu(t_*^{(1)} - t_1) + k(\sin t_*^{(1)} - \sin t_1), \quad (7.1)$$

where $t_*^{(1)}$ is the root of the equation

$$\cos t_*^{(1)} - \cos t_1 + k \left[\mu(t_1 + 2\pi - t_*^{(1)}) + \sin t_1 - \sin t_*^{(1)} \right] = 0, \quad (7.2)$$

Fig. 8. The space of solutions to Eq. (2.2) in region III ($k = 0.15, \mu = 1.5$).

which belongs to the interval $[t_3, t_4]$. The initial condition defining the integral curve $u_+^{(3)}(t)$ is calculated, as in region II, by the formulae (6.3) and (6.4), and the integral curve $u_-^{(3)}(t)$ is given by the initial condition (5.8).

We also recall that by virtue of Lemma 2 Eq. (2.2) admits a 2π -periodic solution $u_*(t)$ for parameter values from region III. It is given by the initial condition

$$u_*(t_1) = \cos t_* - \cos t_1 - k\mu(t_* - t_1) - k(\sin t_* - \sin t_1), \quad (7.3)$$

where t_* is calculated by the formula (4.15). In Fig. 8a, the integral curve $u_*(t)$ is shown as a heavy line.

Based on the analysis of the behavior of the integral curves of Eq. (2.2), we can make the following statements on the motion of the body on the time interval $(t_1, t_1 + 2\pi]$.

1. If the inequality $u(t_1) \geq 2\pi k\mu$ is satisfied at the initial instant, then the body moves on the entire time interval $(t_1, t_1 + 2\pi]$ in the positive direction, and its velocity on this interval decreases by $2\pi k\mu$. In Fig. 8a, such a motion corresponds to a region lying above the integral curve $u_+^{(1)}(t)$. If the inequality $u(t_1) \leq u_-^{(3)}(t_1)$ is satisfied, the rigid body moves on the entire time interval $(t_1, t_1 + 2\pi]$ in the negative direction, and its velocity on this time interval decreases in absolute value by $2\pi k\mu$. In Fig. 8a, such a motion corresponds to a region lying below the integral curve $u_-^{(3)}(t)$.
2. If the inequality $u_-^{(1)}(t_1) < u(t_1) < 2\pi k\mu$ is satisfied at the initial instant, then the body does not stop in the upper deceleration zone, but comes to a stop in the lower deceleration zone and remains at rest until time $t = t_1 + 2\pi$. In Fig. 8a, such a motion corresponds to a region bounded by the integral curves $u_-^{(1)}(t)$ and $u_+^{(1)}(t)$. For $t \geq t_1 + 2\pi$ these integral curves coincide ($u_-^{(1)}(t) \equiv u_+^{(1)}(t)$) and this region degenerates to one integral curve.
3. If the inequality $u_-^{(3)}(t_1) \leq u(t_1) < u_+^{(3)}(t_1)$ is satisfied at the initial instant, then the body comes to a stop in the upper deceleration zone and remains at rest until time $t = t_3$, and then it starts moving in the negative direction with velocity $u_-^{(3)}(t)$. In Fig. 8a, such a motion corresponds to a region bounded by the integral curves $u_-^{(3)}(t)$ and $u_+^{(3)}(t)$, for $t \geq t_3$ these integral curves coincide, ($u_+^{(3)}(t) \equiv u_-^{(3)}(t)$), and this region degenerates to one integral curve.
4. If the inequality $u_+^{(3)}(t_1) \leq u(t_1) < u_-^{(1)}(t_1)$ is satisfied at the initial instant, then on the interval $(t_1, t_1 + 2\pi]$ the body moves without a stop in the lower deceleration zone and has at time $t = t_1 + 2\pi$ nonzero (negative) velocity. In Fig. 8a, such a motion corresponds to a region bounded by the integral curves $u_-^{(1)}(t)$ and $u_+^{(3)}(t)$; this region is shown in grey. This region contains also the integral curve $u_*(t)$, which corresponds to motion of the body with 2π -periodically changing velocity without stops in the deceleration zones.

The statements formulated above allow us to draw important conclusions on the motion of the body on an infinite time interval. In particular, we show that for parameter values from region III, irrespective of the value of the initial velocity of motion, the body asymptotically approaches the periodic mode of motion corresponding to the integral curve $u_*(t)$.

First of all, we note that if the initial velocity of the body satisfies the inequality $u(t_1) > u_-^{(1)}(t_1)$ or the inequality $u(t_1) < u_-^{(3)}(t_1)$, then the inequality $u_-^{(1)}(t_1) \leq u(t_1 + 2\pi n_*) \leq u_+^{(3)}(t_1)$ is satisfied for some natural n_* . Thus, it suffices to show that if $u_+^{(3)}(t_1) \leq u(t_1) \leq u_-^{(1)}(t_1)$, then $\lim_{n \rightarrow \infty} u(t_1 + 2\pi n) = u_*(t_1)$.

We first prove that $\lim_{n \rightarrow \infty} u_+^{(3)}(t_1 + 2\pi n) = u_*(t_1)$. Indeed, as was shown in Section 6, the inequality $u_+^{(3)}(t_1) < u_+^{(3)}(t_1 + 2\pi)$ holds. Therefore, by virtue of Lemma 1, the values $u_+^{(3)}(t_1 + 2\pi n)$, $n \in \mathbf{N}$, form a nondecreasing sequence. This sequence is obviously bounded by the quantity $u_*(t_1)$, therefore it has some limit, which we denote by $\tilde{u}$. Then the solution to Eq. (2.2), given by the initial condition $u(t_1) = \tilde{u}$, is 2π -periodic. On the other hand, by Lemma 2 Eq. (2.2) in region III has a unique 2π -periodic solution $u_*(t)$ and hence $\tilde{u} = u_*(t_1)$.

Reasoning in a similar way, it is easy to show that $\lim_{n \rightarrow \infty} u_-^{(1)}(t_1 + 2\pi n) = u_*(t_1)$. Indeed, in region III the inequality $u_0(t_1 + 2\pi) < 0$ is satisfied on the solution $u_0(t)$ (with the initial condition $u_0(t_1) = 0$). Therefore, since $u_-^{(1)}(t_1 + 2\pi) = 0$, we have $u_-^{(1)}(t_1 + 2\pi) > u_0(t_1 + 2\pi)$. Consequently, $u_-^{(1)}(t_1) > u_0(t_1) = 0$, and hence $u_-^{(1)}(t_1) > u_-^{(1)}(t_1 + 2\pi)$. Therefore, by virtue of Lemma 1, the values $u_-^{(1)}(t_1 + 2\pi n)$, $n \in \mathbf{N}$, form a nondecreasing sequence. This sequence is bounded from below and has a limit equal to $u_*(t_1)$.

Now let $u(t)$ be a solution to Eq. (2.2) for which the inequality $u_+^{(3)}(t_1) < u(t_1) < u_-^{(1)}(t_1)$ is satisfied at the initial time instant $t = t_1$. Then the inequality $u_+^{(3)}(t_1 + 2\pi n) < u(t_1 + 2\pi n) < u_-^{(1)}(t_1 + 2\pi n)$ is satisfied for any natural n and hence $\lim_{n \rightarrow \infty} u(t_1 + 2\pi n) = u_*(t_1)$.

Thus, in region III all solutions to Eq. (2.2) asymptotically approach a 2π -periodic solution $u_*(t)$ as $t \rightarrow \infty$.

The main result of this section can be formulated as follows. *If the system parameters take values from region III, then after a finite time interval the rigid body will move at any initial velocity without sticking, stopping outside the deceleration zones and asymptotically approaching the 2π -periodic mode of motion.*

8. Conclusions

The plane of system parameters k, μ is divided into three regions I, II and III (see Fig. 3), in each of which there exists a unique mode of motion with 2π -periodically changing velocity. In these regions, periodic motion is qualitatively different.

In region I the rigid body, moving with 2π -periodically changing velocity, comes to a stop twice on the time interval 2π , after which it remains at rest for a finite time interval (sticks), and then continues moving in the opposite direction. In this case, both the velocity of the body and its coordinate changes periodically, so that during the time interval 2π the body performs zero displacement. The intervals of rest of the body take place when the internal mass passes the so-called (upper and lower) deceleration zones, in which the horizontal component of the inertial force applied to the internal mass does not exceed the sliding friction force applied to the body. Irrespective of the value of the initial velocity of the body, its motion reaches in finite time the above-mentioned periodic mode.

In region II, the body, moving with 2π -periodically changing velocity, does not stop when the internal mass passes the upper deceleration zone; therefore, during the time interval 2π the body sticks only once, when the internal mass is in the lower deceleration zone. In this case the coordinate of the body is no longer a periodic function of time, and the body moves in the positive direction during a complete revolution of the internal mass in a circle. As in region I, irrespective of the value of the initial velocity of the body, its motion reaches in finite time the above-mentioned periodic mode.

In region III, a characteristic feature of motion with 2π -periodically changing velocity is that the body performs stops outside the deceleration zones. After each stop the body does not stick, but immediately starts moving in the opposite direction, performing motions in the positive direction during a complete revolution of the internal mass in a circle. This periodic motion occurs in the limiting mode, which the motion of the body approaches asymptotically irrespective of the value of its initial velocity.

References

- [1] Vartholomeos, P. and Papadopoulos, E., Dynamics, Design and Simulation of a Novel Microrobotic Platform Employing Vibration Microactuators, *J. Dyn. Sys., Meas., Control*, 2005, vol. 128, no. 1, pp. 122–133.
- [2] Vartholomeos, P. and Papadopoulos, E., Analysis and Experiments on the Force Capabilities of Centripetal-Force-Actuated Microrobotic Platforms, *IEEE Trans. on Robotics*, 2008, vol. 24, no. 3, pp. 588–599.
- [3] Vartholomeos, P., Papadopoulos, E., and Vlachos, K., Analysis and Motion Control of a Centrifugal-Force Microrobotic Platform, *IEEE Trans. Autom. Sci. Eng.*, 2013, vol. 10, no. 3, pp. 545–553.
- [4] Vlachos, K., Papadimitriou, D., and Papadopoulos, E., Vibration-Driven Microrobot Positioning Methodologies for Nonholonomic Constraint Compensation, *Engineering*, 2015, vol. 1, no. 1, pp. 66–72.
- [5] Jatsun, S. F., Mischenko, V. Ya., and Safarov, D. I., Investigation of a Two-Mass Vibrating Motion of the Robot, *Izv. Vyssh. Uchebn. Zaved. Mashinostr.*, 2006, no. 5, pp. 32–42 (Russian).
- [6] Jatsun, S. F., Razin'kova, A. V., and Grankin, A. N., Investigation of the Vibrating Motion of the Robot with Electromagnetic Drive, *Izv. Vyssh. Uchebn. Zaved. Mashinostr.*, 2006, no. 10, pp. 53–64 (Russian).
- [7] Yatsun, S. F., Lupekhina, I. V., and Sapronov, K. A., Modeling of Motion of Hopping Vibration Driven Microrobot, *Izv. Kursk. Gos. Tekh. Univ.*, 2009, no. 2(27), pp. 25–31 (Russian).
- [8] Yatsun, S. F., Bezmen, P. A., Sapronov, K. A., and Rublev, S. B., Dynamics of the Vibration Mobile Robot with Internal Movable Mass, *Izv. Kursk. Gos. Tekh. Univ.*, 2010, no. 2(31), pp. 21a–31 (Russian).
- [9] Volkova, L. Yu. and Yatsun, S. F., Simulation of the Plane Controlled Motion of a Three-Mass Vibration System, *J. Comput. Syst. Sci. Int.*, 2012, vol. 51, no. 6, pp. 859–878; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2012, no. 6, pp. 122–141.
- [10] Yatsun, S. F. and Volkova, L. Yu., Simulation of Dynamic Modes of Vibration Robot Moving along the Surface of the Viscous Resistance, *Spektakhnika i svyaz*, 2012, no. 3, pp. 25–29 (Russian).
- [11] Volkova, L. Yu. and Yatsun, S. F., Studying of Regularities of Movement of the Jumping Robot at Various Positions of a Point of Fixing of a Foot, *Nelin. Dinam.*, 2013, vol. 9, no. 2, pp. 327–342 (Russian).
- [12] Wang, Q. M., Zhang, W. M., and Ju, J. Ch., Kinematics and Dynamics Analysis of a Micro-Robotic Platform Driven by Inertial-Force Propulsion, in *Engineering Decisions for Industrial Development*, J. Xu, P. Wang, Zh. Fang (Eds.), Applied Mechanics and Materials, vol. 733, Stäfa: TransTech, 2015, pp. 531–534.
- [13] Chernous'ko, F. L., On the Motion of a Body Containing a Movable Internal Mass, *Dokl. Phys.*, 2005, vol. 50, no. 11, pp. 593–597; see also: *Dokl. Ross. Akad. Nauk*, 2005, vol. 405, no. 1, pp. 56–60.
- [14] Chernous'ko, F. L., Analysis and Optimization of the Motion of a Body Controlled by Means of a Movable Internal Mass, *J. Appl. Math. Mech.*, 2006, vol. 70, no. 6, pp. 819–842; see also: *Prikl. Mat. Mekh.*, 2006, vol. 70, no. 6, pp. 915–941.

- [15] Bolotnik, N. N., Zeidis, I. M., Zimmermann, K., and Yatsun, S. F., Dynamics of Controlled Motion of Vibration-Driven Systems, *J. Comput. Syst. Sci. Int.*, 2006, vol. 45, no. 5, pp. 831–840; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2006, no. 5, pp. 157–167.
- [16] Bolotnik, N. N. and Figurina, T. Yu., Optimal Control of the Rectilinear Motion of a Rigid Body on a Rough Plane by Means of the Motion of Two Internal Masses, *J. Appl. Math. Mech.*, 2008, vol. 72, no. 2, pp. 126–135; see also: *Prikl. Mat. Mekh.*, 2008, vol. 72, no. 2, pp. 216–229.
- [17] Sobolev, N. A. and Sorokin, K. S., Experimental Investigation of a Model of a Vibration-Driven Robot with Rotating Masses, *J. Comput. Syst. Sci. Int.*, 2007, vol. 46, no. 5, pp. 826–835; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2007, no. 5, pp. 161–170.
- [18] Figurina, T. Yu., Optimal Motion Control for a System of Two Bodies on a Straight Line, *J. Comput. Syst. Sci. Int.*, 2007, vol. 46, no. 2, pp. 227–233; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2007, no. 2, pp. 65–71.
- [19] Sorokin, K. S., Motion of a Mechanism along a Rough Inclined Plane Using the Motion of Internal Oscillating Masses, *J. Comput. Syst. Sci. Int.*, 2009, vol. 48, no. 6, pp. 993–1001; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2009, no. 6, pp. 150–158.
- [20] Ivanov, A. P. and Sakharov, A. V., Dynamics of Rigid Body, Carrying Moving Masses and Rotor, on a Rough Plane, *Nelin. Dinam.*, 2012, vol. 8, no. 4, pp. 763–772 (Russian).
- [21] Zhan, X. and Xu, J., Locomotion Analysis of a Vibration-Driven System with Three Acceleration Controlled Internal Masses, *Adv. Mech. Eng.*, 2015, vol. 7, no. 3, 12 pp.
- [22] Chernousko, F. L., Motion of a Body along a Plane under the Influence of Movable Internal Masses, *Dokl. Phys.*, 2016, vol. 61, no. 10, pp. 494–498; see also: *Dokl. Akad. Nauk*, 2016, vol. 470, no. 4, pp. 406–410.
- [23] Bardin, B. S., On Non-Impact Jumps of a Body Carrying Movable Masses, in *Proc. of the 18th Internat. Symp. “Dynamics of Vibroimpact (Strong Nonlinear) Systems” (Moscow, 2015)*, pp. 42–49 (Russian).
- [24] Bilchenko, G. G., The Influence of Mobile Load on the Carrier Motion, in *Proc. of the 11th Int. Chetaev Conf. “Analytical Mechanics, Stability and Control”: Vol. 1, Sect. 1. Analytical Dynamics*, S. N. Vassilyev et al. (Eds.), pp. 37–44 (Russian).
- [25] Fang, H. and Xu, J., Stick-Slip Effect in a Vibration-Driven System with Dry Friction: Sliding Bifurcations and Optimization, *J. Appl. Mech.*, 2013, vol. 81, no. 5, 051001, 10 pp.
- [26] Chernous’ko, F. L. and Bolotnik, N. N., Mobile Robots Controlled by the Motion of Internal Bodies, *Tr. Inst. Mat. i Mekh. UrO RAN*, 2010, vol. 16, no. 5, pp. 213–222 (Russian).
- [27] Bolotnik, N. N., Figurina, T. Yu., and Chernous’ko, F. L., Optimal Control of the Rectilinear Motion of a Two-Body System in a Resistive Medium, *J. Appl. Math. Mech.*, 2012, vol. 76, no. 1, pp. 1–14; see also: *Prikl. Mat. Mekh.*, 2012, vol. 76, no. 1, pp. 3–22.
- [28] Bolotnik, N. N., Nunuparov, A. M., and Chashchukhin, V. G., Capsule-Type Vibration-Driven Robot with an Electromagnetic Actuator and an Opposing Spring: Dynamics and Control of Motion, *J. Comput. Syst. Sci. Int.*, 2016, vol. 55, no. 6, pp. 986–1000; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2016, no. 6, pp. 146–160.
- [29] Golitsyna, M. V., Periodic Mode of Movement of the Vibration-Driven Robot with Limited Control, *Prikl. Mat. Mekh.*, 2018, vol. 82, no. 1, pp. 3–15 (Russian).
- [30] Golitsyna, M. V. and Samsonov, V. A., Estimating the Domain of Admissible Parameters of a Control System of a Vibratory Robot, *J. Comput. Syst. Sci. Int.*, 2018, vol. 57, no. 2, pp. 255–272; see also: *Izv. Akad. Nauk. Teoriya i Sistemy Upravleniya*, 2018, no. 2, pp. 85–101.
- [31] Chernous’ko, F. L., Optimal Control of the Motion of a Two-Mass System, *Dokl. Math.*, 2018, vol. 97, no. 3, pp. 295–299; see also: *Dokl. Akad. Nauk*, 2018, vol. 480, no. 5, pp. 528–532.
- [32] Bardin, B. S. and Panev, A. S., On Periodic Motions of the Body with Movable Internal Mass over a Horizontal Surface, *Trudy MAI*, 2015, vol. 84, Art. 5 (Russian).
- [33] Bardin, B. S. and Panev, A. S., On Dynamics of a Rigid Body Moving on a Horizontal Plane by Means of Motion of an Internal Particle, *Vibroeng. Procedia*, 2016, vol. 8, pp. 135–141.

- [34] Panev, A. S., On Motion of a Rigid Body with Mobile Internal Mass on a Horizontal Plane in a Viscous Medium, *Trudy MAI*, 2018, vol. 98, Art. 2 (Russian).
- [35] Bardin, B. S. and Panev, A. S., On the Motion of a Rigid Body with an Internal Moving Point Mass on a Horizontal Plane, *AIP Conf. Proc.*, 2018, vol. 1959, no. 1, 030002.
- [36] Filippov, A. F., Differential Equations with Discontinuous Right-Hand Side, *Mat. Sb. (N. S.)*, 1960, vol. 51(93), no. 1, pp. 99–128 (Russian).
- [37] Filippov, A. F., *Differential Equations with Discontinuous Righthand Sides*, Math. Appl., vol. 18, Dordrecht: Springer, 1988.