NONLINEAR PHYSICS AND MECHANICS

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Phase Transition to Quadrupolar Vortices in a Spherical Model of the Energy-Enstrophy Theory — Exact Solution

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A new energy-enstrophy model for the equilibrium statistical mechanics of barotropic flow on a sphere is introduced and solved exactly for phase transitions to quadrupolar vortices when the kinetic energy level is high. Unlike the Kraichnan theory, which is a Gaussian model, we substitute a microcanonical enstrophy constraint for the usual canonical one, a step which is based on sound physical principles. This yields a spherical model with zero total circulation, a microcanonical enstrophy constraint and a canonical constraint on energy, with angular momentum fixed to zero. A closed-form solution of this spherical model, obtained by the Kac–Berlin method of steepest descent, provides critical temperatures and amplitudes of the symmetry-breaking quadrupolar vortices. This model and its results differ from previous solvable models for related phenomena in the sense that they are not based on a mean-field assumption.

Keywords: energy-enstrophy theory, long-range spherical model, phase transition, rotating atmospheres

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1. Introduction

This paper contains new results on the application of equilibrium statistical mechanics to geophysical and astrophysical flows [1, 6, 15, 18, 23, 25–28, 30–33]. Our results complement recent results on the statistical mechanics of atmospheric flows on rotating planets and related problems connected to the energy-*enstrophy* model [5, 6, 11, 22] including a previous paper by this author on the case of nonzero angular momentum [21]. The main difference between the current paper and [21] is that, in the zero angular momentum case here, we will show that the phase transition from the well-mixed state breaks symmetry to a new coherent structure, called the quadrupolar vortex state. Experimental data on self-organized macroscopic angular momentum in 2D flows available for the planar geometry and numerical results [27, 28], in particular, those concerning planar vortex flows in a box, have shown that domain-scale coherent vortices that spin up spontaneously from small eddies are robust and sometimes intermittent. This motivated the previous nonzero angular momentum study. For a (near) perfect spherical surface, the inviscid barotropic (nondivergent) flow conserves fluid angular momentum. This situation, in turn, motivates the current study.

In brief, we outline here the main physical ideas on which the formulation of the spherical models of energy-*enstrophy* theory are based. Firstly, Stokes' theorem implies the conservation of zero total circulation of the flows, which then leads naturally to fixing it microcanonically in the statistical mechanics model. Next, *enstrophy* and angular momentum are conserved in ideal barotropic vorticity equations. Assuming this, we impose microcanonical constraints on the *enstrophy* or square-norm of the vorticity, and the angular momentum of the flow. Lastly, the kinetic energy of the flow in the atmosphere is generated by insolation and gravitational-thermal overturning. This suggests that, as a first step, a plausible equilibrium statistical model is one where the kinetic energy is constrained canonically, hence a conjugate variable, namely, the temperature, and the angular momentum are fixed at zero. Unlike other theories [25] which do not have simple closed-form solutions, these spherical models do not constrain any higher vorticity moments. The phase spherical models do not refer to the fact that the flow is on a sphere, but to the microcanonical *enstrophy* constraint which has been shown in the author's previous computationally-based works [12–14, 20, 24] to be precisely the Kac–Berlin spherical constraint on the square-norm of the barotropic vorticity.

The main objectives of this paper are as follows: (I) discretized implementation of the microcanonical *enstrophy*, total circulation and angular momentum constraints leading to a spin-lattice model for the energy-*enstrophy* theory that is grounded in the physics of ideal barotropic fluid flows on a perfect spherical surface, and (II) exact solutions of the resulting spherical model with finite long-range interactions using the Kac–Berlin method [17].

The main point is that the exact (non-meanfield) solutions of the resulting spherical model support high-energy phase transitions to quadrupolar vortices. Effective computer simulations and mean-field approximations of these energy-*enstrophy* models have been discussed elsewhere [6, 9, 12–14, 20, 24, 32]. It is well known that in 2D the mean-field approximation is exact for equilibrium statistical mechanics provided that the interactions in the model are of infinite range. Indeed, rigorous results for the exactness of mean field have been obtained for some specific flow geometries even when the interactions are of long but finite range. However, we do not know of any previous work that establishes the exactness of the mean field for the barotropic vorticity model, which will be shown below to have finite long-range interactions.

We note here several important (but well-known) facts concerning the statistical mechanics approach: (A) the spin-lattice models derived here are viewed as Hamiltonian discrete approxi-

mations to the barotropic kinetic energy, and a key parameter here is the number N of lattice sites uniformly spaced on the surface of the sphere, (B) as in all calculations of phase transitions in equilibrium statistical mechanics theory, we must take the so-called thermodynamic limit, as N tends to infinity in this family of spin-lattice models, (C) unlike the bulk or extensive thermodynamic limit in condensed matter physics where the volume of the physical domain of interest tends to infinity with N , phase transitions in fluid dynamics must instead be based on a nonextensive thermodynamic limit, one in which the physical domain volume is fixed as the number N of lattice sites tends to infinity, and (D) this nonextensive thermodynamic limit is taken below by a characteristic scaling of the inverse temperature, $\beta = \beta' N$.

Finally, contrary to the popular belief, there are in the literature several spherical models with finite long-range interactions which support nontrivial phase transitions to ordered phases at sufficiently small but nonzero numerical values of the temperature [4, 17]. We will calculate in closed form the free energy, critical temperature and the energy amplitudes in the quadrupolar modes as a function of the fixed amount of enstrophy in the flow. It is not surprising that the critical temperature is linear in the fixed enstrophy.

2. Background — Dynamics and Statistical Mechanics

2.1. Statistical Mechanics of 2D Fluid Flows

Fjortoft's and, later, Kraichnan's study of energy inverse cascades in nearly inviscid quasi-2D turbulence — a non-equilibrium result — renewed interest in Onsager's approach [16] to 2D turbulence which is based on equilibrium statistical mechanics [29]. The mean-field sinh-Poisson equations arising from the Lagrangian vortex gas methodology have yielded notable results on the emergence of domain-scale flows consisting of one or several large coherent vortices [16, 29].

At the same time, the equilibrium statistical mechanics of the 2D Euler equations and the barotropic vorticity model in spectral and lattice forms have been studied extensively [1, 8, 15, 25]. The classical theory in this field is Kraichnan's energy-enstrophy theory [1, 8], which is based on the key assumption of existence of an inverse energy cascade that supports relatively short equilibration times during which the total flow energy and enstrophy are approximately fixed. Indeed, before the successful works of Miller and Robert [25, 26], which support phase transitions involving the emergence of macroscopic angular momentum, Kraichnan's energy-enstrophy theory was considered to be the end of the line of thought begun by Onsager. Recent applications of the Miller–Robert theory include [6, 32].

Although Onsager introduced the notion of negative temperatures to vortical flows in the 1940s with his seminal paper [16], Kraichnan and others solved the Gaussian energy-enstrophy theories for nearly inviscid 2D flows and showed that they *did not* support any interesting phase transitions to large-scale coherent flow structures even at negative temperatures. This inability of the classical energy-enstrophy theories to predict phase transitions to domain-scale flows is not due to the use of an incorrect energy functional, nor is it because of a fundamental shortcoming of the basic assumptions used to formulate these theories. The reasons for this inability are simple and lie in the incorrect choice of constraints in the partition function as discussed below. It will require the right number and correct choice of statistical mechanics constraints to produce an exactly solvable, predictive scientific theory for basic atmospheric models.

Due to the doubly canonical form of its Gibbs ensemble or partition function, the classical energy-enstrophy theories are exactly-solvable Gaussian models which are not well defined at low numerical values of statistical temperatures [9]. However, most of the interesting physics

of transitions to domain-scale flows are expected to occur at sufficiently low numerical values of the relevant temperature. Note the important fact that the critical temperatures could be either positive or negative in the equilibrium statistical mechanics of many quasi-2D flows. In the latter case, the ordered phase usually coincides with negative temperatures that have numerical values smaller than the critical ones, which corresponds to flow states with extremely high kinetic energy. A non-Gaussian energy-entropy model is therefore the first requirement for a model that supports interesting transition physics in atmospheric flows.

What we offer here differs from previous works in three key ways: (A) formulation of a scientifically correct and solvable statistical mechanics theory of barotropic flows based on the canonical constraint on energy, the microcanonical constraint on entropy, total circulation and angular momentum, (B) this theory is not a mean-field theory and its partition function is non-Gaussian, (C) exact closed-form solutions of this model which predict qualitatively the phenomena of condensation at high enough energy to quadrupole vortices when the angular momentum is fixed at zero.

2.2. Canonical energy constraint

As explained above, the microcanonical entropy constraint in the spherical model is introduced to derive a non-Gaussian model that remains exactly solvable; the microcanonical constraint on total circulation follows from the topology of vorticity fields on a sphere. The canonical energy constraint here is associated with a reservoir that exchanges energy with intermediate and smaller scales in the flow that are bounded below by and widely separated from molecular scales. The smallest scales where viscous dissipation acts are not treated in our model. This is a crude first attempt to model the realistic energy exchange mechanism of planetary atmospheres which is largely based on insolation coupled to gravitational-thermal instability. Our adoption here of a simple barotropic model precludes any meaningful modeling of these effects beyond precisely that of a canonical constraint in a statistical mechanics setting. The logarithmic Green's function in the barotropic model, having finite long-range interactions, is not ideally treated by a reservoir, but the alternative choice of a microcanonical constraint on energy yields a model that cannot be solved in closed form.

Nonetheless, the role of constraints has been debated in the context of mean-field models — in summary, there can be significant physical and mathematical differences between canonical versus microcanonical constraints on energy when the interactions are of long range. For a review of the connections between several variational principles that have been introduced in statistical theories of 2D flows depending on the choice of the constraints, see [7].

Moreover, in this discussion of the physics of the canonical constraint on energy, it should be emphasized that the largest scales (near the domain length scale) in the barotropic flow are excluded by design from this exchange of energy with the reservoir — these scales comprise the long-range order from the phase transition and are treated in our application of the large N saddle point method as nonergodic scales separately from the smaller so-called ergodic ones.

2.3. Model equations

We will use spherical coordinates $\cos \theta$, where θ is the colatitude and longitude ϕ . The vorticity is given by

$$q(t; \cos \theta, \phi) = \Delta \Psi$$

in terms of the velocity stream function Ψ , where Δ is the negative of the Laplace–Beltrami operator on the unit sphere S^2 . Thus, a barotropic vorticity field, by Stokes' theorem, has the

following expansion in terms of spherical harmonics:

$$q(x, t) = \sum_{l \geq 1, m} \alpha_{lm}(t) \psi_{lm}(x). \quad (2.1)$$

Reference [21] establishes the simple fact that all the three spherical harmonics $\alpha_{1m}\psi_{1m}(x)$ contain the angular momentum, AM , in the barotropic flow, and conservation of angular momentum in the situation studied here, therefore, this implies that the amplitudes α_{1m} are fixed to be zero wlog. Similarly, the zero total circulation constraint is represented by setting the amplitude $\alpha_{00} = 0$.

2.4. Physical quantities of the barotropic vorticity model

The total kinetic energy of the fluid is

$$H[q] = \frac{1}{2} \int_{S^2} dx [u^2 + v^2] = -\frac{1}{2} \int_{S^2} dx \Psi(x) q(x),$$

where $u(x)$, $v(x)$ are the zonal and meridional components of the velocity and $\Psi(x)$ is the stream function for the velocity. Circulation in the model is fixed to be $\int q dx = 0$, which is a direct consequence of Stokes' theorem on a sphere. It is easy to see that the kinetic energy functional H does not have a minimum, without the further requirement of a constraint on the size of its argument, the vorticity field $q(x)$. A natural constraint for this quantity is therefore its square norm or enstrophy, which gives a further justification for fixing microcanonically the enstrophy in the spherical models below.

3. Spherical Model for Energy-Enstrophy Theory

There is a natural vectorial formulation of the above physical quantities on a 2D mesh over the sphere that leads to a spherical model for barotropic flows on a massive sphere [21]. We represent the normal vorticity q at site j by the vector

$$\vec{s}_j = s_j \vec{n}_j,$$

where $\vec{n}_j$ denotes the outward unit normal to the sphere S^2 at x_j . Denoting by γ_{jk} the angle subtended at the center of S^2 by the lattice sites $x_j = (\theta_j, \phi_j)$ and $x_k = (\theta_k, \phi_k)$, we obtain the following lattice energy functional for the total kinetic energy of a barotropic flow:

$$H_N = -\frac{1}{2} \sum_{j \neq k}^N J_{jk} \vec{s}_j \cdot \vec{s}_k, \quad (3.1)$$

where the interaction matrix is now given by the *long (finite) range*

$$J_{jk}(x_j, x_k) = \frac{16\pi^2}{N^2} \frac{\ln(1 - \cos \gamma_{jk})}{\cos \gamma_{jk}},$$

where the dot denotes the inner product in R^3 . Although long range in the sense that antipodal sites j and k on the planetary sphere have nonvanishing interaction J_{jk} , it is *finite* in the sense

that, as the size N of the mesh tends to infinity, H_N tends to the well-defined and finite limiting expression of the kinetic energy of relative flow, namely,

$$KE[q] = -\frac{1}{2} \int_{S^2} dx \, \psi(x) q(x) = -\frac{1}{2} \int_{S^2} dx' \int_{S^2} dx \, \Delta^{-1}(q(x')) q(x)$$

provided that its argument q is constrained to have fixed square-norm or enstrophy, $Q = \int_{S^2} dx \, q^2(x) > 0$.

The resulting spin-lattice spherical model parametrized by the number N of lattice sites in the mesh consists of a canonical Gibbs ensemble with the following microcanonical constraints:

(i) enstrophy

$$\frac{4\pi}{N} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j = Q > 0,$$

(ii) zero total circulation

$$\frac{4\pi}{N} \sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j = 0,$$

which will be imposed by dropping the lowest spherical harmonic ψ_{00} from the eigenfunction representation of the vorticity, and (iii) zero angular momentum AM which is fixed by setting the amplitudes with azimuthal wavenumber $l = 1$, $\alpha_{1m} = 0$. We note that the factor $\frac{4\pi}{N}$ is essentially the area of the unit sphere per lattice site.

We also note that the vector

$$\Gamma = \frac{4\pi}{N} \sum_{j=1}^N \vec{s}_j$$

is the so-called magnetization which turns out to be a natural order parameter in numerical simulations for the statistics of barotropic flows on a massive sphere.

4. Solution of the Spherical model

The partition function of the above spherical models is calculated using Laplace's integral form for path integrals

$$\begin{aligned} Z_N &\propto \int D(\vec{s}) \exp(-\beta H_N(\vec{s})) \delta\left(N - \frac{4\pi}{Q} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right) \delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM) \\ &= \int_{\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)} D(\vec{s}) \exp(-\beta H_N(\vec{s})) \left(\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta \left(N - \frac{4\pi}{Q} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right) \right) \\ &= \int_{\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)} D(\vec{s}) \exp\left(\frac{\beta}{2} \sum_{j \neq k}^N J_{jk} \vec{s}_j \cdot \vec{s}_k\right) \left(\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta \left(N - \frac{4\pi}{Q} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right) \right), \end{aligned}$$

where $a > 0$ is chosen to be large enough for the calculation below. We have replaced the microcanonical enstrophy constraint, $\delta\left(Q - \frac{4\pi}{N} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)$ by the mathematically equivalent $\delta\left(N - \frac{4\pi}{Q} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)$, and then substituted it by the mathematically equivalent Laplace integral $\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta\left(N - \frac{4\pi}{Q} \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right)$. Thus, we get

$$\begin{aligned} Z_N &= \int_{\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)} D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(N \left(\eta - \frac{4\pi}{QN} \eta \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j \right. \right. \\ &\quad \left. \left. + \frac{\beta}{2N} \sum_{j \neq k} J_{jk} \vec{s}_j \cdot \vec{s}_k \right) \right) \\ &= \int_{\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)} D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(N \left(\eta - \frac{1}{N} \sum_{j,k} K_{jk}(Q, \beta, \eta') \vec{s}_j \cdot \vec{s}_k \right) \right), \end{aligned}$$

where

$$K_{jk}(Q, \beta, \eta; x_j, x_k) = \begin{cases} \frac{4\pi}{Q} \eta & j = k \\ -\frac{\beta}{2} J_{jk}(x_j, x_k) & j \neq k \end{cases}.$$

To evaluate the Gaussian integrals in Z_N , we expand the vorticity vector field in terms of the spherical harmonics,

$$\vec{q}(x) = \sum_{l=2}^{\infty} \sum_{m=-l}^l \alpha_{lm} \psi_{lm}(x) \vec{n}(x),$$

where $\vec{n}(x)$ is the outward unit normal to S^2 at x . We stress that this expansion need not include the harmonic $\psi_{00}(x)$ because of the zero circulation condition on vorticity $\vec{s}$, nor $\psi_{1m}(x)$, $m = -1, 0, 1$ due to the zero angular momentum microcanonical constraint.

Solution of the Gaussian integrals requires diagonalizing the interaction in H_N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=2}^{\infty}$, which are natural Fourier modes for Laplace–Beltrami eigenvalue problems on S^2 with zero circulation and angular momentum:

$$\begin{aligned} \vec{s}_j &= \vec{n}_j \sum_{l=2}^{\infty} \sum_{m=-l}^l \alpha_{lm} \psi_{lm}(x_j) \\ -\frac{1}{2} \sum_{j \neq k} J_{jk} \vec{s}_j \cdot \vec{s}_k &= \frac{1}{2} \sum_{l=2}^{\infty} \sum_{m=-l}^l \lambda_{lm} \alpha_{lm}^2, \end{aligned}$$

where the eigenvalues of the Green's function for the Laplace–Beltrami operator on S^2 are

$$\lambda_{lm} = \frac{1}{l(l+1)}, \quad l > 1, \quad m = -l, \dots, 0, \dots, l$$

and α_{lm} are the corresponding amplitudes. Thus,

$$\frac{1}{N} \sum_{j,k}^N K_{jk}(Q, \beta, \eta) \vec{s}_j \cdot \vec{s}_k = \sum_{l=2}^{\infty} \sum_{m=-l}^l \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2$$

and

$$\begin{aligned} Z_N &= \int_{\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)} D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp \left(N \left(\eta - \frac{1}{N} \sum_{j,k}^N K_{jk}(Q, \beta, \eta') \vec{s}_j \cdot \vec{s}_k \right) \right) \\ &= \int \prod_{m=-2}^2 d\alpha_{2m} \int D_{l \geq 3}(\alpha) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp \left\{ N \left[\begin{aligned} &\eta - \sum_{l=3}^{\infty} \sum_{m=-l}^l \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \\ &- \left(\frac{\beta}{12N} + \frac{\eta}{Q} \right) \sum_{m=-2}^2 \alpha_{2m}^2 \end{aligned} \right] \right\} \end{aligned}$$

where we have split off the integral over the ground state spherical harmonics ψ_{2m} , $m = -2, -1, 0, 1, 2$. They represent the so-called nonergodic mode which motivates this split. The microcanonical constraints $\delta\left(\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j\right) \delta(AM)$ have been realized by neglecting the spherical harmonics ψ_{lm} with $l = 0, 1$, $m = -l, \dots, l$, from the partition function in the last line of the equation.

By choosing $\text{Re}(\eta) = a > 0$ large enough, we interchange the order of integration of the term $\exp\left(-N \sum_{l=3}^{\infty} \sum_{m=-l}^l \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2\right)$ to obtain

$$\begin{aligned} Z_N &\propto \int \prod_{m=-2}^2 d\alpha_{2m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp \left\{ N \left[\eta - \left(\frac{\beta}{12N} + \frac{\eta}{Q} \right) \sum_{m=-2}^2 \alpha_{2m}^2 \right] \right\} \\ &\quad \int D_{l \geq 3}(\alpha) \exp \left(-N \sum_{l=3}^{\infty} \sum_{m=-l}^l \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right). \end{aligned}$$

4.1. Restricted partition function and nonergodic modes

Next, we write the problem in terms of the restricted partition function $Z_N(\alpha_{2m}; \beta, Q)$, that is,

$$\begin{aligned} Z_N(\beta, Q) &\propto \int \prod_{m=-2}^2 d\alpha_{2m} Z_N(\alpha_{2m}; \beta, Q) \\ &= \int \prod_{m=-2}^2 d\alpha_{2m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp \left\{ N \left[\eta - \left(\frac{\beta}{12N} + \frac{\eta}{Q} \right) \sum_{m=-2}^2 \alpha_{2m}^2 \right] \right\} \\ &\quad \int D_{l \geq 3}(\alpha) \exp \left(-N \sum_{l \geq 3} \sum_{m=-l}^l \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right). \end{aligned}$$

Due to nonergodicity of the ordered domain-scale nature of modes ψ_{2m} , we do not integrate over these ordered modes whose amplitudes are denoted by α_{2m} . We also note that all the higher harmonics will turn out to have zero amplitudes in the ordered phase of this problem. The statistics of the problem are therefore completely determined by the restricted partition function $Z_N(\alpha_{2m}; \beta, Q)$. The amplitudes of the ordered modes appear as parameters in this restricted partition function, and will have to be evaluated separately.

Standard Gaussian integration is used to evaluate the last integral, which yields, after using the nonextensive thermodynamic limit scaling, $\beta'N = \beta$,

$$\int_{l \geq 3} D(\alpha) \exp \left(- \sum_{l=3} \sum_{m=-l}^l \left(\frac{\beta'N \lambda_{lm}}{2} + \frac{N\eta}{Q} \right) \alpha_{lm}^2 \right) = \prod_{l=3}^{\sqrt{N}} \prod_{m=-l}^l \left(\frac{\pi}{\frac{N\eta}{Q} + \frac{\beta'N}{2} \lambda_{lm}} \right)^{1/2},$$

provided that the physically significant Gaussian conditions hold: for $l \geq 3$,

$$\frac{\beta' \lambda_{lm}}{2} + \frac{\eta}{Q} = \frac{\beta'}{2l(l+1)} + \frac{\eta}{Q} > 0. \quad (4.1)$$

Substituting these Gaussian expressions into the partition function gives

$$Z_N(\alpha_{2m}; \beta, Q) \propto \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp \left\{ N \left[\eta - \left(\frac{\beta'}{12} + \frac{\eta}{Q} \right) \sum_{m=-2}^2 \alpha_{2m}^2 - \frac{1}{2N} \sum_{l=3} \sum_m \ln \left(\frac{N\eta}{Q} + \frac{\beta'N}{2} \lambda_{lm} \right) \right] \right\},$$

where the partition function is now in the form where the saddle point method can be applied for large N . Doing that yields the free energy per site, f , evaluated at the most probable macrostate, to be given by

$$f(\beta', Q) = U - TS = -\frac{1}{\beta'} \ln Z_N(\alpha_{2m}; \beta', Q) \quad (4.2)$$

$$= -\frac{1}{\beta'} F(\eta(\beta'), Q, \beta') \quad (4.3)$$

with

$$\begin{aligned} F(\eta(\beta'), Q, \beta') &= \eta'(\beta') \left[1 - \frac{1}{Q} \sum_{m=-2}^2 \alpha_{2m}^2 \right] - \frac{\beta'}{12} \sum_{m=-2}^2 \alpha_{2m}^2 \\ &\quad - \frac{1}{2N} \sum_{l=3} \sum_m \ln \left(\frac{N\eta(\beta')}{Q} + \frac{\beta'N}{2} \lambda_{lm} \right). \end{aligned}$$

Note that “the free energy per site evaluated exactly at the most probable macrostate” is precisely extracting the dominant term, as $N \rightarrow \infty$, in the α_{2m} -parametrized integral $Z_N(\alpha_{2m}; \beta, Q)$, to be that given by the saddle point, $\eta = \eta(\beta')$. In other words,

$$(Z_N(\alpha_{2m}; \beta, Q))^{1/N} \rightarrow_{N \rightarrow \infty} \exp [F(\eta(\beta'), Q, \beta')]. \quad (4.4)$$

4.2. Saddle Points and Energy Threshold for Phase Transitions

Provided that the saddle point $\eta(\beta')$ can be determined at the given inverse temperature β' , the stable (most probable) macrostate is given by the extremum of the expression $F(\eta(\beta'), Q, \beta')$. The saddle point condition gives one equation for the determination of six variables η, α_{2m} in terms of inverse temperature β' and enstrophy Q ,

$$0 = \frac{\partial F}{\partial \eta'} = \left(1 - \frac{1}{Q} \sum_{m=-2}^2 \alpha_{2m}^2\right) - \frac{1}{2NQ} \sum_{l=3} \sum_m \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2} \lambda_{lm}\right)^{-1}, \quad (4.5)$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point.

We need five more conditions to determine the amplitudes α_{2m} and the saddle point $\eta(\beta') > 0$. They are provided by the following equations of state representing the variational principle for free energy at stable states:

$$0 = \frac{\partial F}{\partial \alpha_{2m}} = - \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{6}\right) \alpha_{2m}. \quad (4.6)$$

Thus, a coupled system of 5 algebraic equations (4.5), (4.6) determines six unknowns in terms of the given enstrophy $Q > 0$ and the scaled inverse temperature β' . The equations of state for α_{2m} imply that for $m = -2, -1, 0, 1, 2$ either

$$\alpha_{2m} = 0 \text{ or } \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{6}\right) = 0.$$

Since the saddle point is on the line with real part, $a > 0$, we deduce $\eta(\beta') > 0$, and with enstrophy $Q > 0$ we obtain

$$\beta' = -\frac{12\eta}{Q} < 0$$

when $|\alpha_{2m}| > 0$ for at least one value of m . This proves that in order to have positive energy, $\alpha_{2m} \neq 0$, in the condensed modes, the system's kinetic energy must be sufficiently high to make T' not only negative but numerically small (and inverse temperature $\beta' < 0$ and numerically large). Thus, we conclude that the critical temperature $T_c < 0$.

In addition, the above Gaussian conditions imply that for $l > 2$

$$\frac{\beta'}{2l(l+1)} + \frac{\eta(\beta')}{Q} > 0,$$

which implies that the large N limit in the RHS of the saddle point condition is well-defined and finite:

$$\left(1 - \frac{1}{Q} \sum_{m=-2}^2 \alpha_{2m}^2\right) = \lim_{N \rightarrow \infty} \frac{1}{2NQ} \sum_{l=3} \sum_m \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \quad (4.7)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2NQ} \sum_{l=3} \sum_m \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1}. \quad (4.8)$$

When either $T' > 0$ or $T' < 0$ and numerically large, there is no energy in the condensed modes with $l = 2$, that is,

$$\left(1 - \frac{1}{Q} \sum_{m=-2}^2 \alpha_{2m}^2\right) = 1.$$

But for sufficiently high energy or hot enough negative temperatures, given by a threshold $T'_c < T' < 0$, the energy in these modes increases until they contain all of the fixed enstrophy Q at absolute zero $T' = 0^-$:

$$\left(1 - \frac{1}{Q} \sum_{m=-2}^2 \alpha_{2m}^2\right) \searrow 0.$$

So to calculate $T'_c(Q) < 0$, we need to find the most negative value of inverse temperature, denoted by β'_c , for which

$$1 = \lim_{N \rightarrow \infty} \frac{1}{2NQ} \sum_{l=3}^{\sqrt{N}} \sum_{m=-l}^l \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1}, \quad (4.9)$$

and after which, for values of $\beta' < \beta'_c < 0$, $\lim_{N \rightarrow \infty} \frac{1}{2NQ} \sum_{l=3}^{\sqrt{N}} \sum_{m=-l}^l \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} < 1$.

Inserting $\eta = -\frac{\beta'Q}{12}$ into RHS of the above equation, we calculate β'_c to be given by

$$-\infty < \beta'_c = \lim_{N \rightarrow \infty} \frac{1}{2NQ} \sum_{l=3}^{\sqrt{N}} \sum_{m=-l}^l \left(-\frac{1}{12} + \frac{1}{2l(l+1)} \right)^{-1} \quad (4.10)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{QN} \sum_{l=3}^{\sqrt{N}} \sum_{m=-l}^l \left(\lambda_{lm} - \frac{1}{6} \right)^{-1} < 0 \quad (4.11)$$

and check that, for $\beta' < \beta'_c(Q) < 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{\beta'NQ} \sum_{l=3}^{\sqrt{N}} \sum_m \left(-\frac{1}{6} + \lambda_{lm} \right)^{-1} < 1.$$

We note the significance of the linear dependence of the critical temperature on the given enstrophy Q of the flow.

In other words, the extreme saddle point

$$\eta^* = -\frac{\beta'Q}{12}$$

is no longer adequate to solve (4.9) for $\beta' < \beta'_c(Q) < 0$, which is the so-called *sticking of the saddle point* at the critical point β'_c .

Moreover, the saddle point equation gives us a way to compute the equilibrium amplitudes of the superrotating modes for temperatures hotter than the negative critical temperature T_c . For temperatures T such that $T_c < T < 0$,

$$\sum_{m=-2}^2 \alpha_{2m}^2(T) = Q \left(1 - \frac{T}{T_c} \right).$$

5. Conclusion

We have given a physically natural formulation of the constraints for an energy-enstrophy theory of barotropic flows on a spherical surface which led to an exactly solvable spherical model. In the nonextensive thermodynamic limit (the correct one for fixed domain fluid flow problems), we gave closed-form expressions for the free energy, critical temperature $T'_c < 0$, and the critical exponent of condensation into the quadrupolar modes, where the spherical harmonics ψ_{2m} , $m = -2, \dots, 2$, have nonzero amplitudes $\alpha_{2m}(Q, \beta')$.

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Conflict of Interest

The author declares that he has no conflict of interest.

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